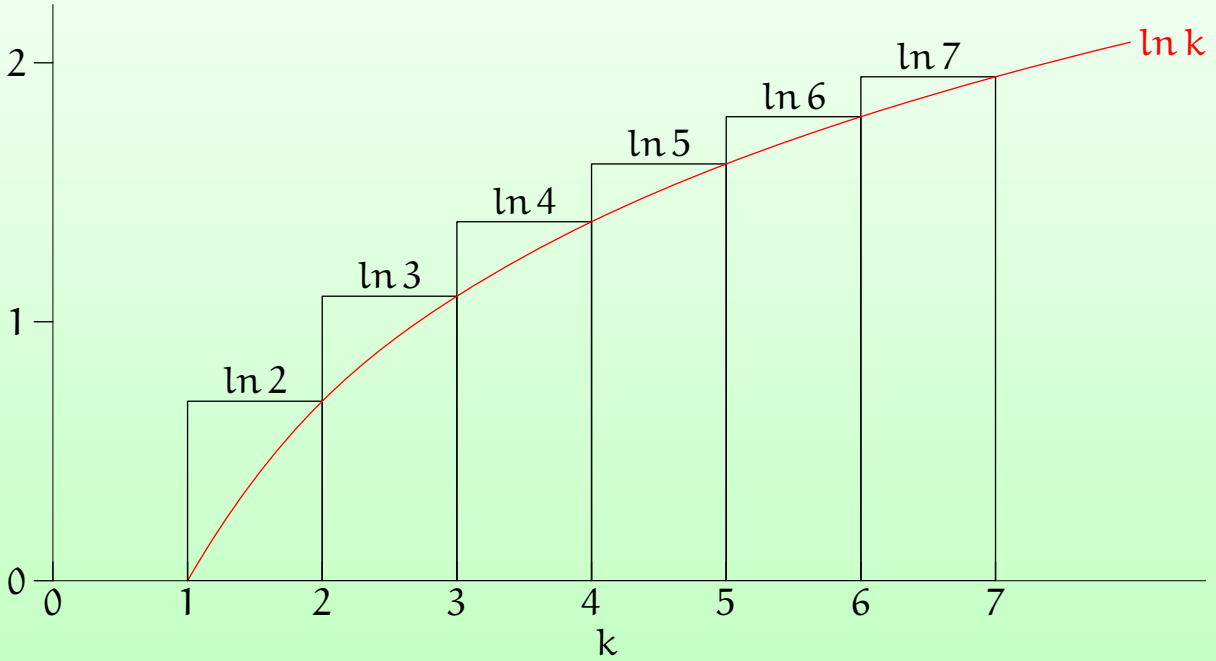
Pictures help to accurately approximate $\ln n!$

$$3! = 3 \times 2 \times 1;$$
$$4! = 4 \times 3 \times 2 \times 1;$$
$$5! = 5 \times 4 \times 3 \times 2 \times 1;$$
$$\ldots.$$

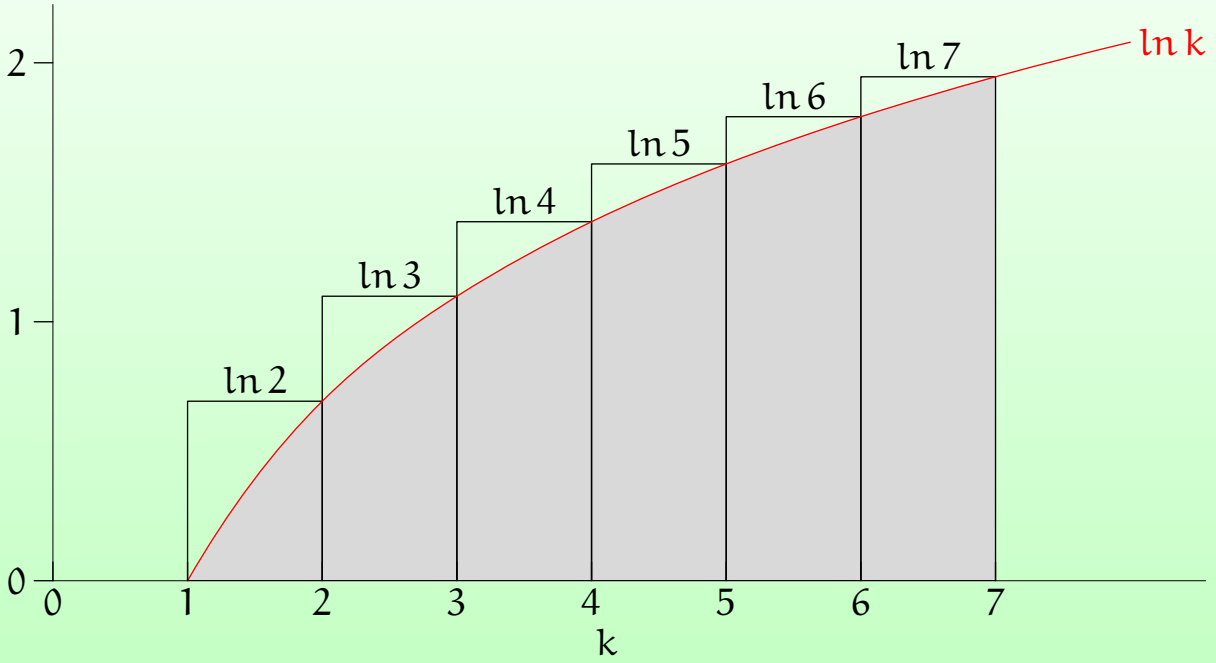
$n!$ is the most important function in statistical mechanics.

$\ln n!$ is the area of the rectangles



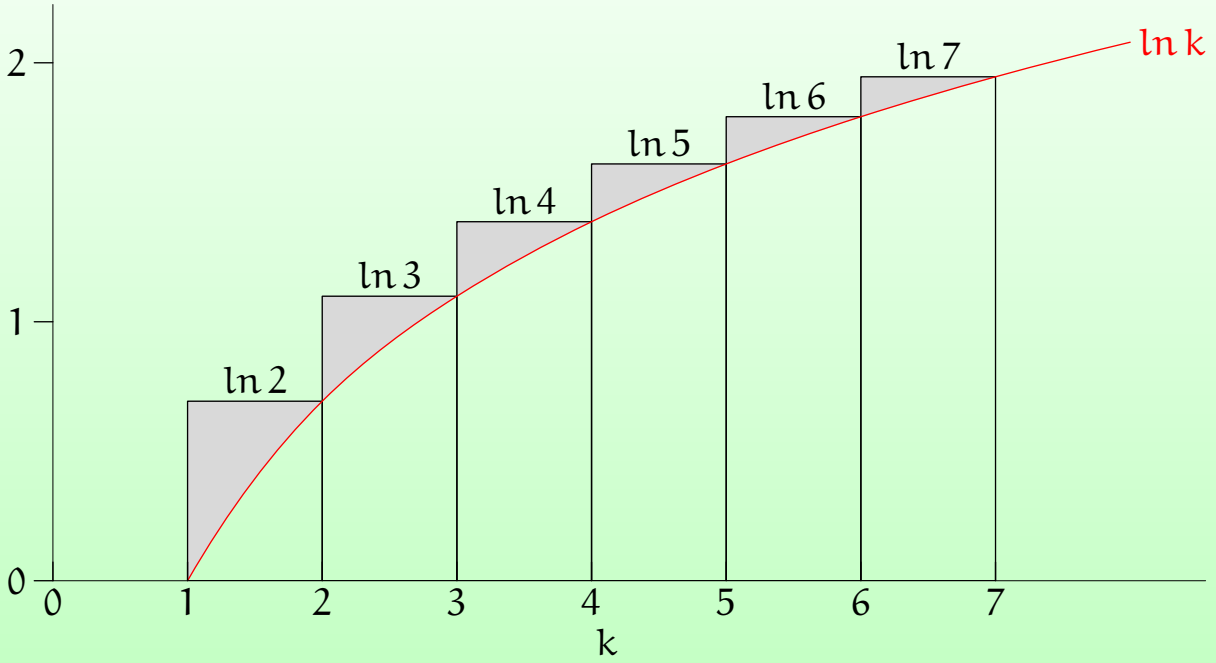
$$\ln n! = \sum_1^n \ln k$$

The area under $\ln k$ is the first approximation

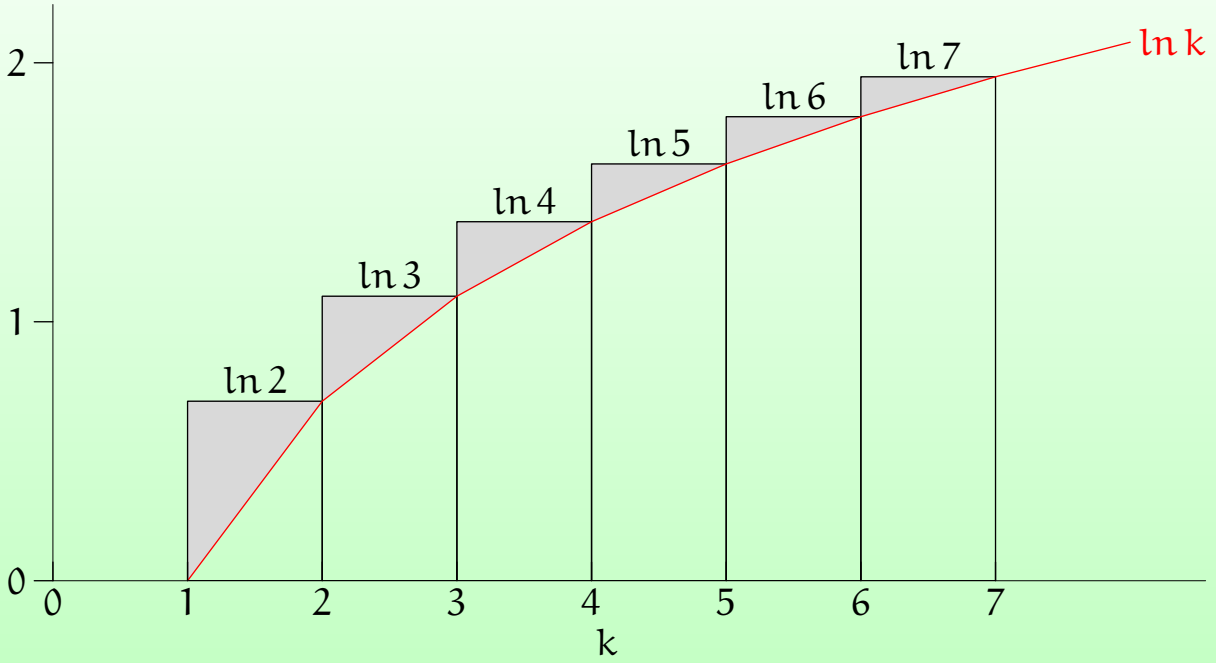


$$\ln n! \approx \int_1^n \ln k \, dk = n \ln n - n + 1$$

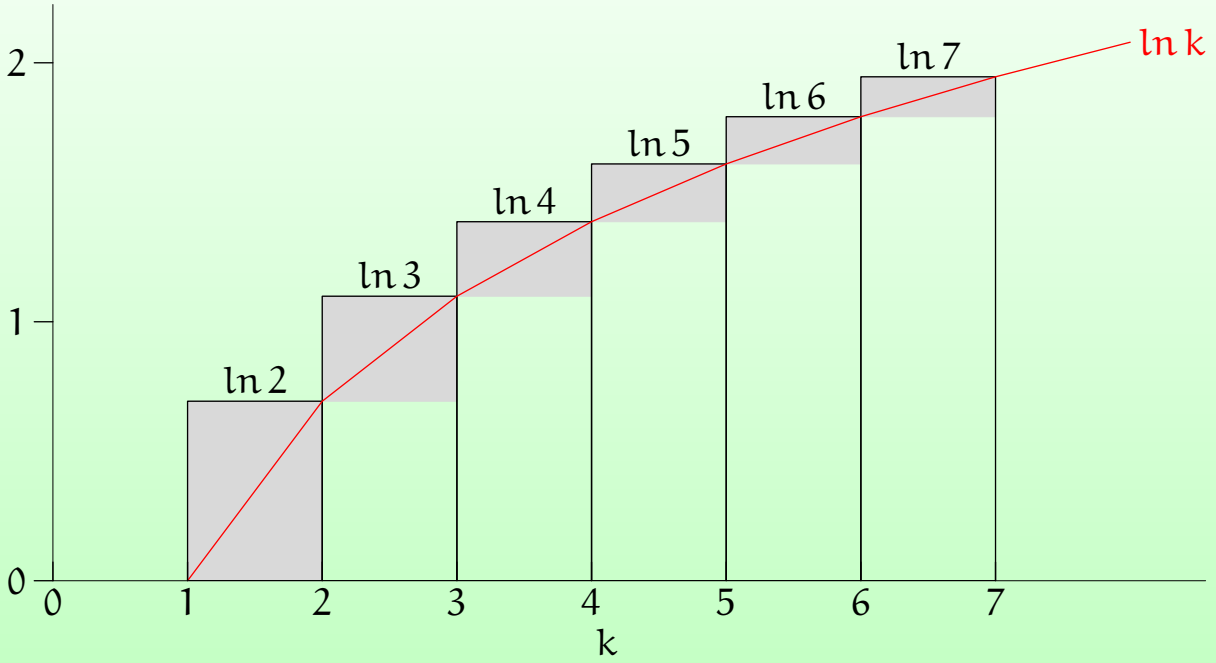
The error is the protruing pieces



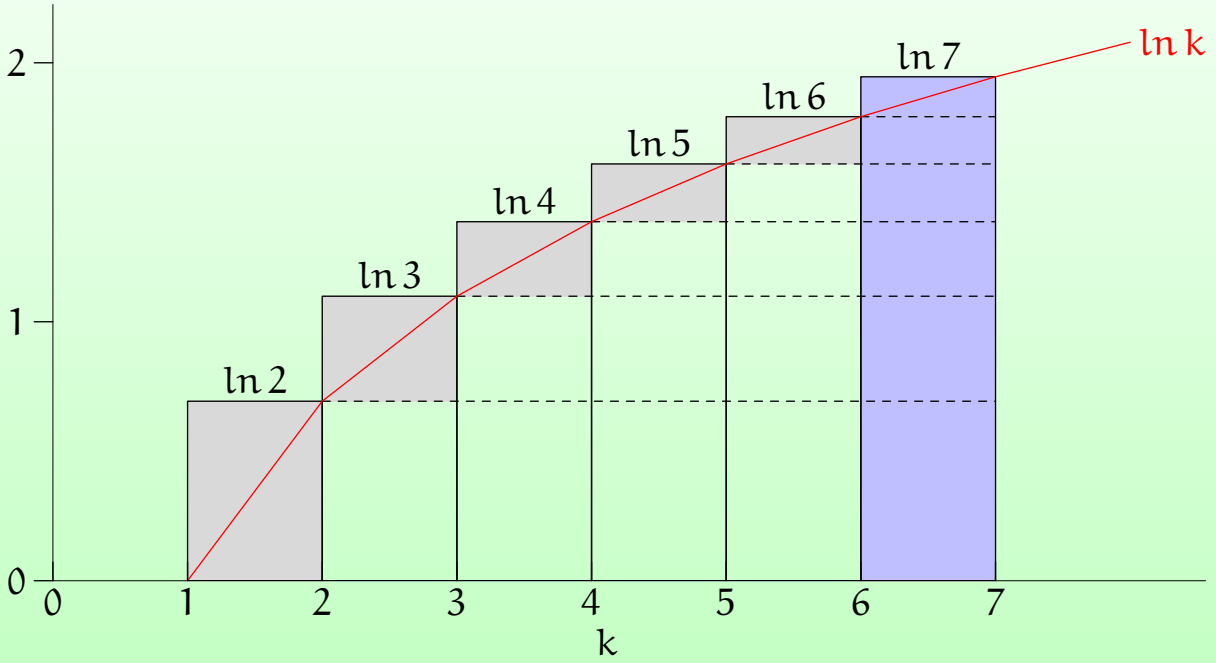
Each piece is almost a triangle



Doubling the 'triangles' makes them easier to add



The rectangles slide across and stack at the end

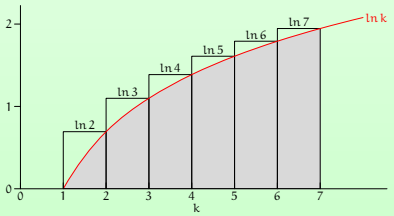


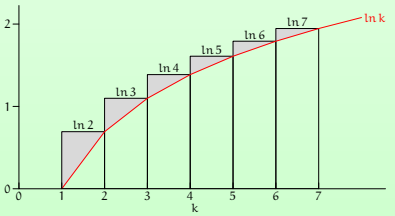
Sum of doubled protrusions $= \ln n$

Combine the integral and approximated protrusion

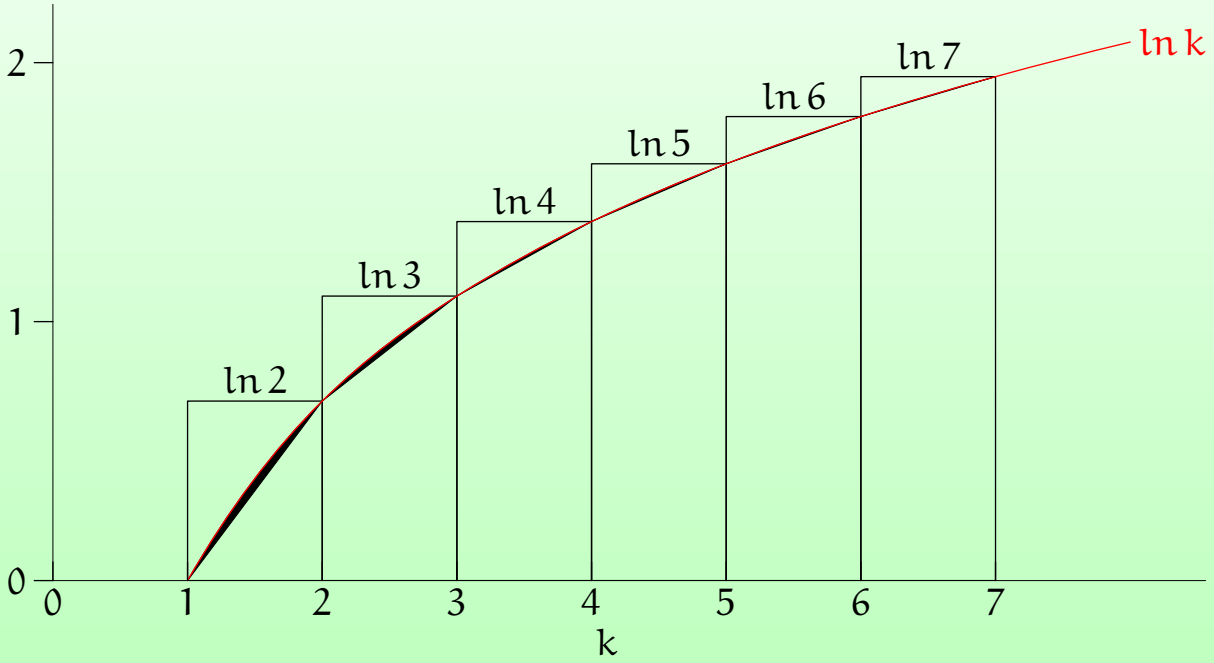
$$\ln n! = \sum_{1}^{n} \ln k$$

$$\approx \underbrace{n \ln n - n + 1}_{} + \underbrace{\frac{1}{2} \ln n}_{}$$





The preceding pictorial approximation ignores only a tiny region



Numerical calculation confirms the accuracy

Picture: $7 \times (\ln 7 - 1) + 1 + \frac{1}{2} \ln 7 = 8.594 \ldots\ldots$

Exact: $\sum_{1}^{7} \ln k = 8.525 \ldots\ldots$

The approximation makes error of 0.07 in $\ln 7!$ (which results in a 7% error in $7!$).