Problem 10.1 Answer:

(A) From previous problems, we know that \( i_L(0^+) = 0 \) because the inductor current will not change instantaneously. We have also shown in lecture that the peak inductor current will be twice the source current \( I_S \), and the circuit will ring at \( \omega = \frac{1}{\sqrt{LC}} \). Given this information, we can write

\[
i_L(t) = I_S \left[ 1 - \cos \left( \frac{t}{\sqrt{LC}} \right) \right]
\]

(B) Again, we can write the answer for this problem down directly. We know that \( \omega = \frac{1}{\sqrt{LC}} \), \( i_L(0^+) = I_L \), and that the peak value of \( i_L(t) = I_L \). Given this we can write

\[
i_L(t) = I_L \left[ \cos \left( \frac{t}{\sqrt{LC}} \right) \right]
\]

(C) The solution for \( i_L(t) \) has the general form

\[
i_L(t) = Ae^{\frac{j}{\sqrt{LC}}} + Be^{-\frac{j}{\sqrt{LC}}} + I_S
\]

We know that \( i_L(0^+) = I_L \) because the inductor current is continuous. Additionally, from \( t = 0^- \rightarrow 0^+ \) the capacitor voltage stays constant at \( t = 0 \) because there is not a source of infinite current in the circuit. So \( \frac{d}{dt}i_L(0^+) = 0 \) because \( v_C(t) = v_L(t) = L \frac{di_L(t)}{dt} \). Given this, we can write

\[
I_L - I_S = A + B \quad \text{and} \quad 0 = \frac{jA}{\sqrt{LC}} - \frac{jB}{\sqrt{LC}}
\]

From this we know that \( A = B \), and so \( A = B = \frac{I_L-I_S}{2} \). We can now expand the general solution above using Euler’s formula and the facts that \( \cos(x) = \cos(-x) \) and \( \sin(x) = -\sin(-x) \) to find

\[
i_L(t) = \frac{I_L-I_S}{2} \left[ \cos \left( \frac{t}{\sqrt{LC}} \right) + j \sin \left( \frac{t}{\sqrt{LC}} \right) + \cos \left( -\frac{t}{\sqrt{LC}} \right) + j \sin \left( -\frac{t}{\sqrt{LC}} \right) \right] + I_S
\]

\[
= \frac{I_L-I_S}{2} \left[ 2 \cos \left( \frac{t}{\sqrt{LC}} \right) + j \sin \left( \frac{t}{\sqrt{LC}} \right) - j \sin \left( \frac{t}{\sqrt{LC}} \right) \right] + I_S
\]

\[
= (I_L-I_S) \cos \left( \frac{t}{\sqrt{LC}} \right) + I_S
\]

\[
= I_S \left[ 1 - \cos \left( \frac{t}{\sqrt{LC}} \right) \right] + I_L \left[ \cos \left( \frac{t}{\sqrt{LC}} \right) \right]
\]

This is the sum of the answers we arrived at in Parts (A) and (B).
Problem 10.2 Answer:

(A) Writing KCL where $R_1$, $R_2$ and $C$ meet gives

\[ \frac{v_s - v_o}{R_1} + C \frac{d}{dt}(v_s - v_o) = \frac{v_o}{R_2} \]

This can be re-arranged to give

\[ \frac{dv_o}{dt} + \frac{v_o}{C(R_1||R_2)} = \frac{dv_s}{dt} + \frac{v_s}{CR_1} \]

(B) Substituting $v_{Sf}(t) = V_S e^{j\omega t}$ into the differential equation above and guessing a solution of the form $v_{Of}(t) = V_o e^{j\omega t}$ we find

\[ V_o j\omega e^{j\omega t} + \frac{V_o e^{j\omega t}}{C(R_1||R_2)} = V_S j\omega e^{j\omega t} + \frac{V_S e^{j\omega t}}{CR_1} \]

\[ V_o \left( j\omega + \frac{1}{C(R_1||R_2)} \right) = V_S \left( j\omega + \frac{1}{CR_1} \right) \]

\[ V_o = \frac{V_S}{j\omega + \frac{1}{C(R_1||R_2)}} \]

\[ V_o = \frac{V_S}{j\omega + \frac{1}{CR_1}} \]

\[ V_o = V_S \frac{R_2 + j\omega CR_1 R_2}{(R_1 + R_2) + j\omega CR_1 R_2} \]

(C) The solution is $v_O(t) = |V_O| \cos(\omega t + \angle V_O)$ where $V_O$ is the complex quantity obtained above.

The forced response to $v_S(t) = V_S \cos(\omega t)$ is

\[ V_S \frac{\sqrt{R_2^2 + (\omega CR_1 R_2)^2}}{\sqrt{(R_1 + R_2)^2 + (\omega CR_1 R_2)^2}} \cos \left( \omega t + \tan^{-1} \left( \frac{\omega CR_1 R_2}{R_1 + R_2} \right) \right) \]

This is in the form requested, where

\[ V_o = V_S \frac{\sqrt{R_2^2 + (\omega CR_1 R_2)^2}}{\sqrt{(R_1 + R_2)^2 + (\omega CR_1 R_2)^2}} \] and \[ \theta = \tan^{-1} \left( \frac{\omega CR_1 R_2}{R_1 + R_2} \right) \]

Problem 10.3 Answer:

(A) (i) The total impedance is the parallel combination of $R$ and $\frac{1}{Cj\omega}$. This is

\[ Z = \frac{R}{RCj\omega + 1} \]

(ii) Admittances in parallel add, so

\[ Y = Cj\omega + \frac{1}{R_2} + \frac{1}{\frac{1}{R_1 + Lj\omega}} \]
(iii) We can use the current divider relationship to find that
\[ I_2 = I_1 \frac{R + L_j \omega}{R + L_j \omega + \frac{1}{L_j \omega}}. \]
From this:
\[ H(j\omega) = \frac{C_j \omega (R + L_j \omega)}{C_j \omega (R + L_j \omega) + 1} \]

(iv) Let \( V_2 = V_2^+ - V_2^- \). Using the voltage divider relationship we can write
\[ V_2^+ = \frac{L_j \omega}{L_j \omega + R_1} V_1 \]
\[ V_2^- = \frac{R_2 C_j \omega}{R_2 C_j \omega + 1} V_1 \]

From this we find
\[ H(j\omega) = \frac{L_j \omega}{L_j \omega + R_1} - \frac{R_2 C_j \omega}{R_2 C_j \omega + 1} = \frac{j\omega (L - R_1 R_2 C)}{R_1 - R_2 L C \omega^2 + j\omega (L + R_1 R_2 C)} \]

(B) Express the input current \( i_1(t) \) as \( \Re \{I e^{j\omega t}\} \). We can express \( i_2(t) \) then as \( \Re \{H(j\omega)I e^{j\omega t}\} \).
\[ i_2(t) = \Re \{H(j\omega)I e^{j\omega t}\} \]
\[ = I \Re \left\{ \frac{C_j \omega (R + L_j \omega)}{C_j \omega (R + L_j \omega) + 1} e^{j\omega t} \right\} \]
\[ = I \Re \left\{ \frac{-L C \omega^2 + R C j \omega}{1 - L C \omega^2 + R C j \omega} e^{j\omega t} \right\} \]
\[ = I \frac{\sqrt{(L C \omega^2)^2 + (R C \omega)^2}}{\sqrt{(1 - L C \omega^2)^2 + (R C \omega)^2}} \Re \left\{ e^{j \tan^{-1}(-R/L) - j \tan^{-1}\left(\frac{R C \omega}{1 - L C \omega^2}\right)} e^{j\omega t} \right\} \]
\[ = I \frac{\sqrt{(L C \omega^2)^2 + (R C \omega)^2}}{\sqrt{(1 - L C \omega^2)^2 + (R C \omega)^2}} \cos \left( \omega t - \tan^{-1}\left(\frac{R}{L}\right) - \tan^{-1}\left(\frac{R C \omega}{1 - L C \omega^2}\right) \right) \]

(C) Again, we can re-express \( v_1(t) \) as \( \Im \{V e^{j\omega t}\} \). So
\[ v_2(t) = V \Im \left\{ H(j\omega) e^{j\omega t} \right\} \]
\[ = V \Im \left\{ \left( \frac{j \omega (L - R_1 R_2 C)}{R_1 - R_2 L C \omega^2 + j \omega (L + R_1 R_2 C)} \right) e^{j\omega t} \right\} \]
\[ = V \frac{\omega (L - R_1 R_2 C)}{\sqrt{(R_1 - R_2 L C \omega^2)^2 + (\omega (L + R_1 R_2 C))^2}} \Im \left\{ e^{j \omega t + j \tan^{-1}\left(\frac{L - R_1 R_2 C}{R_1 - R_2 L C \omega^2}\right)} - j \tan^{-1}\left(\frac{\omega (L + R_1 R_2 C)}{R_1 - R_2 L C \omega^2}\right) \right\} \]
\[ = V \frac{\omega (L - R_1 R_2 C)}{\sqrt{(R_1 - R_2 L C \omega^2)^2 + (\omega (L + R_1 R_2 C))^2}} \sin \left( \omega t + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega (L + R_1 R_2 C)}{R_1 - R_2 L C \omega^2}\right) \right) \]

Given the special case \( \frac{L}{R_1} = R_2 C \), we can simplify the above expression. The numerator in the magnitude becomes
\[ \omega (L - R_1 \frac{L}{R_1}) = 0 \]
So \( v_2(t) = 0 \).
Problem 10.4 Answer:

(A) When the terminal pair is open-circuited \( V_t \) is present at the output terminals, so

\[
V_t = 10V \sin(10^4 t)
\]

When the terminal pair is short-circuited, we can find \( Z_t \) in the following manner. First, represent the Thévenin voltage as \( V_t = 10V \{ e^{j10^4 t} \} \). Now, we can write:

\[
5mA \sin \left(10^4 t - \frac{\pi}{6}\right) = 10V \sin \left\{ -\frac{1}{Z_t} e^{j10^4 t} \right\} = -\frac{10V}{|Z_t|} \sin \left(10^4 t - \angle Z_t\right) = \frac{10V}{|Z_t|} \sin \left(10^4 t + \pi - \angle Z_t\right)
\]

Given this, we can find \(|Z_t|\) and \( \angle Z_t \) as

\[
|Z_t| = 2k\Omega \quad \text{and} \quad \angle Z_t = -\frac{5\pi}{6}
\]

So we can write \( Z_t \) as

\[
Z_t = 2k\Omega e^{-j\frac{5\pi}{6}} = -1000(\sqrt{3} + j)
\]

(B) The impedance of the capacitor at \( 10^4 \text{rad/s} \) is \( Z_C = -2000j\Omega \). The total impedance connected to \( V_t \) is then

\[
Z_t + Z_C = -1000(\sqrt{3} + j)
\]

The magnitude and phase of this impedance is

\[
|Z_t + Z_C| = 2000\sqrt{3} \quad \text{and} \quad \angle(Z_t + Z_C) = \tan^{-1}(\sqrt{3}) - \pi
\]

Given this, \( i(t) \) is given by

\[
i(t) = -\frac{10V}{|Z_t + Z_C|} \sin \left(10^4 t - \angle(Z_t + Z_C)\right) = \frac{5\sqrt{3}}{3} \text{mA} \sin \left(10^4 t + \pi + \pi - \tan^{-1}(\sqrt{3})\right) = \frac{5\sqrt{3}}{3} \sin \left(10^4 t - \frac{\pi}{3}\right) \text{mA}
\]
Problem 10.5 Answer:

(A) The power delivered to the resistor is equal to

\[ P_{RL} = \frac{1}{2} \left( \frac{v_S}{R_S + R_L} \right)^2 R_L \]

Differentiating this with respect to \( R_L \) and setting it equal to zero gives:

\[
0 = \frac{d}{dR_L} \left[ \frac{1}{2} \left( \frac{v_S}{R_S + R_L} \right)^2 R_L \right] \\
0 = \frac{v_S^2 (R_S + R_L)^2 - 2 v_S^2 R_L (R_S + R_L)}{2 (R_S + R_L)^2} \\
0 = (R_S + R_L)^2 - 2 R_L (R_S + R_L) \\
0 = R_S - R_L
\]

So, when \( R_L = R_S \) the power delivered to the resistor is at its maximum value.

(B) Let \( \theta = \omega t \). The time average power dissipated by a resistor is

\[
\overline{P}_R = \frac{1}{2\pi} \int_0^{2\pi} |I| \sin(\theta) \cdot |I| R \sin(\theta) d\theta \\
= \frac{|I|^2 R}{2\pi} \int_0^{2\pi} \sin^2(\theta) d\theta \\
= \frac{|I|^2 R}{2\pi} \cdot \pi \\
= \frac{1}{2} |I|^2 R
\]

(C) Only the real components of impedances dissipate energy, the complex pieces correspond to energy storage, not dissipation. The average power dissipation of \( Z_L \) is then

\[
\overline{P}_{Z_L} = \frac{\Re\{Z_L\}}{2\pi} \int_0^{2\pi} i_L(\theta) d\theta \\
= \frac{\Re\{Z_L\}}{2\pi} \int_0^{2\pi} \left[ \frac{V_S}{R_S + Z_L} \sin(\theta - \angle(R_S + Z_L)) \right]^2 d\theta \\
= \frac{\Re\{Z_L\}}{2\pi} \left[ \frac{V_S}{R_S + Z_L} \right]^2 \int_0^{2\pi} [\sin(\theta - \angle(R_S + Z_L))]^2 d\theta \\
= \frac{\Re\{Z_L\}}{2} \left| \frac{V_S}{R_S + Z_L} \right|^2 \\
= \frac{R_L V_S^2}{2} \left| \frac{1}{(R_S + R_L) + jX_L} \right|^2 \\
= \frac{R_L V_S^2}{2} \frac{1}{(R_S + R_L)^2 + X_L^2}
\]
From the expression above, we can see that $X_L = 0$ maximizes the time-average power dissipated by the load. Now, find the value of $R_L$ that maximizes $P_Z$ is the following maximized:

$$P_{ZL} = \frac{1}{2} \left( \frac{V_S}{R_S + R_L} \right)^2 R_L$$

This is the same problem we were asked to solve in Part (A), so we know the answer is $R_L = R_S$.

(D) (i) The parallel combination of the impedances of the resistor and capacitor is

$$Z = R_L || \frac{1}{jC\omega}$$

$$= \frac{R_L}{1 + jR_L C\omega}$$

$$= \frac{R_L}{1 + jR_L C\omega} * \frac{1 - jR_L C\omega}{1 - jR_L C\omega}$$

$$= \frac{R_L (1 - jR_L C\omega)}{1 + (R_L C\omega)^2}$$

$$= \frac{R_L}{1 + (R_L C\omega)^2} - j\frac{R_L^2 C\omega}{1 + (R_L C\omega)^2}$$

(ii) Set $R_S$ equal to the real part of the answer above, and solve for $\omega C$.

$$R_S = \frac{R_L}{1 + (R_L C\omega)^2}$$

$$(R_L C\omega)^2 = \frac{R_L}{R_S} - 1$$

$$C\omega = \frac{1}{R_L} \sqrt{\frac{R_L}{R_S} - 1}$$

$$C = \frac{1}{\omega R_L} \sqrt{\frac{R_L}{R_S} - 1}$$

(iii) We want the sum of the inductor’s impedance and the load impedance to equal $R_S$. The inductor’s impedance is purely imaginary, so for the given $C\omega$ value in Part (ii) above, we want

$$L\omega = \frac{R_L^2 C\omega}{1 + (R_L C\omega)^2}$$

Substituting in our answer from above gives

$$L\omega = \frac{R_L}{\frac{R_L}{R_S} - 1}$$

$$L = \frac{R_S}{\frac{R_L}{R_S} - 1}$$
If we wanted to match a load whose resistance is less than the source resistance, we need to “flip” the inductor and capacitor, as shown in the figure below.