Recitation Notes.

Linear Harmonic oscillator.
Quantum mechanical solution

A linear harmonic oscillator is a potential well with the following formula:
\[ V(x) = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2 \] with \( k = m\omega^2 \). This is a nice example of how the correct choice of operator can simplify the solution of a problem and enhance the physical insight we gain from it.

First we write the Hamiltonian of the system
\[ \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \] then we write the eigenvalue problem
\[ \hat{H}\phi = E\phi \]

It can be shown rigorously that if we call \( V_m \) the minimum value of the potential energy, the eigenvalue problem has solutions only for \( E=V_m \). In this particular case \( E=0 \).

Because we will work with operators to solve this problem it is convenient to change our operators with dimensionless operators so not to have to drag around many constants. This is the only reason for these substitutions.

\[ \hat{H}' = \frac{1}{\hbar\omega} \hat{H} \]
\[ \hat{X}' = \sqrt{\frac{m\omega}{\hbar}} \hat{X} \]
\[ \hat{P}' = \frac{1}{\sqrt{m\omega\hbar}} \hat{P} \]

From now on we will indicate the new operators as \( H, X \) and \( P \).
The first thing to notice is that: \( [X, P] = i \)

The new Hamiltonian can be written as:
\[ H = \frac{1}{2} (X^2 + P^2) \]

We can rewrite the eigenvalue problem using this new Hamiltonian as follows:
\( H\phi = \epsilon\phi \) keeping in mind the newly introduced eigenvalue \( \epsilon \) is a number related to the energy \( E \) by \( E = \hbar \omega \epsilon \)

Let's now start solving the eigenvalues problems introducing the best operators for this specific problem:
The first operators to be introduced are \( a \) and its adjoint defines as follows:
\[
a = \frac{1}{\sqrt{2}} (X + iP)
\]
\[
a^+ = \frac{1}{\sqrt{2}} (X - iP)
\]
\[
X = \frac{1}{\sqrt{2}} (a + a^+)
\]
\[
P = \frac{1}{\sqrt{2}} (a - a^+)
\]

First we have to notice that \(a\) and \(a^+\) are not hermitian, thus they do not have to commute, in fact we can see that:

\[
[a,a^+] = \frac{1}{2} [X + iP, X - iP] = \frac{1}{2} \{[X,X] + [P,P] + iP[X] - iP[X,P]\} = \frac{i}{2} \{[P,X] - [X,P]\} = 1
\]

We can now easily calculate the product of these two operators:

\[
a^+a = \frac{1}{\sqrt{2}} (X + iP) \cdot \frac{1}{\sqrt{2}} (X - iP) = \frac{1}{2} (X^2 + P^2 + iP[X,P]) = \frac{1}{2} (X^2 + P^2 - 1)
\]

And the define a new operator as:

\[N = a^+a\]

We can see that this new operator is related to the Hamiltonian by the following equation:

\[H = \frac{1}{2} (X^2 + P^2) = \frac{1}{2} (X^2 + P^2 - 1 + 1) = \frac{1}{2} (X^2 + P^2 - 1) + \frac{1}{2} = N + \frac{1}{2}\]

And we can rewrite the eigenvalue problem:

\[(N + \frac{1}{2})\phi = \varepsilon\phi \rightarrow N\phi = (\varepsilon - \frac{1}{2})\phi \rightarrow N\phi = \nu\phi\]

\[\varepsilon = \nu + \frac{1}{2}\]

\[E = (\nu + \frac{1}{2})\hbar\omega\]

We have a new eigenvalue problem that is related to the original one. Solving it will allow us to solve the original problem. Note that the eigenfunctions staid the same through all these transformations.

Let's now analyze the physical properties of the newly introduced operators:

A) we know that for any function \(|\psi|^2 \geq 0\) so we can consider the function \((\psi' = a\phi_\nu)\) the is derived by the operator \(a\) applied on eigenfunction that solves our eigenvalue problem \(\phi_\nu \rightarrow N\phi_\nu = \nu\phi_\nu\).
In that case:

\[ 0 \leq |a\phi_o|^2 = \langle a\phi_o|a\phi_o \rangle = \langle \phi_o|a^+a|\phi_o \rangle = \langle \phi_o|N|\phi_o \rangle = \nu \langle \phi_o|\phi_o \rangle \geq 0 \]

\[ \langle \phi_o|\phi_o \rangle = |\phi_o|^2 \geq 0 \quad \Rightarrow \quad \nu \geq 0 \]

So the first conclusion is that \( \nu = 0 \).

B) Now if in the previous set for equation we would have chosen the particular eigenfunction that depends on the eigenvalue \( \nu = 0 \), we would have had:

\[ |a\phi_o|^2 = \langle a\phi_o|a\phi_o \rangle = \langle \phi_o|a^+a|\phi_o \rangle = \langle \phi_o|N|\phi_o \rangle = \nu \langle \phi_o|\phi_o \rangle = 0 \]

\[ |a\phi_o|^2 = 0 \quad \Rightarrow \quad |a\phi_o| = 0 \]

This equation allows us to find the first eigenfunction independently from the Hamiltonian of the system.

C) Now let us assume that \( \nu > 0 \). Knowing that \( [N,a] = -a \) we can write:

\[ [N,a]\psi = -a\psi \]

for any function. If we do it for the eigenfunction that solves our eigenvalue problem

\[ \phi_o \rightarrow N\phi_o = \nu \phi_o \]

we have:

\[ [N,a]\phi_o = -a\phi_o \]

\[ N(a\phi_o) - aN\phi_o = -a\phi_o \]

\[ N\phi_o = a\nu \phi_o - a\phi_o = a\nu \phi_o - a\phi_o = (\nu - 1)a\phi_o \]

\[ N(a\phi_o) = (\nu - 1)(a\phi_o) \]

\[ a\phi_o = C\phi_{0-1} \]

What this means is that applying the operator \( a \) on an eigenfunction, within a normalization constant, we get the eigenfunction that depends on the eigenvalue \( \nu - 1 \).

For this reason \( a \) is called the destruction operator.

Note that this proof is not valid for \( \nu = 0 \).

D) With argument similar to the ones in the points B and C we can prove that

\[ |a^+\phi_o| > 0 \]

and

\[ a^+\phi_o = C\phi_{0+1} \]

What this means is that applying the operator \( a^+ \) on an eigenfunction, within a normalization constant, we get the eigenfunction that depends on the eigenvalue \( \nu + 1 \).

For this reason \( a \) is called the creation operator.

E) It can be shown that not to break condition A the ONLY value that \( \nu \) can assume are integers.

Thus we know at this point without having done any calculation the eigenvalues of our problem and they are:
\[ E = (\nu + \frac{1}{2})\hbar\omega \]
\[ \nu = 0,1,2,3,\ldots, n \]

Let's know go back to the final equation in point B and solve it to find the first eigenfunction of our problem:

\[ |a\varphi_0\rangle = 0 \]
\[ \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{m\omega}{\hbar}} x + \frac{i}{\sqrt{m\omega\hbar}} \left( -\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \right] \varphi_0 = 0 \rightarrow \left( \frac{m\omega}{\hbar} x + \frac{\partial}{\partial x} \right) \varphi_0 = 0 \]
\[ \varphi_0 = C e^{-\frac{m\omega}{2\hbar} x^2} \rightarrow \text{with} \langle \varphi_0 | \varphi_0 \rangle = 1 \rightarrow C = \left( \frac{m\omega}{\pi\hbar} \right)^{\frac{1}{2}} \]

It's that easy!

Now all we have to do is apply the creation operator n times to this eigenfunction and find all the solutions, following the formula:

\[ \varphi_n = C_n (a^+)^n \varphi_0 \]

It can be shown that in order to have all the eigenfunctions normalized
\[ C_n = \frac{1}{\sqrt{n!}} \]

Now we can write the general expression for the eigenfunction of our system:

\[ \varphi_n = \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{2^n}} \left( \sqrt{\frac{m\omega}{\hbar}} x - \sqrt{\frac{\hbar}{m\omega}} \frac{\partial}{\partial x} \right)^n \left( \frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{1}{2} \frac{m\omega}{\hbar} x^2} \]