Angular momentum $L$ was introduced in classical mechanics together with the moment of inertia $I$ to have for rotations the same role that momentum $(p)$ and mass $m$ have for linear motion. These pair of quantities the only once needed to calculate the kinetic energy for rotation and linear motion of a body. The angular momentum classically is defined as $\vec{L} = \vec{r} \times \vec{p}$ where $O$ is the origin of the system. It can be shown that the angular momentum for a system of particles is invariant in respect of the choice of origin. Angular momentum is the only measurable that has not a perfect quantum mechanical counter part. In fact a quantum mechanical angular momentum is an operator that has the following properties:

$$
\hat{J}_i = \hat{J}_x + \hat{J}_y + \hat{J}_z
$$

$$
[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z
$$

$$
[\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x
$$

$$
[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y
$$

Defining $\hat{J}^2 = \hat{J} \cdot \hat{J} = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

Just form the analysis of the defined properties it can be shown that:

$$
[\hat{J}^2, \hat{J}_i] = 0
$$

$i = x, y, z$

Introducing two new operators $J_i = J_x \pm iJ_y$ and studying their properties in a similar way to the way it is done for the solution of the linear harmonic oscillator we can find the mathematical expression for both the eigenvalues and the eigenfunctions of this operators, they are $m, l,$ and $Y_i^m(\theta, \phi)$ respectively. It can be further proved that

$$-l \leq m \leq l$$

and $l = \frac{n}{2}$ with $n$ integer or zero.

We can now introduce the orbital angular momentum the is a particular angular momentum and is the quantum mechanical counter part to the classical angular momentum:

$$\hat{L} = \hat{r} \times \hat{p} = \frac{\hbar}{i} (r \times \nabla)$$

Now recalling the definition of the laplacian in spherical coordinates:

$$\nabla^2 = \frac{1}{r} \frac{\partial^2}{\partial r^2} \frac{r}{r^2} + \frac{(r \times \nabla)^2}{r^2}$$

it is easy to see the expression of the Hamiltonian in spherical coordinates is:

$$H = \frac{\hbar^2}{2m} \nabla^2 + V = \frac{\hbar^2}{2m} \frac{1}{r} \frac{\partial^2}{\partial r^2} \frac{r}{r^2} + \frac{\hbar^2}{2m} \frac{(r \times \nabla)^2}{r^2} + V(r, \theta, \phi) = -\frac{\hbar^2}{2m} \frac{1}{r} \frac{\partial^2}{\partial r^2} \frac{r}{r^2} + \frac{\hat{L}^2}{2m} + V(r, \theta, \phi)$$

Now under the assumption of $V(r, \theta, \phi) = V(r)$ (central potential) it can be shown that:
\[
\left[ \hat{H}, \hat{L}^2 \right] = 0
\]
This means that in a central potential the Hamiltonian and the square of the angular momentum always commute, i.e. have the same eigenfunction within a constant that does not depend on \( \theta, \phi \).

We can write the eigenfunction of the Hamiltonian as follows:
\[
\psi(r, \theta, \phi) = R(r) \Phi_{lm}^m(\theta, \phi)
\]
introducing the following new function: \( u(r) = rR(r) \) and knowing that \( \hat{L}^2 Y_{lm}^m(\theta, \phi) = l(l+1)\hat{h}^2 Y_{lm}^m(\theta, \phi) \) it is easy to show that the eigenvalue problem can be rewritten as follows:
\[
\hat{H} \Psi = E \Psi \longrightarrow \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E \right) u(r) = 0
\]
This equations are true for all central potentials.

Now we start analyzing the hydrogen atom, giving an expression to the potential:
\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + \frac{e^2}{r} - E \right) u(r) = 0
\]
The best way to solve this equation is to look at the asymptotic behavior.
For \( r \to 0 \)
We can easily see that \( \frac{\hbar^2 l(l+1)}{2mr^2} \to \infty \) thus in order to have a solution for the differential equation we need to impose \( u(0) = 0 \) with this condition in mind and knowing that \( 1/r^2 \) goes to infinity faster then \( 1/r \) we can assume the close to the origin \( u(r) \) will be a positive power function of \( r \) for example we can assume \( u(r) = Cr^{s+1} \). We can thus rewrite the eigenvalue problem as:
\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2 l(l+1)}{2mr^2} \right) Cr^{s+1} = 0
\]
With simple calculations we find that in order for this equation to be satisfied we need to impose \( s = l \) or \( s = -(l+1) \). This second condition violated the initial assumption that \( s > 0 \) (being \( l \) always positive) so it needs to be discarded.
For \( r \to \infty \) we can rewrite the eigenvalue problem as:
\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} - E \right) u(r) = 0 \] whose only physical solution is \( u(r) = Ae^{-kr} \) with \( k = \sqrt{\frac{2mE}{\hbar^2}} \)
Now in order to solve completely the full equation knowing out asymptotic behavior we can assume that the general for of our equation will be the product of an exponentials for a polynomial: \( u(r) = e^{-kr} \sum C_q r^q \)
Substituting this form of the equation into the complete Hamiltonian and using the asymptotic behavior at zero as found previously we can solve analytically the equation for the hydrogen atom.