Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

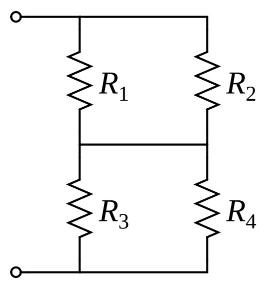
Homework #2 Solutions

Problem 2.1 Answer:

(A) (i) To find the Thévenin resistance, first combine R_2 and R_3. Their resistances add because they are in series with each other. This new resistor is in parallel with R_1, so the total equivalent resistance is

$$R_{\text{Th}} = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

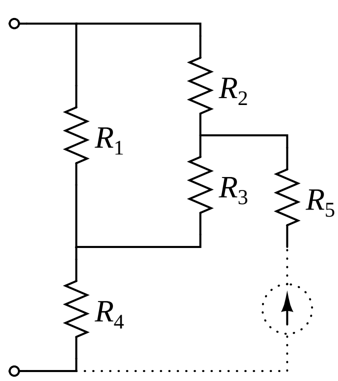
(ii) We're only asked to find the Thévenin resistance of the network. Any independent sources in the network don't affect the Thévenin resistance, so we can remove them. A voltage source, when set to 0, becomes a wire, as shown in the figure below.



The Thévenin resistance of this new circuit is just the series combination of $R_1||R_2$ and $R_3||R_4$ which is equal to

$$R_{\text{Th}} = \frac{R_1R_2}{R_1 + R_2} + \frac{R_3R_4}{R_3 + R_4}$$

(iii) Like in part (*ii*), we can remove the independent source from this circuit. The resulting circuit is drawn below. It's been rearranged a little for clarity.



The resistor R_5 does not impact the Thévenin resistance, because one of its terminals is not connected to the rest of the network now that the current source has been removed. R_2 and R_3 are in series, in parallel with R_1. All of this is in series with R_4. The resulting resistance is

$$R_{\text{Th}} = R_4 + \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

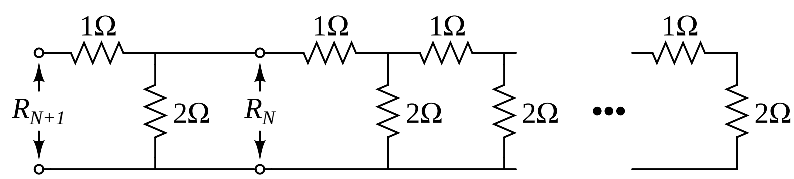
(iv) The 1Ω and 2Ω resistors on the right in series combine to form a 3Ω resistor in parallel with the remaining 2Ω resistor. Their combined resistance is $\frac{6}{5}\Omega$. This is in series with the other 1Ω resistor, making the Thévenin resistance equal

$$R_{\text{Th}} = \frac{11}{5}\Omega$$

(v) We know from part (*iv*) that the Thévenin equivalent of the four resistors on the right side of the circuit are $\frac{11}{5}\Omega$. This is in parallel with the third 2Ω resistor, which is in series with the 1Ω resistor. The total resistance is

$$R_{\text{Th}} = \frac{\frac{11}{5} * 2}{\frac{11}{5} + 2} + 1 = 1 + \frac{22}{21}\Omega$$

(B) (i) The circuit for R_{N+1} is shown in the figure below.



R_N is in parallel with the new 2Ω resistor, which is then in series with the 1Ω resistor. R_{N+1} is then

$$R_{N+1} = 1 + \frac{2R_N}{2 + R_N}$$

(ii) We know that the resistance of the ladder network is not going to change if we add one more 1Ω-2Ω section, so $\lim_{N\to\infty} R_{N+1} = R_N$. We can write, from part (*i*) above:

$$R_N = 1 + \frac{2R_N}{2 + R_N}$$

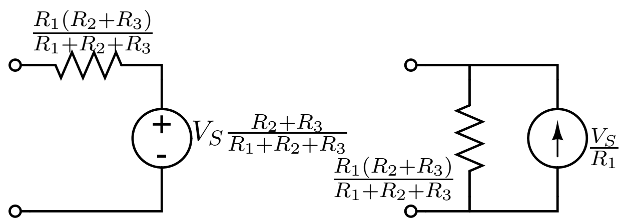
Multiplying both sides by $2 + R_N$ yields a quadratic equation for R_N:

$$R_N^2 - R_N - 2 = 0$$

Its roots are $R_{\text{N}} = -1$ and $R_{\text{N}} = 2$. R_N cannot be negative, so $\lim_{N\to\infty} R_N = 2\Omega$. Notice when we another 1Ω-2Ω section was added in part (*v*) above, the resistance got closer to 2Ω, which is, as we have just shown, it's limit as $N \to \infty$.

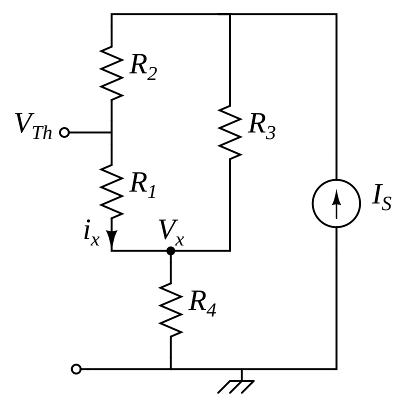
Problem 2.2 Answer:

(i) The Thévenin resistance for this circuit is the result obtained from Problem 2.1 Part A(i), which is $\frac{R_1(R_2+R_3)}{R_1+R_2+R_3}$. The Thévenin voltage is determined by V_S and the voltage divider made by the two resistances R_1 and R_2+R_3. So, $V_{Th} = V_S\frac{R_2+R_3}{R_1+R_2+R_3}$. The Norton equivalent current can be found using Ohm's law. $I_N = \frac{V_{Th}}{R_{\text{Th}}} = \frac{V_S}{R_1}$. The resulting equivalent circuits are shown below.



(ii) This circuit is more complicated than the one above. R_{Th} can be taken from Problem 2.1 Part A(iii). It is $R_{\text{Th}} = R_4 + \frac{R_1(R_2+R_3)}{R_1+R_2+R_3}$. We can either find V_{Th} or I_N next, and use the one to find the other.

The first important step in solving for V_{Th} is realizing that R_5 does not impact it. Whether or not R_5 is present, I_S still flows into the R_2-R_3 node of the circuit. This means we can remove R_5 from the circuit without changing V_{Th}. This new circuit is drawn below, with the resistors rearranged and a few new variables named.



It is clear that the current running through R_4 is equal to I_S. We know, then, that $V_x = I_S R_4$. We also know that $V_{Th} = V_x + i_x R_1$. The current i_x can be found using the current divider relationship:

$$i_x = \frac{R_3}{R_1 + R_2 + R_3} I_S$$

The Thévenin voltage is then

$$V_{Th} = \frac{R_4(R_1 + R_2 + R_3) + R_1 R_3}{R_1 + R_2 + R_3} I_S$$

The Norton current can be found by dividing this by R_{Th}, which gives

$$
\begin{aligned}
I_N &= \frac{V_{Th}}{R_{\mathrm{Th}}} = \frac{R_4(R_1+R_2+R_3)+R_1R_3}{R_1+R_2+R_3} I_S * \frac{R_1+R_2+R_3}{R_4(R_1+R_2+R_3)+R_1(R_2+R_3)} \\
I_N &= \frac{R_4(R_1+R_2+R_3)+R_1R_3}{R_4(R_1+R_2+R_3)+R_1(R_2+R_3)} I_S
\end{aligned}
$$

Problem 2.3 Answer:

(A) We are given 2 v-i data points for the network. These are (4 V, 2 mA), and (6 V, -1 mA). $R_{\mathrm{Th}} = \left|\frac{V_1-V_2}{I_1-I_2}\right| \left|\frac{6-4}{-1-2}\right| k\Omega = \frac{2}{3} k\Omega$. Now that we know R_{Th}, we can find V_{Th} by solving the following equation:

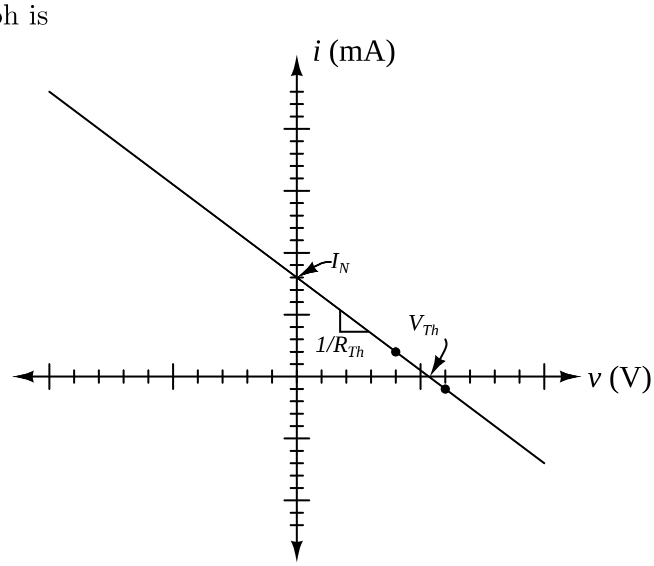
$$v = -R_{\mathrm{Th}} i + V_{Th}$$

Substituting in one of the v-i points given in the problem, we find that $V_{Th} = 5\frac{1}{3}$V. This is the voltage across the network's terminals when no current is flowing in or out of the network, so it is V_{Th}.

(B) The graph is just determined by the equation from above, rewritten as a function of voltage rather than of current.

$$i = -\frac{1}{R_{\mathrm{Th}}} v + 8 \times 10^{-3}$$

The resulting graph is



Problem 2.4 Answer:

(A) Summing the currents out of the node e, we can write the node equation

$$\frac{e}{R_2} + \frac{e-V_2}{R_3} + \frac{e-V_1}{R_1} = 0$$

Solving for e gives

$$e = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{V_1 R_2 R_3 + V_2 R_1 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

(B) Using superposition, each voltage source only sees a resistive divider network, and we can write out e directly as

$$e = V_1 \frac{R_2||R_3}{R_2||R_3 + R_1} + V_2 \frac{R_1||R_2}{R_1||R_2 + R_3} = \frac{V_1 \frac{R_2 R_3}{R_2 + R_3}}{\frac{R_2 R_3}{R_2 + R_3} + R_1} + \frac{V_2 \frac{R_1 R_2}{R_1 + R_2}}{\frac{R_1 R_2}{R_1 + R_2} + R_3} = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

(C) Writing KCL at the two nodes, summing currents *out* of the nodes yields

$$\begin{aligned}
\frac{e_1}{R_1} + \frac{e_1 - V_S}{R_2} - I_S &= 0 \\
\frac{e_2}{R_4} + \frac{e_2 - V_S}{R_3} + I_S &= 0
\end{aligned}$$

Solving these equations for e_1 and e_2 gives

$$e_1 = \left(\frac{V_S}{R_2} + I_S \right) \frac{R_1 R_2}{R_1 + R_2} \quad e_2 = \left(\frac{V_S}{R_3} - I_S \right) \frac{R_3 R_4}{R_3 + R_4}$$

(D) Using superposition, we can write down the expressions for the node voltages e_1 and e_2 directly.

$$\begin{aligned}
e_1 &= \frac{R_1}{R_1 + R_2} V_S + I_S \frac{R_1 R_2}{R_1 + R_2} = \left(\frac{V_S}{R_2} + I_S \right) \frac{R_1 R_2}{R_1 + R_2} \\
e_2 &= \frac{R_4}{R_3 + R_4} V_S - I_S \frac{R_3 R_4}{R_3 + R_4} = \left(\frac{V_S}{R_3} - I_S \right) \frac{R_3 R_4}{R_3 + R_4}
\end{aligned}$$

Problem 2.5 Answer:

(A) Summing the currents out of each node yields the following equations

$$\begin{aligned}
\frac{e_1 - e_3}{R_5} + \frac{e_1 - e_2}{R_1} - I_S &= 0 \\
\frac{e_2 - e_1}{R_1} + \frac{e_2 - e_3}{R_3} + \frac{e_2}{R_2} &= 0 \\
\frac{e_3 - e_1}{R_5} + \frac{e_3 - e_2}{R_3} + \frac{e_3}{R_4} &= 0
\end{aligned}$$

Replacing the reciprocal resistances with conductances, and rearranging some terms, we can write the above equations as

$$\begin{aligned}
(G_1 + G_5)e_1 + (-G_1)e_2 + (-G_5)e_3 &= I_S \\
(-G_1)e_1 + (G_1 + G_2 + G_3)e_2 + (-G_3)e_3 &= 0 \\
(-G_5)e_1 + (-G_3)e_2 + (G_3 + G_4 + G_5)e_3 &= 0
\end{aligned}$$

which can be re-written in matrix form as

$$\begin{bmatrix} G_1+G_5 & -G_1 & -G_5 \\ -G_1 & G_1+G_2+G_3 & -G_3 \\ -G_5 & -G_3 & G_3+G_4+G_5 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} I_S \\ 0 \\ 0 \end{bmatrix}$$

(B) If $G_1 = G_2 = G_3 ... = G_5 = 1mho = 1\frac{1}{\Omega}$, then the matrix equation above becomes:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} I_S \\ 0 \\ 0 \end{bmatrix}$$

We can use Matlab, or a calculator, or our hands to solve this equation for e_1, e_2 and e_3, and find that

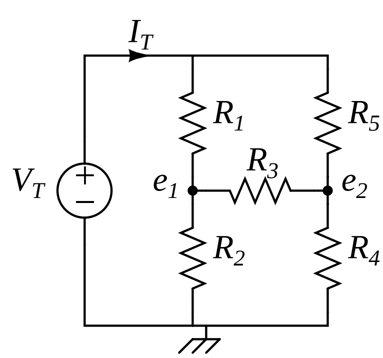
$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} I_S \\ \frac{I_S}{2} \\ \frac{I_S}{2} \end{bmatrix}$$

(C) The equivalent resistance of the network can be found by dividing the potential e_1 by the current entering the network, I_S. From Part (B) above, we know that $e_1 = I_S$, so $R_{\text{Th}} = 1\Omega$.

Another way to solve this problem is through a symmetry argument. If all the resistances are 1Ω, then R_3 carries no current. To prove this, assume it. The node voltages e_2 and e_3 will be equal to each other, because $\frac{R_1}{R_2} = \frac{R_5}{R_4}$. If e_2 and e_3 are equal, then no current flows through R_3, and we've verified our assumption. We can remove R_3, and realize that the total resistance seen by the source is 2Ω in parallel with 2Ω, which is equal to 1Ω.

General Case:

In the general case, where we can't use any symmetry tricks to simplify the problem, we have to solve the circuit using the node method. Let's apply a test voltage V_T to the resistor network where the source is, and find an expression for the current into the resistor network I_T. We can then evaluate $\frac{V_T}{I_T}$ to find the equivalent resistance of the network. The circuit is drawn below, rearranged a bit for clarity. The node voltages that we will be solving for are labeled as well.



We know that the current I_T is equal to the sum of the currents down through the resistors R_2 and R_4. This can be written as $I_T = G_2e_1 + G_4e_2$, where G_2 and G_4 are the conductances associated with R_2 and R_4, respectively. To find e_1 and e_2 we can use the node method to write

$$\begin{aligned} \frac{e_1 - V_T}{R_1} + \frac{e_1 - e_2}{R_3} + \frac{e_1}{R_2} &= 0 \\ \frac{e_2 - V_T}{R_5} + \frac{e_2 - e_1}{R_3} + \frac{e_2}{R_4} &= 0 \end{aligned}$$

which can be rewritten in matrix form:

$$\begin{bmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 + G_5 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} B_T G_1 \\ V_T G_5 \end{bmatrix}$$

We can solve this for the node voltages using simple matrix algebra, and find e_1 and e_2 in terms of the conductances and V_T. Substitute these back into the equation for R_{Th} above and we find that:

$$R_{\text{Th}} = \frac{(G_1 + G_2 + G_3)(G_3 + G_4 + G_5) - G_3^2}{G_2[(G_3 + G_4 + G_5)G_1 + G_3 G_5] + G_4[(G_1 + G_2 + G_3)G_5 + G_1 G_3]}$$

For the special case in Part (B), this reduces down to 1Ω.