Problem 2.1 Answer:

(A) (i) To find the Thévenin resistance, first combine $R_2$ and $R_3$. Their resistances add because they are in series with each other. This new resistor is in parallel with $R_1$, so the total equivalent resistance is

$$R_{Th} = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

(ii) We’re only asked to find the Thévenin resistance of the network. Any independent sources in the network don’t affect the Thévenin resistance, so we can remove them. A voltage source, when set to 0, becomes a wire, as shown in the figure below.

![Diagram](image1)

The Thévenin resistance of this new circuit is just the series combination of $R_1 || R_2$ and $R_3 || R_4$ which is equal to

$$R_{Th} = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}$$

(iii) Like in part (ii), we can remove the independent source from this circuit. The resulting circuit is drawn below. It’s been rearranged a little for clarity.

![Diagram](image2)
The resistor $R_5$ does not impact the Thévenin resistance, because one of its terminals is not connected to the rest of the network now that the current source has been removed. $R_2$ and $R_3$ are in series, in parallel with $R_1$. All of this is in series with $R_4$. The resulting resistance is

$$R_{Th} = R_4 + \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

(iv) The 1Ω and 2Ω resistors on the right in series combine to form a 3Ω resistor in parallel with the remaining 2Ω resistor. Their combined resistance is $\frac{6}{5}$Ω. This is in series with the other 1Ω resistor, making the Thévenin resistance equal

$$R_{Th} = \frac{11}{5} \Omega$$

(v) We know from part (iv) that the Thévenin equivalent of the four resistors on the right side of the circuit are $\frac{11}{5}$Ω. This is in parallel with the third 2Ω resistor, which is in series with the 1Ω resistor. The total resistance is

$$R_{Th} = \frac{11}{5} \cdot \frac{2}{1 + \frac{2}{21}}$$

(B) (i) The circuit for $R_{N+1}$ is shown in the figure below.

\[ R_N \text{ is in parallel with the new 2Ω resistor, which is then in series with the 1Ω resistor. } R_{N+1} \text{ is then} \]

$$R_{N+1} = 1 + \frac{2R_N}{2 + R_N}$$

(ii) We know that the resistance of the ladder network is not going to change if we add one more 1Ω-2Ω section, so $\lim_{N \to \infty} R_{N+1} = R_N$. We can write, from part (i) above:

$$R_N = 1 + \frac{2R_N}{2 + R_N}$$

Multiplying both sides by $2 + R_N$ yields a quadratic equation for $R_N$:

$$R_N^2 - R_N - 2 = 0$$

Its roots are $R_N = -1$ and $R_N = 2$. $R_N$ cannot be negative, so $\lim_{N \to \infty} R_N = 2Ω$. Notice when we another 1Ω-2Ω section was added in part (v) above, the resistance got closer to 2Ω, which is, as we have just shown, it’s limit as $N \to \infty$. 
Problem 2.2 Answer:

(i) The Thévenin resistance for this circuit is the result obtained from Problem 2.1 Part A(i), which is $R_1(R_2 + R_3) / (R_1 + R_2 + R_3)$. The Thévenin voltage is determined by $V_S$ and the voltage divider made by the two resistances $R_1$ and $R_2 + R_3$. So, $V_{Th} = V_S (R_2 + R_3) / (R_1 + R_2 + R_3)$. The Norton equivalent current can be found using Ohm’s law. $I_N = \frac{V_{Th}}{R_{Th}} = \frac{V_S}{R_1}$. The resulting equivalent circuits are shown below.

(ii) This circuit is more complicated than the one above. $R_{Th}$ can be taken from Problem 2.1 Part A(iii). It is $R_{Th} = R_4 + R_1(R_2 + R_3) / (R_1 + R_2 + R_3)$. We can either find $V_{Th}$ or $I_N$ next, and use the one to find the other.

The first important step in solving for $V_{Th}$ is realizing that $R_5$ does not impact it. Whether or not $R_5$ is present, $I_S$ still flows into the $R_2$-$R_3$ node of the circuit. This means we can remove $R_5$ from the circuit without changing $V_{Th}$. This new circuit is drawn below, with the resistors rearranged and a few new variables named.

It is clear that the current running through $R_4$ is equal to $I_S$. We know, then, that $V_x = I_S R_1$. We also know that $V_{Th} = V_x + i_x R_1$. The current $i_x$ can be found using the current divider relationship:

$$i_x = \frac{R_3}{R_1 + R_2 + R_3} I_S$$

The Thévenin voltage is then

$$V_{Th} = \frac{R_4(R_1 + R_2 + R_3) + R_1 R_3}{R_1 + R_2 + R_3} I_S$$
The Norton current can be found by dividing this by $R_{\text{Th}}$, which gives

\[
I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}} = \frac{R_4(R_1 + R_2 + R_3) + R_1R_3}{R_4(R_1 + R_2 + R_3) + R_1(R_2 + R_3)}I_S + \frac{R_1 + R_2 + R_3}{R_4(R_1 + R_2 + R_3) + R_1(R_2 + R_3)}I_S
\]

Problem 2.3 Answer:

(A) We are given 2 v-i data points for the network. These are (4 V, 2 mA), and (6 V, -1 mA).

\[
R_{\text{Th}} = \left| \frac{V_1 - V_2}{I_1 - I_2} \right| = \left| \frac{6 - 4}{-1 - 2} \right| k\Omega = \frac{2}{3} k\Omega.
\]

Now that we know $R_{\text{Th}}$, we can find $V_{\text{Th}}$ by solving the following equation:

\[
v = -R_{\text{Th}}i + V_{\text{Th}}
\]

Substituting in one of the v-i points given in the problem, we find that $V_{\text{Th}} = 5\frac{1}{3} V$. This is the voltage across the network’s terminals when no current is flowing in or out of the network, so it is $V_{\text{Th}}$.

(B) The graph is just determined by the equation from above, rewritten as a function of voltage rather than of current.

\[
i = -\frac{1}{R_{\text{Th}}}v + 8 \times 10^{-3}
\]

The resulting graph is

Problem 2.4 Answer:

(A) Summing the currents out of the node $e$, we can write the node equation

\[
\frac{e}{R_2} + \frac{e - V_2}{R_3} + \frac{e - V_1}{R_1} = 0
\]
Solving for $e$ gives

$$e = \frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{V_1 R_2 R_3 + V_2 R_1 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

(B) Using superposition, each voltage source only sees a resistive divider network, and we can write out $e$ directly as

$$e = V_1 \frac{R_2}{R_2 || R_3 + R_1} + V_2 \frac{R_1}{R_1 || R_2 + R_3} = \frac{V_1 R_2 R_3 + V_2 R_1 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} = \frac{V_1}{R_1} + \frac{V_2}{R_3}$$

(C) Writing KCL at the two nodes, summing currents out of the nodes yields

$$\frac{e_1}{R_1} + \frac{e_1 - V_S}{R_2} - I_S = 0$$
$$\frac{e_2}{R_4} + \frac{e_2 - V_S}{R_3} + I_S = 0$$

Solving these equations for $e_1$ and $e_2$ gives

$$e_1 = \left(\frac{V_S}{R_2} + I_S\right) \frac{R_1 R_2}{R_1 + R_2} \quad e_2 = \left(\frac{V_S}{R_3} - I_S\right) \frac{R_3 R_4}{R_3 + R_4}$$

(D) Using superposition, we can write down the expressions for the node voltages $e_1$ and $e_2$ directly.

$$e_1 = \frac{R_1}{R_1 + R_2} V_S + \frac{R_1 R_2}{R_1 + R_2} = \left(\frac{V_S}{R_2} + I_S\right) \frac{R_1 R_2}{R_1 + R_2}$$
$$e_2 = \frac{R_4}{R_3 + R_4} V_S - \frac{R_3 R_4}{R_3 + R_4} = \left(\frac{V_S}{R_3} - I_S\right) \frac{R_3 R_4}{R_3 + R_4}$$

Problem 2.5 Answer:

(A) Summing the currents out of each node yields the following equations

$$\frac{e_1 - e_3}{R_5} + \frac{e_1 - e_2}{R_1} - I_S = 0$$
$$\frac{e_2 - e_1}{R_1} + \frac{e_2 - e_3}{R_3} + \frac{e_2}{R_2} = 0$$
$$\frac{e_3 - e_1}{R_5} + \frac{e_3 - e_2}{R_3} + \frac{e_3}{R_4} = 0$$

Replacing the reciprocal resistances with conductances, and rearranging some terms, we can write the above equations as

$$(G_1 + G_5)e_1 + (-G_1)e_2 + (-G_5)e_3 = I_S$$
$$(-G_1)e_1 + (G_1 + G_2 + G_3)e_2 + (-G_3)e_3 = 0$$
$$(-G_5)e_1 + (-G_3)e_2 + (G_3 + G_4 + G_5)e_3 = 0$$
which can be re-written in matrix form as

\[
\begin{bmatrix}
G_1 + G_5 & -G_1 & -G_5 \\
-G_1 & G_1 + G_2 + G_3 & -G_3 \\
-G_5 & -G_3 & G_3 + G_4 + G_5
\end{bmatrix}
\cdot
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
=
\begin{bmatrix}
I_S \\
0 \\
0
\end{bmatrix}
\]

(B) If \( G_1 = G_2 = G_3 = \cdots = G_5 = 1 \text{mho} = 1 \frac{1}{\Omega} \), then the matrix equation above becomes:

\[
\begin{bmatrix}
2 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{bmatrix}
\cdot
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
=
\begin{bmatrix}
I_S \\
0 \\
0
\end{bmatrix}
\]

We can use Matlab, or a calculator, or our hands to solve this equation for \( e_1, e_2 \) and \( e_3 \), and find that

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
=
\begin{bmatrix}
\frac{I_S}{2} \\
\frac{I_S}{2} \\
\frac{I_S}{2}
\end{bmatrix}
\]

(C) The equivalent resistance of the network can be found by dividing the potential \( e_1 \) by the current entering the network, \( I_S \). From Part (B) above, we know that \( e_1 = I_S \), so \( R_{Th} = 1 \Omega \).

Another way to solve this problem is through a symmetry argument. If all the resistances are \( 1 \Omega \), then \( R_3 \) carries no current. To prove this, assume it. The node voltages \( e_2 \) and \( e_3 \) will be equal to each other, because \( \frac{R_1}{R_2} = \frac{R_5}{R_4} \). If \( e_2 \) and \( e_3 \) are equal, then no current flows through \( R_3 \), and we’ve verified our assumption. We can remove \( R_3 \), and realize that the total resistance seen by the source is \( 2 \Omega \) in parallel with \( 2 \Omega \), which is equal to \( 1 \Omega \).

**General Case:**

In the general case, where we can’t use any symmetry tricks to simplify the problem, we have to solve the circuit using the node method. Let’s apply a test voltage \( V_T \) to the resistor network where the source is, and find an expression for the current into the resistor network \( I_T \). We can then evaluate \( \frac{V_T}{I_T} \) to find the equivalent resistance of the network. The circuit is drawn below, rearranged a bit for clarity. The node voltages that we will be solving for are labeled as well.

![Resistor Network Diagram](image)

We know that the current \( I_T \) is equal to the sum of the currents down through the resistors \( R_2 \) and \( R_4 \). This can be written as \( I_T = G_2 e_1 + G_4 e_2 \), where \( G_2 \) and \( G_4 \) are the conductances.
associated with $R_2$ and $R_4$, respectively. To find $e_1$ and $e_2$ we can use the node method to write

\[
\frac{e_1 - V_T}{R_1} + \frac{e_1 - e_2}{R_3} + \frac{e_1}{R_2} = 0
\]

\[
\frac{e_2 - V_T}{R_5} + \frac{e_2 - e_1}{R_3} + \frac{e_2}{R_4} = 0
\]

which can be rewritten in matrix form:

\[
\begin{bmatrix}
G_1 + G_2 + G_3 & -G_3 \\
-G_3 & G_3 + G_4 + G_5
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} =
\begin{bmatrix}
B_T G_1 \\
V_T G_5
\end{bmatrix}
\]

We can solve this for the node voltages using simple matrix algebra, and find $e_1$ and $e_2$ in terms of the conductances and $V_T$. Substitute these back into the equation for $R_{Th}$ above and we find that:

\[
R_{Th} = \frac{(G_1 + G_2 + G_3)(G_3 + G_4 + G_5) - G_3^2}{G_2[(G_3 + G_4 + G_5)G_1 + G_3G_5] + G_4[(G_1 + G_2 + G_3)G_5 + G_1G_3]}
\]

For the special case in Part (B), this reduces down to $1\Omega$. 