Problem 5.1 Answer:

(A) In the saturation regime, the equation for the drain current, $i_D$, is

$$i_D = \frac{K}{2} \left(v_{GS} - V_T \right)^2$$

Looking at the MOSFET characteristics, we see that for $v_{GS} = 6V$, the $i_{DS}$ vs $v_{DS}$ curve passes through the point $i_D = 8mA$ and $v_{DS} = 4V$. This occurs along the parabola that separates the triode region from the saturation region, so we additionally know that $v_{DS} = v_{GS} - V_T$. Substitute numbers into the above equations to find

$$v_{DS} = v_{GS} - V_T$$
$$4V = 6V - V_T$$
$$V_T = 2V$$

and

$$i_D = \frac{K}{2} \left(v_{GS} - V_T \right)^2$$
$$8mA = \frac{K}{2} \left(4V \right)^2$$
$$K = \frac{1mA}{V^2}$$

(B) (a) For this circuit, $v_{DS} = v_O = V_S - R_1i_{DS}$. When $i_{DS} = 0$, $v_{DS} = V_S$. When $v_{DS} = 0$, $i_{DS} = \frac{V_S}{R_1}$. The slope of this line is $-\frac{1}{R_1}$. It is plotted on the figure below.
(b) For this circuit, \( v_{DS} = V_S - v_O = V_S - R_2i_{DS} \). When \( i_{DS} = 0 \), \( v_{DS} = V_S \). When \( v_{DS} = 0 \), \( i_{DS} = \frac{V_S}{R_2} \). The slope of this line is \(-\frac{1}{R_2}\). It is plotted on the figure below.

![Graph of \( v_{DS} \) vs. \( v_{dd} \)]

(c) For this circuit, \( v_{DS} = V_S - (R_1 + R_2)i_{DS} \). When \( i_{DS} = 0 \), \( v_{DS} = V_S \). When \( v_{DS} = 0 \), \( i_{DS} = \frac{V_S}{R_1 + R_2} \). The slope of this line is \(-\frac{1}{R_1 + R_2}\). It is plotted on the figure below.

![Graph of \( v_{DS} \) vs. \( v_{dd} \)]

(C) If the amplifier is operating in its saturation regime, we know that \( i_{DS} = \frac{K}{2}(v_{GS} - V_T)^2 \), where \( K \) and \( V_T \) are the parameters we found in Part (A).

(a) For this amplifier, \( v_{GS} = v_I \). The output voltage is \( v_O = V_S - i_{DS}R_1 \). We can write

\[
v_O = V_S - \frac{KR_1}{2}(v_I - V_T)^2
\]

(b) This amplifier is more complicated to analyze. The output voltage is only \( v_O = R_2i_{DS} \). The gate-source voltage is no longer \( v_I \), it is \( v_{GS} = v_I - v_O \). We can write

\[
v_O = \frac{KR_2}{2}(v_I - V_T - v_O)^2
\]
For simplicity, let \( v_X = v - V_T \). Expand the quadratic above to get
\[
\frac{K R_2}{2} v_O^2 - (K R_2 v_X + 1) v_O + \frac{K R_2}{2} v_X^2 = 0
\]
Use the quadratic equation to solve for \( v_O \) and find:
\[
v_O = \frac{K R_2 v_X + 1 \pm \sqrt{(K R_2 v_X + 1)^2 - (K R_2 v_X)^2}}{K R_2}
\]
Substitute \( v_I - V_T \) back in for \( v_X \) and simplify:
\[
v_O = v_I - V_T + \frac{1}{K R_2} \pm \sqrt{\left(v_I - V_T + \frac{1}{K R_2}\right)^2 - (v_I - V_T)^2}
\]
This gives us two possible answers for \( v_O \), but which one is correct? We know that \( v_{GS} > V_T \). We also know that \( v_{GS} = v_I - v_O \), consequently \( v_I - V_T > v_O \). Given the expression for \( v_O \) above, this means that
\[
v_I - V_T > v_I - V_T + \frac{1}{K R_2} \pm \sqrt{\left(v_I - V_T + \frac{1}{K R_2}\right)^2 - (v_I - V_T)^2}
\]
or
\[
0 > \frac{1}{K R_2} \pm \sqrt{\left(v_I - V_T + \frac{1}{K R_2}\right)^2 - (v_I - V_T)^2}
\]
Clearly, the \( \pm \) should be a minus sign for our expression for \( v_O \) to make sense. The final answer is then
\[
v_O = v_I - V_T + \frac{1}{K R_2} - \sqrt{\left(v_I - V_T + \frac{1}{K R_2}\right)^2 - (v_I - V_T)^2}
\]
(c) The analysis of this amplifier is similar. We can see that \( v_O = V_S - R_1 i_{DS} \). Also note that \( v_{GS} = v_I - R_2 i_{DS} \). We can’t just substitute into the equation for the drain current and solve, though, because \( v_{GS} \) above is written in terms of the drain current. Realize, though, that the voltage across \( R_2 \) does not depend on \( R_1 \) as long as the MOSFET is in the saturation region. From the voltage across \( R_2 \) we can find \( i_{DS} \), which gives \( v_O \) from the equation above. We know the voltage across \( R_2 \) from circuit (b) above. It is
\[
v_{R_2} = v_I - V_T + \frac{1}{K R_2} - \sqrt{\left(v_I - V_T + \frac{1}{K R_2}\right)^2 - (v_I - V_T)^2}
\]
Using this, \( v_O \) for this circuit is
\[
\begin{align*}
v_O &= V_S - R_1 i_{DS} \\
v_O &= V_S - R_1 \frac{v_{R_2}}{R_2} \\
v_O &= V_S - \frac{R_1}{R_2} \left( v_I - V_T + \frac{1}{K R_2} - \sqrt{\left(v_I - V_T + \frac{1}{K R_2}\right)^2 - (v_I - V_T)^2} \right)
\end{align*}
\]
For each amplifier find $\frac{dv_o}{dv_i}$ by differentiating the expressions found for $v_O$ in Part (C). Simplify this with the assumption that $v_I - V_T >> \frac{1}{2KR_2}$. F

(b) First, differentiate the expression for $v_O$ in terms of $v_I$ from Part (C) above for circuit (b)

$$\frac{dv_O}{dv_I} = \frac{d}{dv_I} \left[ v_I - V_T + \frac{1}{KR_2} - \sqrt{\left( v_I - V_T + \frac{1}{KR_2} \right)^2 - (v_I - V_T)^2} \right]$$

$$= 1 - \frac{d}{dv_I} \left[ \sqrt{\frac{2(v_I - V_T)}{KR_2}} + \left( \frac{1}{KR_2} \right)^2 \right]$$

$$= 1 - \frac{1}{KR_2} \left( \frac{2(v_I - V_T)}{KR_2} + \left( \frac{1}{KR_2} \right)^2 \right)^{-5}$$

$$= 1 - (2KR_2(v_I - V_T) + 1)^{-5}$$

If we assume that $v_I - V_T >> \frac{1}{2KR_2}$, then $2KR_2(v_I - V_T) >> 1$ and the expression above simplifies to

$$\frac{dv_O}{dv_I} = 1$$

(c) Recall that the drain current of this circuit, $i_{DS}$ is equal to the output voltage of circuit (b) divided by the resistance $R_2$. We can write the following, where $v_{O,b}$ is the expression for the output voltage of circuit (b)

$$\frac{dv_O}{dv_I} = \frac{d}{dv_I} \left[ V_S - \frac{R_1}{R_2} v_{O,b} \right]$$

$$= - \frac{R_1}{R_2} \frac{dv_{O,b}}{dv_I}$$

We found $\frac{dv_{O,b}}{dv_I}$ above, it is just 1. So for circuit (c) the small signal gain $\frac{dv_O}{dv_I}$ is

$$\frac{dv_O}{dv_I} = \frac{R_1}{R_2}$$

(E) For all of the amplifiers, when $v_I < V_T$ the MOSFETS are all in the cutoff region, and the drain currents are zero. So, the lower bound on $v_I$ is $V_T$. That is, the MOSFETS are in saturation for $v_I > V_T$. The upper bound for each circuit depends on keeping $v_{GS} < v_{DS} + V_T$.

(a) For this amplifier, $v_{DS} = v_O$ and $v_{GS} = v_I$. Using our results from Part (C) for $v_O$, the MOSFET leaves the saturation regime when

$$v_I - V_T = V_S - \frac{KR_1}{2} (v_I - V_T)^2$$

Solve this quadratic (using the quadratic formula) to find that

$$v_I = V_T + \frac{-1 + \sqrt{1 + 2Kr_1 V_S}}{KR_1}$$
So the MOSFET operates in the saturation region for

\[ V_T < v_I < V_T + \frac{-1 + \sqrt{1 + 2KR_1V_S}}{KR_1} \]

and

\[ \frac{1 + \sqrt{-1 + 2KR_1V_S}}{KR_1} < v_O < V_S \]

(b) For this circuit, \( v_{DS} = V_S - v_O \) and \( v_{GS} = v_I - v_O \). The MOSFET is in saturation for

\[ v_I - v_O < V_T + V_S - v_O \]

or

\[ v_i < V_T + V_S \]

So the MOSFET is in saturation for

\[ V_T < v_I < V_T + V_S \]

and

\[ 0 < v_O < V_S + \frac{1}{KR_2} - \sqrt{\left( \frac{1}{V_S} + \frac{1}{KR_2} \right)^2 - V_S^2} \]

(c) Realize that at the saturation region’s boundary that \( v_{GS} - V_T = v_{DS} \). We also know that \( i_D \) will be equal to \( \frac{V_S - v_{DS}}{R_1 + R_2} \). This gives

\[ \frac{K}{2} \frac{v_{DS}^2}{R_1 + R_2} = \frac{V_S - v_{DS}}{R_1 + R_2} \]

Use the quadratic equation to find this value of \( v_{DS} \), and call it \( v_{DS}^* \).

\[ v_{DS}^* = -\frac{1}{K(R_1 + R_2)} + \sqrt{\left( \frac{1}{K(R_1 + R_2)} \right)^2 + \frac{2V_S}{K(R_1 + R_2)}} \]

Given this, the output voltage at the saturation-triode boundary is

\[ v_O = v_{DS}^* + R_2i_{DS} \]

\[ v_O = v_{DS}^* + R_2 \left( \frac{V_S - v_{DS}}{R_1 + R_2} \right) \]

\[ v_O = \frac{R_1}{R_1 + R_2} v_{DS}^* + \frac{R_2}{R_1 + R_2} V_S \]

\[ v_O = \frac{R_1}{R_1 + R_2} \left( -\frac{1}{K(R_1 + R_2)} + \sqrt{\left( \frac{1}{K(R_1 + R_2)} \right)^2 + \frac{2V_S}{K(R_1 + R_2)}} \right) + \frac{R_2}{R_1 + R_2} V_S \]

What about \( v_I \) at the boundary? Since \( v_{GS} - V_T = v_{DS}^* \) at the boundary:

\[ v_I^* - v_{R_2} - V_T = v_{DS}^* \]
\[v_I^* = v_O^* + V_T\]
\[v_I^* = V_T + \frac{R_1}{R_1 + R_2} \left( -\frac{1}{K(R_1 + R_2)} + \sqrt{\left( \frac{1}{K(R_1 + R_2)} \right)^2 + \frac{2V_S}{K(R_1 + R_2)}} \right) + \frac{R_2}{R_1 + R_2} V_S\]

**Problem 5.2 Answer:**

(A) Assume that \(v_{DS}\) of M3 is sufficient to keep it operating in the saturation region. The gate-source voltage of M3 is the voltage drop across the resistor \(R\), which is \(\frac{R}{R + 8k\Omega}\) - 20V. The current \(i_C\) is then

\[i_C = \frac{1mA/V^2}{2} \left( \frac{20R}{R + 8k\Omega} - V_T \right)^2\]

We can solve this for \(R\):

\[
2mA = \frac{1mA/V^2}{2} \left( \frac{20R}{R + 8k\Omega} - V_T \right)^2
\]

\[
4V^2 = \left( \frac{20R}{R + 8k\Omega} - 2V \right)^2
\]

\[
4 = 20 \frac{R}{R + 8k\Omega}
\]

\[
8k\Omega = 4R
\]

\[
2k\Omega = R
\]

We will validate the assumption that M3 operates in saturation for \(v_I = 0\) in Part (B) below.

(B) With \(v_I = 0\), both M1 and M2 will have identical \(v_{GS}\) voltages. The currents through the two transistors will be identical (as long as they are both in the saturation region – we will prove this below). The sum of \(i_A\) and \(i_B\) is \(i_C = 2mA\), so \(i_A = i_B = 1mA\). The gate-source voltage of M1 and M2 is equal to \(-e_X\). We can solve the drain current equation for \(v_{GS}\) for M1 and find \(e_X\):

\[
1mA = \frac{2mA/V^2}{2} (e_X - V_T)^2
\]

\[
1 = -e_X - 2V
\]

\[
-3V = e_X
\]

Note that the \(e_X = -1\) also satisfies the above quadratic, but it would mean that \(v_{GS} < V_T\) for both MOSFETs, putting them in the cutoff region.

Let’s confirm that both M2 and M3 are operating, then, in their saturation regimes. For M3, \(v_{DS} = -3 + 10 = 7V\), which is greater than \(v_{GS} - V_T = 2V\), so it is operating in the saturation regime. For M2, \(v_O = 10 - 5k\Omega * 1mA = 5V\). Then \(v_{DS} = 5 - 3 = 8V\) which is greater than \(v_{GS} - V_T = 1V\).

(C) We know that the sum of the currents \(i_A\) and \(i_B\) must be 2mA. We can write

\[
\frac{K}{2} (v_{GS1} - V_T)^2 + \frac{K}{2} (v_{GS2} - V_T)^2 = 2mA
\]
where \( v_{GS1} = v_I - e_X \) and \( v_{GS2} = -e_X \). Take the Taylor-series expansion of this to arrive at

\[
\frac{K}{2}(v_{GS1} - V_T)^2 + K(v_{GS1} - V_T)v_{gs1} + \frac{K}{2}(v_{GS2} - V_T)^2 + K(v_{GS2} - V_T)v_{gs2} = 2mA
\]

Now, subtract the first equation evaluated at the operating point determined by \( V_{GS1} \) and \( V_{GS2} \) to get

\[
K(v_{GS1} - V_T)v_{gs1} + K(v_{GS2} - V_T)v_{gs2} = 0
\]

We know that \( V_{GS1} = V_I - e_X = -E_X \) and that \( V_{GS2} = -E_X \). We also know that \( v_{gs1} = v_I - e_x \) and \( v_{gs2} = -e_x \). Substitute these into the above equation:

\[
K(-E_X - V_T)(v_I - e_x) + K(-E_X - V_T)(-e_x) = 0
\]

Solving this for \( e_x \) yields

\[
e_x = \frac{v_I}{2}
\]

The output voltage \( v_O \) is

\[
v_O = V_S - \frac{K \times 5k\Omega}{2}(-e_X - V_T)^2
\]

Take the Taylor-series expansion of this and subtract out the operating point to get

\[
v_o = -K \times 5k\Omega(-E_X - V_T)(-e_x)
\]

Substitute values into this equation and take the derivative with respect to \( v_I \) to find the small signal gain:

\[
\frac{dv_o}{dv_I} = \frac{d}{dv_I} \left[ -K \times 5k\Omega(-E_X - V_T)(-e_x) \right]
\]

\[
= \frac{d}{dv_I} \left[ -2mA \times \frac{5k\Omega}{2^2}(3 - 2) \left( -\frac{v_I}{2} \right) \right]
\]

\[
= \frac{d}{dv_I} \left[ -10 \left( -\frac{v_I}{2} \right) \right]
\]

\[
= 5
\]

(D) When \( M2 \) enters the cutoff region, its \( v_{GS} \) voltage will be equal to \( V_T \). This means that \( e_X = -V_T \). All of the current \( i_C \) will have to flow through \( M1 \) (because \( M2 \) is cutoff). Recall that \( M1's \) \( v_{GS} = v_I - e_X = v_I + V_T \). So, solve \( M1's \) drain current equation for \( v_I \):

\[
2mA = \frac{2mA/V^2}{2}(v_{GS} - V_T)^2
\]

\[
\sqrt{2V} = v_I
\]

When \( v_I = 1V \) MOSFET \( M2 \) will enter the cutoff region, and our model for the circuit is no longer valid.
Problem 5.3 Answer:

(A) MOSFET M1’s gate and drain terminals are tied together. This means that \( v_{GS} = v_{DS} \) for M1. This ensures that \( v_{DS} > v_{GS} - V_T \), so M1 is always operating in its saturation region. We can express its drain current, then, as a function of \( v_{GS} \) as follows:

\[
i_S = \frac{K}{2} (v_{GS} - V_T)^2
\]

Solve this equation for \( v_{GS} \) to find

\[
v_{GS} = V_T + \sqrt{\frac{2i_S}{K}}
\]

From the circuit diagram we can see that \( v_{GS} \) of M1 is equal to \( v_{GS} \) of M2. As long as \( v_{GS} - V_T < 10V \) MOSFET M2 will operate in the saturation regime. This means that

\[
\begin{align*}
v_{GS} - V_T &< 10V \\
\sqrt{\frac{2i_S}{K}} &< 10V \\
i_S &< 50mA
\end{align*}
\]

So as long as \( i_S < 50mA \) M2 operates in the saturation region. Substitute the expression for \( v_{GS} \) above to find \( i_O \).

\[
i_O = \frac{K}{2} (v_{GS} - V_T)^2
\]

\[
i_O = \frac{K}{2} \left( V_T + \sqrt{\frac{2i_S}{K}} - V_T \right)^2
\]

\[
i_O = \frac{K}{2} \frac{2i_S}{K}
\]

\[
i_O = i_S
\]

This circuit is called a “current mirror” because the output current is always equal to the input current. It is “mirrored” around the circuit by the MOSFETs. A graph of \( i_O \) vs. \( i_S \) is shown below.
(B) With a 1kΩ load resistor between the drain of M2 and the $V_S$ rail, the $v_{DS}$ of M2 is given by

$$v_{DS} = 10V - i_O \times 1k\Omega$$

As long as this value stays greater than $v_{GS} - V_T$ (where $v_{GS}$ is determined by $i_S$ and M1) M2 will operate in its saturation regime, and $i_O = i_S$. Substituting in the values for $K$ and $V_T$ found from Problem 1 we find

$$v_{DS} > v_{GS} - V_T$$

$$10 - i_S \times 1k\Omega > \sqrt{\frac{2i_S}{K}}$$

$$100 - i_S \times 2 \times 10^4 + i_S^2 \times 10^6 > i_S \times 2 \times 10^3$$

This inequality becomes false when $i_S$ satisfies the polynomial

$$i_S^2 \times 10^6 - i_S(2 \times 10^4 + 2 \times 10^3) + 100 = 0$$

Solving for $i_S$ gives

$$i_S = 15.583mA \quad \text{or} \quad 6.417mA$$

Only the second answer to the quadratic above satisfies the inequality. For any current $i_S$ less than this, the inequality will also be true. So, $i_O = i_S$ as long as

$$i_S < 6.417mA$$

A graph of $i_O$ vs. $i_S$ is shown below.