Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Homework #10 Solutions

Problem 10.1 Answer:

(A) From previous problems, we know that $i_L(0^+) = 0$ because the inductor current will not change instantaneously. We have also shown in lecture that the peak inductor current will be twice the source current I_S, and the circuit will ring at $\omega = \frac{1}{\sqrt{LC}}$. Given this information, we can write

$$i_L(t) = I_S\left[1 - \cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

(B) Again, we can write the answer for this problem down directly. We know that $\omega = \frac{1}{\sqrt{LC}}$, $i_L(0^+) = I_L$, and that the peak value of $i_L(t) = I_L$. Given this we can write

$$i_L(t) = I_L\left[\cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

(C) The solution for $i_L(t)$ has the general form

$$i_L(t) = \underbrace{Ae^{j\frac{t}{\sqrt{LC}}} + Be^{-j\frac{t}{\sqrt{LC}}}}_{\text{Homogeneous Solution}} + \underbrace{I_S}_{\text{Particular Solution}}$$

We know that $i_L(0^+) = I_L$ because the inductor current is continuous. Additionally, from $t = 0^- \rightarrow 0^+$ the capacitor voltage stays constant at $t = 0$ because there is not a source of infinite current in the circuit. So $\frac{d}{dt}i_L(0^+) = 0$ because $v_C(t) = v_L(t) = L\frac{di_L(t)}{dt}$. Given this, we can write

$$I_L - I_S = A + B \quad \text{and} \quad 0 = \frac{jA}{\sqrt{LC}} - \frac{jB}{\sqrt{LC}}$$

From this we know that $A = B$, and so $A = B = \frac{I_L - I_S}{2}$. We can now expand the general solution above using Euler's formula and the facts that $\cos(x) = \cos(-x)$ and $\sin(x) = -\sin(-x)$ to find

$$\begin{aligned}
i_L(t) &= \frac{I_L - I_S}{2}\left[\cos\left(\frac{t}{\sqrt{LC}}\right) + j\sin\left(\frac{t}{\sqrt{LC}}\right) + \cos\left(-\frac{t}{\sqrt{LC}}\right) + j\sin\left(-\frac{t}{\sqrt{LC}}\right)\right] + I_S \\
&= \frac{I_L - I_S}{2}\left[2\cos\left(\frac{t}{\sqrt{LC}}\right) + j\sin\left(\frac{t}{\sqrt{LC}}\right) - j\sin\left(\frac{t}{\sqrt{LC}}\right)\right] + I_S \\
&= (I_L - I_S)\cos\left(\frac{t}{\sqrt{LC}}\right) + I_S \\
&= I_S\left[1 - \cos\left(\frac{t}{\sqrt{LC}}\right)\right] + I_L\left[\cos\left(\frac{t}{\sqrt{LC}}\right)\right]
\end{aligned}$$

This is the sum of the answers we arrived at in Parts (A) and (B).

Problem 10.2 Answer:

(A) Writing KCL where R_1, R_2 and C meet gives

$$\frac{v_S - v_O}{R_1} + C\frac{d}{dt}(v_S - v_O) = \frac{v_O}{R_2}$$

This can be re-arranged to give

$$\frac{dv_O}{dt} + \frac{v_O}{C(R_1||R_2)} = \frac{dv_S}{dt} + \frac{v_S}{CR_1}$$

(B) Substituting $v_{Sf}(t) = V_S e^{j\omega t}$ into the differential equation above and guessing a solution of the form $v_{Of}(t) = V_O e^{j\omega t}$ we find

$$\begin{aligned}
V_O j\omega e^{j\omega t} + \frac{V_O e^{j\omega t}}{C(R_1||R_2)} &= V_S j\omega e^{j\omega t} + \frac{V_S e^{j\omega t}}{CR_1} \\
V_O \left(j\omega + \frac{1}{C(R_1||R_2)} \right) &= V_S \left(j\omega + \frac{1}{CR_1} \right) \\
V_O &= V_S \frac{j\omega + \frac{1}{CR_1}}{j\omega + \frac{R_1+R_2}{CR_1R_2)}} \\
V_O &= V_S \frac{R_2 + j\omega CR_1R_2}{(R_1+R_2) + j\omega CR_1R_2}
\end{aligned}$$

(C) The solution is $v_O(t) = |V_O|\cos(\omega t + \angle V_O)$ where V_O is the complex quantity obtained above. The forced response to $v_S(t) = V_S\cos(\omega t)$ is

$$V_S \frac{\sqrt{R_2^2 + (\omega CR_1R_2)^2}}{\sqrt{(R_1+R_2)^2 + (\omega CR_1R_2)^2}} \cos\left(\omega t + \tan^{-1}(\omega CR_1) - \tan^{-1}\left(\frac{\omega CR_1R_2}{R_1+R_2} \right) \right)$$

This is in the form requested, where

$$V_o = V_S \frac{\sqrt{R_2^2 + (\omega CR_1R_2)^2}}{\sqrt{(R_1+R_2)^2 + (\omega CR_1R_2)^2}} \quad \text{and} \quad \theta = \tan^{-1}(\omega CR_1) - \tan^{-1}\left(\frac{\omega CR_1R_2}{R_1+R_2} \right)$$

Problem 10.3 Answer:

(A) (i) The total impedance is the parallel combination of R and $\frac{1}{Cj\omega}$. This is

$$Z = \frac{R}{RCj\omega + 1}$$

(ii) Admittances in parallel add, so

$$Y = Cj\omega + \frac{1}{R_2} + \frac{1}{R_1 + Lj\omega}$$

(iii) We can use the current divider relationship to find that $I_2 = I_1 \frac{R+Lj\omega}{R+Lj\omega+\frac{1}{Cj\omega}}$. From this:

$$H(j\omega) = \frac{Cj\omega(R+Lj\omega)}{Cj\omega(R+Lj\omega)+1}$$

(iv) Let $V_2 = V_2^+ - V_2^-$. Using the voltage divider relationship we can write

$$\begin{aligned} V_2^+ &= \frac{Lj\omega}{Lj\omega + R_1} V_1 \\ V_2^- &= \frac{R_2 Cj\omega}{R_2 Cj\omega + 1} V_1 \end{aligned}$$

From this we find

$$H(j\omega) = \frac{Lj\omega}{Lj\omega + R_1} - \frac{R_2 Cj\omega}{R_2 Cj\omega + 1} = \frac{j\omega(L - R_1 R_2 C)}{R_1 - R_2 LC\omega^2 + j\omega(L + R_1 R_2 C)}$$

(B) Express the input current $i_1(t)$ as $\Re\left\{Ie^{j\omega t}\right\}$. We can express $i_2(t)$ then as $\Re\{H(j\omega)Ie^{j\omega t}\}$.

$$\begin{aligned} i_2(t) &= \Re\{H(j\omega)Ie^{j\omega t}\} \\ &= I\Re\left\{\frac{Cj\omega(R+Lj\omega)}{Cj\omega(R+Lj\omega)+1} e^{j\omega t}\right\} \\ &= I\Re\left\{\frac{-LC\omega^2 + RCj\omega}{1 - LC\omega^2 + RCj\omega} e^{j\omega t}\right\} \\ &= I\frac{\sqrt{(LC\omega^2)^2 + (RC\omega)^2}}{\sqrt{(1-LC\omega^2)^2 + (RC\omega)^2}} \Re\left\{e^{j\tan^{-1}\left(-\frac{R}{L\omega}\right) - j\tan^{-1}\left(\frac{RC\omega}{1-LC\omega^2}\right)} e^{j\omega t}\right\} \\ &= I\frac{\sqrt{(LC\omega^2)^2 + (RC\omega)^2}}{\sqrt{(1-LC\omega^2)^2 + (RC\omega)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{R}{L\omega}\right) - \tan^{-1}\left(\frac{RC\omega}{1-LC\omega^2}\right)\right) \end{aligned}$$

(C) Again, we can re-express $v_1(t)$ as $\Im\{Ve^{j\omega t}\}$. So

$$\begin{aligned} v_2(t) &= V\Im\left\{H(j\omega)e^{j\omega t}\right\} \\ &= V\Im\left\{\left(\frac{j\omega(L - R_1 R_2 C)}{R_1 - R_2 LC\omega^2 + j\omega(L + R_1 R_2 C)}\right) e^{j\omega t}\right\} \\ &= V\frac{\omega(L - R_1 R_2 C)}{\sqrt{(R_1 - R_2 LC\omega^2)^2 + (\omega(L + R_1 R_2 C))^2}} \Im\left\{e^{j\omega t + j\tan^{-1}\left(\frac{L - R_1 R_2 C}{0}\right) - j\tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_1 - R_2 LC\omega^2}\right)}\right\} \\ &= V\frac{\omega(L - R_1 R_2 C)}{\sqrt{(R_1 - R_2 LC\omega^2)^2 + (\omega(L + R_1 R_2 C))^2}} * \\ &\quad \sin\left(\omega t + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_1 - R_2 LC\omega^2}\right)\right) \end{aligned}$$

Given the special case $\frac{L}{R_1} = R_2 C$, we can simplify the above expression. The numerator in the magnitude becomes

$$\omega(L - R_1 \frac{L}{R_1}) = 0$$

So $v_2(t) = 0$.

Problem 10.4 Answer:

(A) When the terminal pair is open-circuited V_t is present at the output terminals, so

$$V_t = 10\text{V}\sin(10^4 t)$$

When the terminal pair is short-circuited, we can find Z_t in the following manner. First, represent the Thévenin voltage as $V_t = 10\text{V}\Im\left\{e^{j10^4 t}\right\}$. Now, we can write:

$$\begin{aligned}
5\text{mA}\sin\left(10^4 t - \frac{\pi}{6}\right) &= 10\text{V}\Im\left\{-\frac{1}{Z_t}e^{j10^4 t}\right\} \\
&= -\frac{10\text{V}}{|Z_t|}\sin\left(10^4 t - \angle Z_t\right) \\
&= \frac{10\text{V}}{|Z_t|}\sin\left(10^4 t + \pi - \angle Z_t\right)
\end{aligned}$$

Given this, we can find $|Z_t|$ and $\angle Z_t$ as

$$|Z_t| = 2\text{k}\Omega \quad \text{and} \quad \angle Z_t = -\frac{5\pi}{6}$$

So we can write Z_t as

$$Z_t = 2\text{k}\Omega e^{-j\frac{5\pi}{6}} = -1000(\sqrt{3} + j)$$

(B) The impedance of the capacitor at 10^4rad/s is $Z_C = -2000j\Omega$. The total impedance connected to V_t is then

$$Z_t + Z_C = -1000(\sqrt{3} + 3j)$$

The magnitude and phase of this impedance is

$$|Z_t + Z_C| = 2000\sqrt{3} \quad \text{and} \quad \angle(Z_t + Z_C) = \tan^{-1}(\sqrt{3}) - \pi$$

Given this, $i(t)$ is given by

$$\begin{aligned}
i(t) &= -\frac{10\text{V}}{|Z_t + Z_C|}\sin\left(10^4 t - \angle(Z_t + Z_C)\right) \\
i(t) &= \frac{5\sqrt{3}}{3}\text{mA} * \sin\left(10^4 t + \pi + \pi - \tan^{-1}(\sqrt{3})\right) \\
i(t) &= \frac{5\sqrt{3}}{3}\sin\left(10^4 t - \frac{\pi}{3}\right)\text{mA}
\end{aligned}$$

Problem 10.5 Answer:

(A) The power delivered to the resistor is equal to

$$P_{R_L} = \frac{1}{2}\left(\frac{v_S}{R_S + R_L}\right)^2 R_L$$

Differentiating this with respect to R_L and setting it equal to zero gives:

$$\begin{aligned}
0 &= \frac{d}{dR_L}\left[\frac{1}{2}\left(\frac{v_S}{R_S + R_L}\right)^2 R_L\right] \\
0 &= \frac{v_S^2(R_S + R_L)^2 - 2v_S^2 R_L(R_S + R_L)}{2(R_S + R_L)^2} \\
0 &= (R_S + R_L)^2 - 2R_L(R_S + R_L) \\
0 &= R_S - R_L
\end{aligned}$$

So, when $R_L = R_S$ the power delivered to the resistor is at its maximum value.

(B) Let $\theta = \omega t$. The time average power dissipated by a resistor is

$$\begin{aligned}
\overline{P}_R &= \frac{1}{2\pi}\int_0^{2\pi} |I|\sin(\theta) * |I|R\sin(\theta)d\theta \\
&= \frac{|I|^2 R}{2\pi}\int_0^{2\pi} \sin^2(\theta)d\theta \\
&= \frac{|I|^2 R}{2\pi} * \pi \\
&= \frac{1}{2}|I|^2 R
\end{aligned}$$

(C) Only the real components of impedances dissipate energy, the complex pieces correspond to energy storage, not dissipation. The average power dissipation of Z_L is then

$$\begin{aligned}
\overline{P}_{Z_L} &= \frac{\Re\{Z_L\}}{2\pi}\int_0^{2\pi} i_L(\theta)d\theta \\
&= \frac{\Re\{Z_L\}}{2\pi}\int_0^{2\pi}\left[\left|\frac{V_S}{R_S + Z_L}\right|\sin(\theta - \angle(R_S + Z_L))\right]^2 d\theta \\
&= \frac{\Re\{Z_L\}}{2\pi}\left|\frac{V_S}{R_S + Z_L}\right|^2\int_0^{2\pi}\left[\sin(\theta - \angle(R_S + Z_L))\right]^2 d\theta \\
&= \frac{\Re\{Z_L\}}{2}\left|\frac{V_S}{R_S + Z_L}\right|^2 \\
&= \frac{R_L}{2}V_S^2\left|\frac{1}{(R_S + R_L) + jX_L}\right|^2 \\
&= \frac{R_L}{2}V_S^2\frac{1}{(R_S + R_L)^2 + X_L^2}
\end{aligned}$$

From the expression above, we can see that $X_L = 0$ maximizes the time-average power dissipated by the load. Now, find the value of R_L that maximizes of R_L is the following maximized:

$$\overline{P}_{Z_L} = \frac{1}{2}\left(\frac{V_S}{R_S + R_L}\right)^2 R_L$$

This is the same problem we were asked to solve in Part (A), so we know the answer is $R_L = R_S$.

(D) (i) The parallel combination of the impedances of the resistor and capacitor is

$$\begin{aligned}
Z &= R_L || \frac{1}{jC\omega} \\
&= \frac{R_L}{1 + jR_L C\omega} \\
&= \frac{R_L}{1 + jR_L C\omega} * \frac{1 - jR_L C\omega}{1 - jR_L C\omega} \\
&= \frac{R_L(1 - jR_L C\omega)}{1 + (R_L C\omega)^2} \\
&= \frac{R_L}{1 + (R_L C\omega)^2} - j\frac{R_L^2 C\omega}{1 + (R_L C\omega)^2}
\end{aligned}$$

(ii) Set R_S equal to the real part of the answer above, and solve for ωC.

$$\begin{aligned}
R_S &= \frac{R_L}{1 + (R_L C\omega)^2} \\
(R_L C\omega)^2 &= \frac{R_L}{R_S} - 1 \\
C\omega &= \frac{1}{R_L}\sqrt{\frac{R_L}{R_S} - 1} \\
C &= \frac{1}{\omega R_L}\sqrt{\frac{R_L}{R_S} - 1}
\end{aligned}$$

(iii) We want the sum of the inductor's impedance and the load impedance to equal R_S. The inductor's impedance is purely imaginary, so for the given $C\omega$ value in Part (ii) above, we want

$$L\omega = \frac{R_L^2 C\omega}{1 + (R_L C\omega)^2}$$

Substituting in our answer from above gives

$$\begin{aligned}
L\omega &= \frac{R_L\sqrt{\frac{R_L}{R_S} - 1}}{\frac{R_L}{R_S}} \\
L &= \frac{R_S\sqrt{\frac{R_L}{R_S} - 1}}{\omega}
\end{aligned}$$

If we wanted to match a load whose resistance is less than the source resistance, we need to “flip” the inductor and capacitor, as shown in the figure below.