Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
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Homework #5 Solutions

Problem 5.1 Answer:

(A) In the saturation regime, the equation for the drain current, i_D is

$$i_D = \frac{K}{2}(v_{GS} - V_T)^2$$

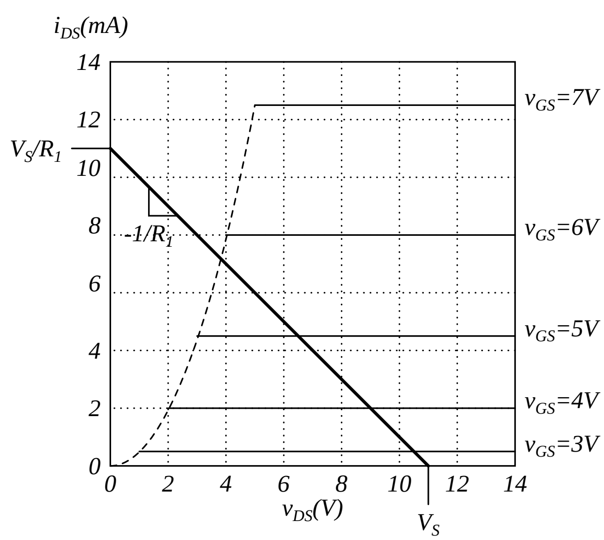
Looking at the MOSFET charecteristics, we see that for $v_{GS} = 6\text{V}$, the i_{DS} vs v_{DS} curve passes through the point $i_D = 8\text{mA}$ and $v_{DS} = 4\text{V}$. This occurs along the parabola that seperates the triode region from the saturation region, so we additionally know that $v_{DS} = v_{GS} - V_T$. Substitute numbers into the above equations to find

$$\begin{aligned} v_{DS} &= v_{GS} - V_T \\ 4\text{V} &= 6\text{V} - V_T \\ V_T &= 2\text{V} \end{aligned}$$

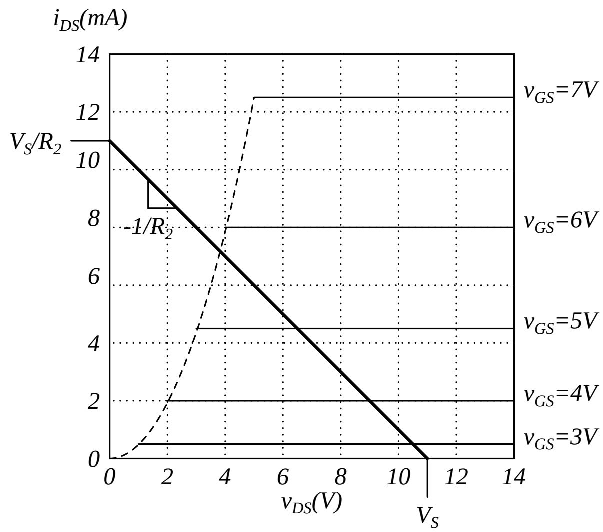
and

$$\begin{aligned} i_D &= \frac{K}{2}(v_{GS} - V_T)^2 \\ 8\text{mA} &= \frac{K}{2}(4\text{V})^2 \\ K &= 1\frac{\text{mA}}{\text{V}^2} \end{aligned}$$

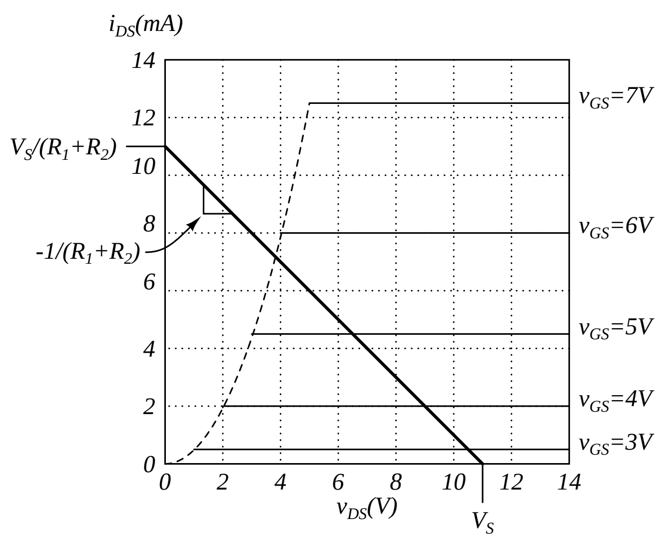
(B) (a) For this circuit, $v_{DS} = v_O = V_S - R_1 i_{DS}$. When $i_{DS} = 0$, $v_{DS} = V_S$. When $v_{DS} = 0$, $i_{DS} = \frac{V_S}{R_1}$. The slope of this line is $-\frac{1}{R_1}$. It is plotted on the figure below.



(b) For this circuit, $v_{DS} = V_S - v_O = V_S - R_2 i_{DS}$. When $i_{DS} = 0$, $v_{DS} = V_S$. When $v_{DS} = 0$, $i_{DS} = \frac{V_S}{R_2}$. The slope of this line is $-\frac{1}{R_2}$. It is plotted on the figure below.



(c) For this circuit, $v_{DS} = V_S - (R_1 + R_2) i_{DS}$. When $i_{DS} = 0$, $v_{DS} = V_S$. When $v_{DS} = 0$, $i_{DS} = \frac{V_S}{R_1+R_2}$. Thes slope of this line is $-\frac{1}{R_1+R_2}$. It is plotted on the figure below.



(C) If the amplifier is operating in its saturation regime, we know that $i_{DS} = \frac{K}{2}(v_{GS} - V_T)^2$, where K and V_T are the parameters we found in Part (A).

(a) For this amplifier, $v_{GS} = v_I$. The output voltage is $v_O = V_S - i_{DS} R_1$. We can write

$$v_O = V_S - \frac{KR_1}{2}(v_I - V_T)^2$$

(b) This amplifier is more complicated to analyze. The output voltage is only $v_O = R_2 i_{DS}$. The gate-source voltage is no longer v_I, it is $v_{GS} = v_I - v_O$. We can write

$$v_O = \frac{KR_2}{2}(v_I - V_T - v_O)^2$$

For simplicity, let $v_X = v_I - V_T$. Expand the quadratic above to get

$$\frac{KR_2}{2}v_O^2 - (KR_2 v_X + 1)v_O + \frac{KR_2}{2}v_X^2 = 0$$

Use the quadratic equation to solve for v_O and find:

$$v_O = \frac{KR_2 v_X + 1 \pm \sqrt{(KR_2 v_X + 1)^2 - (KR_2 v_X)^2}}{KR_2}$$

Substitute $v_I - V_T$ back in for v_X and simplify:

$$v_O = v_I - V_T + \frac{1}{KR_2} \pm \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}$$

This gives us two possible answers for v_O, but which one is correct? We know that $v_{GS} > V_T$. We also know that $v_{GS} = v_I - v_O$, consequently $v_I - V_T > v_O$. Given the expression for v_O above, this means that

$$v_I - V_T > v_I - V_T + \frac{1}{KR_2} \pm \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}$$

or

$$0 > \frac{1}{KR_2} \pm \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}$$

Clearly, the $\pm$ should be a minus sign for our expression for v_O to make sense. The final answer is then

$$v_O = v_I - V_T + \frac{1}{KR_2} - \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}$$

(c) The analysis of this amplifier is similiar. We can see that $v_O = V_S - R_1 i_{DS}$. Also note that $v_{GS} = v_I - R_2 i_{DS}$. We can't just substitute into the equation for the drain current and solve, though, because v_{GS} above is written in terms of the drain current. Realize, though, that the voltage across R_2 does not depend on R_1 as long as the MOSFET is in the saturation region. From the voltage across R_2 we can find i_{DS}, which gives v_O from the equation above. We know the voltage across R_2 from circuit (b) above. It is

$$v_{R_2} = v_I - V_T + \frac{1}{KR_2} - \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}$$

Using this, v_O for this circuit is

$$\begin{aligned}
v_O &= V_S - R_1 i_{DS} \\
v_O &= V_S - R_1 \frac{v_{R_2}}{R_2} \\
v_O &= V_S - \frac{R_1}{R_2}\left(v_I - V_T + \frac{1}{KR_2} - \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}\right)
\end{aligned}$$

(D) For each amplifier find $\frac{dv_o}{dv_i}$ by differentiating the expressions found for v_O in Part (C). Simplify this with the assumption that $v_I - V_T >> \frac{1}{2KR_2}$. F

(b) First, differentiate the expression for v_O in terms of v_I from Part (C) above for circuit (b)

$$\begin{aligned}
\frac{dv_O}{dv_I} &= \frac{d}{dv_I}\left[v_I - V_T + \frac{1}{KR_2} - \sqrt{\left(v_I - V_T + \frac{1}{KR_2}\right)^2 - (v_I - V_T)^2}\right] \\
&= 1 - \frac{d}{dv_I}\left[\sqrt{\frac{2(v_I - V_T)}{KR_2} + \left(\frac{1}{KR_2}\right)^2}\right] \\
&= 1 - \frac{1}{KR_2}\left(\frac{2(v_I - V_T)}{KR_2} + \left(\frac{1}{KR_2}\right)^2\right)^{-.5} \\
&= 1 - (2KR_2(v_I - V_T) + 1)^{-.5}
\end{aligned}$$

If we assume that $v_I - V_T >> \frac{1}{2KR_2}$, then $2KR_2(v_I - V_T) >> 1$ and the expression above simplifies to

$$\frac{dv_O}{dv_I} = 1$$

(c) Recall that the drain current of this circuit, i_{DS} is equal to the output voltage of circuit (b) divided by the resistanct R_2. We can write the following, where $v_{O,b}$ is the expression for the output voltage of circuit (b)

$$\begin{aligned}
\frac{dv_O}{dv_I} &= \frac{d}{dv_I}\left[V_S - \frac{R_1}{R_2}v_{O,b}\right] \\
&= -\frac{R_1}{R_2}\frac{dv_{O,b}}{dv_I}
\end{aligned}$$

We found $\frac{dv_{O,b}}{dv_I}$ above, it is just 1. So for circuit (c) the small signal gain $\frac{dv_O}{dv_I}$ is

$$\frac{dv_O}{dv_I} = -\frac{R_1}{R_2}$$

(E) For all of the amplifiers, when $v_I < V_T$ the MOSFETS are all in the cutoff region, and the drain currents are zero. So, the lower bound on v_I is V_T. That is, the MOSFETS are in saturation for $v_I > V_T$. The upper bound for each circuit depends on keeping $v_{GS} < v_{DS} + V_T$.

(a) For this amplifer, $v_{DS} = v_O$ and $v_{GS} = v_I$. Using our results from Part (C) for v_O, the MOSFET leaves the saturation regime when

$$v_I - V_T = V_S - \frac{KR_1}{2}(v_I - V_T)^2$$

Solve this quadratic (using the quadratic formula) to find that

$$v_I = V_T + \frac{-1 + \sqrt{1 + 2KR_1V_S}}{KR_1}$$

So the MOSFET operates in the saturation region for

$$V_T < v_I < V_T + \frac{-1 + \sqrt{1 + 2KR_1V_S}}{KR_1}$$

and

$$\frac{1 + \sqrt{-1 + 2KR_1V_S}}{KR_1} < v_O < V_S$$

(b) For this circuit, $v_{DS} = V_S - v_O$ and $v_{GS} = v_I - v_O$. The MOSFET is in saturation for

$$v_I - v_O < V_T + V_S - v_O$$

or

$$v_i < V_T + V_S$$

So the MOSFET is in saturation for

$$V_T < v_I < V_T + V_S$$

and

$$0 < v_O < V_S + \frac{1}{KR_2} - \sqrt{\left(V_S + \frac{1}{KR_2}\right)^2 - V_S^2}$$

(c) Realize that at the saturation region's boundary that $v_{GS} - V_T = v_{DS}$. We also know that i_D will be equal to $\frac{V_S - v_{DS}}{R_1 + R_2}$. This gives

$$\frac{K}{2}v_{DS}^2 = \frac{V_S - v_{DS}}{R_1 + R_2}$$

Use the quadratic equation to find this value of v_{DS}, and call it v_{DS}^*.

$$v_{DS}^* = -\frac{1}{K(R_1 + R_2)} + \sqrt{\left(\frac{1}{K(R_1 + R_2)}\right)^2 + \frac{2V_S}{K(R_1 + R_2)}}$$

Given this, the output voltage at the saturation-triode boundary is

$$\begin{aligned}
v_O &= v_{DS}^* + R_2 i_{DS} \\
v_O &= v_{DS}^* + R_2\left(\frac{V_S - v_{DS}}{R_1 + R_2}\right) \\
v_O &= \frac{R_1}{R_1 + R_2}v_{DS}^* + \frac{R_2}{R_1 + R_2}V_S \\
v_O &= \frac{R_1}{R_1 + R_2}\left(-\frac{1}{K(R_1 + R_2)} + \sqrt{\left(\frac{1}{K(R_1 + R_2)}\right)^2 + \frac{2V_S}{K(R_1 + R_2)}}\right) + \frac{R_2}{R_1 + R_2}V_S
\end{aligned}$$

What about v_I at the boundary? Since $v_{GS} - V_T = v_{DS}$ at the boundary:

$$v_I^* - v_{R_2}^* - V_T = v_{DS}^*$$

so

$$
\begin{aligned}
v_I^* &= v_O^* + V_T \\
v_I^* &= V_T + \frac{R_1}{R_1+R_2}\left(-\frac{1}{K(R_1+R_2)} + \sqrt{\left(\frac{1}{K(R_1+R_2)}\right)^2 + \frac{2V_S}{K(R_1+R_2)}}\right) + \frac{R_2}{R_1+R_2}V_S
\end{aligned}
$$

Problem 5.2 Answer:

(A) Assume that v_{DS} of M3 is sufficient to keep it operating in the saturation region. The gate-source voltage of M3 is the voltage drop across the resistor R, which is $\frac{R}{R+8\text{k}\Omega}20\text{V}$. The current i_C is then

$$
i_C = \frac{1\text{mA/V}^2}{2}\left(\frac{20R}{R+8\text{k}\Omega} - V_T\right)^2
$$

We can solve this for R:

$$
\begin{aligned}
2\text{mA} &= \frac{1\text{mA/V}^2}{2}\left(\frac{20R}{R+8\text{k}\Omega} - V_T\right)^2 \\
4\text{V}^2 &= \left(\frac{20R}{R+8\text{k}\Omega} - 2\text{V}\right)^2 \\
4 &= 20\frac{R}{R+8\text{k}\Omega} \\
8\text{k}\Omega &= 4R \\
2\text{k}\Omega &= R
\end{aligned}
$$

We will validate the assumption that M3 operates in saturation for $v_I = 0$ in Part (B) below.

(B) With $v_I = 0$, both M1 and M2 will have identical v_{GS} voltages. The currents through the two transistors will be idential (as long as they are both in the saturation region – we will prove this below). The sum of i_A and i_B is $i_C = 2\text{mA}$, so $i_A = i_B = 1\text{mA}$. The gate-source voltage of M1 and M2 is equal to $-e_X$. We can solve the drain current equation for v_{GS} for M1 and find e_X:

$$
\begin{aligned}
1\text{mA} &= \frac{2\text{mA}/V^2}{2}(-e_X - V_T)^2 \\
1 &= -e_X - 2\text{V} \\
-3\text{V} &= e_X
\end{aligned}
$$

Note that the $e_X = -1$ also satisfies the above quadratic, but it would mean that $v_{GS} < V_T$ for both MOSFETs, putting them in the cutoff region.

Let's confirm that both M2 and M3 are operating, then, in their saturation regimes. For M3, $v_{DS} = -3 + 10 = 7\text{V}$, which is greater than $v_{GS} - V_T = 2\text{V}$, so it is operating in the saturation regime. For M2, $v_O = 10 - 5\text{k}\Omega * 1\text{mA} = 5\text{V}$. Then $v_{DS} = 5 - -3 = 8\text{V}$ which is greater than $v_{GS} - V_T = 1\text{V}$.

(C) We know that the sum of the currents i_A and i_B must be 2mA. We can write

$$
\frac{K}{2}(v_{GS1} - V_T)^2 + \frac{K}{2}(v_{GS2} - V_T)^2 = 2mA
$$

where $v_{GS1} = v_I - e_X$ and $v_{GS2} = -e_X$. Take the Taylor-series expasion of this to arrive at

$$\frac{K}{2}(V_{GS1} - V_T)^2 + K(V_{GS1} - V_T)v_{gs1} + \frac{K}{2}(V_{GS2} - V_T)^2 + K(V_{GS2} - V_T)v_{gs2} = 2mA$$

Now, subtract the first equation evaluated at the operating point determined by V_{GS1} and V_{GS2} to get

$$K(V_{GS1} - V_T)v_{gs1} + K(V_{GS2} - V_T)v_{gs2} = 0$$

We know that $V_{GS1} = V_I - E_X = -E_X$ and that $V_{GS2} = -E_X$. We also know that $v_{gs1} = v_i - e_x$ and $v_{gs2} = -e_x$. Substitute these into the above equation:

$$K(-E_X - V_T)(v_i - e_x) + K(-E_X - V_T)(-e_x) = 0$$

Solving this for e_x yields

$$e_x = \frac{v_i}{2}$$

The output voltage v_O is

$$v_O = V_S - \frac{K * 5\text{k}\Omega}{2}(-e_X - V_T)^2$$

Take the Taylor-series expansion of this and subtract out the operating point to get

$$v_o = -K * 5\text{k}\Omega(-E_X - V_T)(-e_x)$$

Substitute values into this equation and take the derivative with respect to v_i to find the small signal gain:

$$\begin{aligned}
\frac{dv_o}{dv_i} &= \frac{d}{dv_i}\left[-K * 5\text{k}\Omega(-E_X - V_T)(-e_x)\right] \\
&= \frac{d}{dv_i}\left[-2\frac{\text{mA}}{\text{V}^2} * 5\text{k}\Omega(3 - 2)\left(-\frac{v_i}{2}\right)\right] \\
&= \frac{d}{dv_i}\left[-10\left(-\frac{v_i}{2}\right)\right] \\
&= 5
\end{aligned}$$

(D) When M2 enters the cutoff region, its v_{GS} voltage will be equal to V_T. This means that $e_X = -V_T$. All of the current i_C will have to flow through M1 (because M2 is cutoff). Recall that M1's $v_{GS} = v_I - e_X = v_I + V_T$. So, solve M1's drain current equation for v_I:

$$\begin{aligned}
2\text{mA} &= \frac{2\text{mA/V}^2}{2}(v_{GS} - V_T)^2 \\
\sqrt{2}\text{V} &= v_I
\end{aligned}$$

When $v_I = 1\text{V}$ MOSFET M2 will enter the cutoff region, and our model for the circuit is no longer valid.

Problem 5.3 Answer:

(A) MOSFET M1's gate and drain terminals are tied together. This means that $v_{GS} = v_{DS}$ for M1. This ensures that $v_{DS} > v_{GS} - V_T$, so M1 is always operating in its saturation region. We can express its drain current, then, as a function of v_{GS} as follows:

$$i_S = \frac{K}{2}(v_{GS} - V_T)^2$$

Solve this equation for v_{GS} to find

$$v_{GS} = V_T + \sqrt{\frac{2i_S}{K}}$$

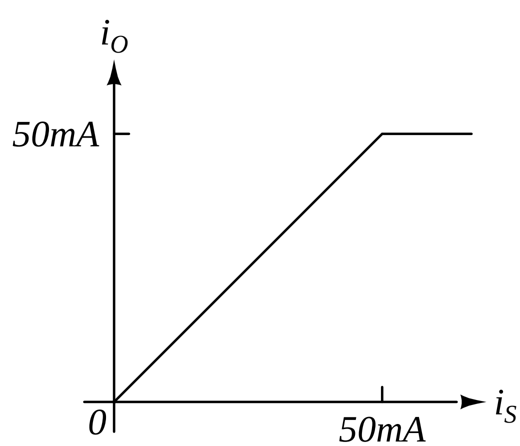
From the circuit diagram we can see that v_{GS} of M1 is equal to v_{GS} of M2. As long as $v_{GS} - v_T < 10\text{V}$ MOSFET M2 will operate in the saturation regime. This means that

$$\begin{aligned} v_{GS} - V_T &< 10\text{V} \\ \sqrt{\frac{2i_S}{K}} &< 10\text{V} \\ i_S &< 50\text{mA} \end{aligned}$$

So as long as $i_S < 50\text{mA}$ M2 operates in the saturation region. Substitute the expression for v_{GS} above to find i_O.

$$\begin{aligned} i_O &= \frac{K}{2}(v_{GS} - V_T)^2 \\ i_O &= \frac{K}{2}\left(V_T + \sqrt{\frac{2i_S}{K}} - V_T\right)^2 \\ i_O &= \frac{K}{2}\frac{2i_S}{K} \\ i_O &= i_S \end{aligned}$$

This circuit is called a "current mirror" because the output current is always equal to the input current. It is "mirrored" around the circuit by the MOSFETS. A graph of i_O vs. i_S is shown below.



(B) With a 1kΩ load resistor between the drain of M2 and the V_S rail, the v_{DS} of M2 is given by

$$v_{DS} = 10\text{V} - i_O * 1\text{k}\Omega$$

As long as this value stays greater than $v_{GS} - V_T$ (where v_{GS} is determined by i_S and M1) M2 will operate in its saturation regime, and $i_O = i_S$. Substituting in the values for K and V_T found from Problem 1 we find

$$\begin{aligned} v_{DS} &> v_{GS} - V_T \\ 10 - i_S * 1\text{k}\Omega &> \sqrt{\frac{2i_S}{K}} \\ 100 - i_S * 2 \times 10^4 + i_S^2 * \times 10^6 &> i_S * 2 \times 10^3 \end{aligned}$$

This inequality becomes false when i_S satisfies the polynomial

$$i_S^2 \times 10^6 - i_S(2 \times 10^4 + 2 \times 10^3) + 100 = 0$$

Solving for i_S gives

$$i_S = 15.583\text{mA} \quad \text{or} \quad 6.417\text{mA}$$

Only the second answer to the quadratic above satisfies the inquality. For any current i_S less than this, the inequality will also be true. So, $i_O = i_S$ as long as

$$i_S < 6.417\text{mA}$$

A graph of i_O vs. i_S is shown below.