Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Homework #6 Solutions

Problem 6.1 Answer:

(A) For M1, $v_{GS} = v_{IN}$. When $v_{IN} < V_T$, M1 is in the cutoff region, and $v_A = 10\text{V}$. When $v_{IN} > V_T$ the MOSFET enters the saturation region. The output voltage of the first stage, v_A is given by

$$v_A = 10\text{V} - \frac{5\text{k}\Omega * 2\text{mA}}{2\text{V}^2}(v_{IN} - 2\text{V})^2$$

The MOSFET enters the saturation region when $v_{DS} = v_{GS} - V_T$. Using the equation for v_A above (remember that $v_A = v_{DS}$) we can write out a quadratic equation for v_{IN}:

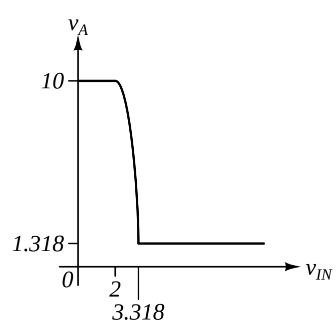
$$v_{IN} - V_T = 10\text{V} - (5\text{V}^{-1})2\text{V}^2(v_{IN} - 2\text{V})^2$$

Using the quadratic equation we find M1 enters the triode region when:

$$v_{IN} \approx 3.318\text{V} \quad \text{and} \quad v_A \approx 1.318\text{V}$$

The triode region approximation locks v_A at this voltage for all greater values of v_{IN}. A sketch of the output voltage is shown below.

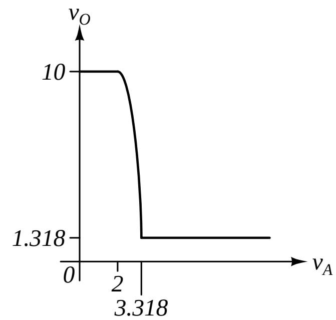
$$v_A = \begin{cases} 10\text{V} & v_{IN} < V_T \\ 10\text{V} - (5\text{V}^{-1})(v_{IN} - 2\text{V})^2 & V_T < v_{IN} < 3.318\text{V} \\ 1.318\text{V} & 3.318\text{V} < v_{IN} \end{cases}$$



(B) This circuit is the same as the one from Part (A), with v_{IN} replaced by v_A and v_A replaced by v_O. We can reuse our answers from above and write:

$$v_O = \begin{cases} 10\text{V} & v_A < V_T \\ 10\text{V} - (5\text{V}^{-1})(v_A - 2\text{V})^2 & V_T < v_A < 3.318\text{V} \\ 1.318\text{V} & 3.318\text{V} < v_A \end{cases}$$

Here's the same graph from above, relabeled.



(C) For M1 to be in saturation, $1.318\text{V} < v_A < 10\text{V}$. For M2 to be in saturation, $2\text{V} < v_A < 3.318\text{V}$. As long as M2 is operating in the saturation region, M1 will be as well (the restrictions on v_A are tighter for M2). We can use the formula we found in Part (A) to find that for this range of v_A, v_{IN} must be:

$$3.156\text{V} < v_{IN} < 3.265\text{V}$$

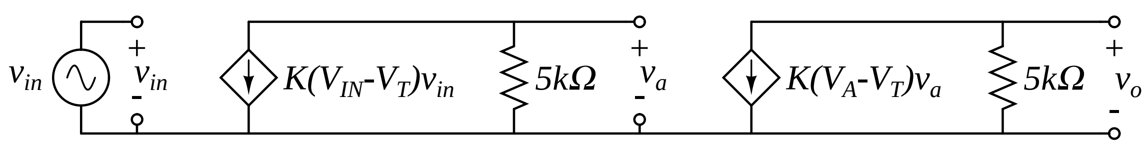
(D) Knowing that both MOSFETs operate in the saturation regime, we can combine the saturation equations from Parts (A) and (B) above to arrive at:

$$v_O = 10\text{V} - (5\text{V}^{-1})\left(10\text{V} - (5\text{V}^{-1})(v_{IN} - 2\text{V})^2 - V_T\right)^2$$

(E) The gain of this amplifier is the derivative of the above expression with respect to v_{IN} evaluated at the operating point $V_{IN} = 3.2\text{V}$. This gives:

$$\begin{aligned}
\frac{v_o}{v_i} &= \frac{d}{dv_I}\left[10\text{V} - (5\text{V}^{-1})\left(8\text{V} - (5\text{V}^{-1})(v_{IN} - 2\text{V})^2\right)^2\right]\Big|_{v_{IN}=3.2\text{V}} \\
&= \left[(10\text{V}^{-1})\left(8\text{V} - (5\text{V}^{-1})(v_{IN} - 2\text{V})^2\right)(10\text{V}^{-1})(v_{IN} - 2\text{V})\right]\Big|_{v_{IN}=3.2\text{V}} \\
&= 96
\end{aligned}$$

(F) The small signal circuit is shown below.



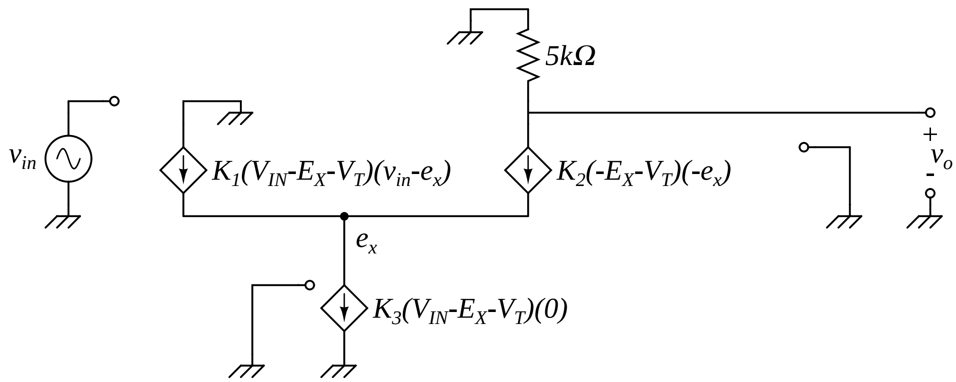
From Part (A) we know that when $v_{IN} = 3.2\text{V}$, $v_A = 2.8\text{V}$. The small-signal output voltage, v_o is

$$\begin{aligned}
v_o &= -v_a * \left(5\text{k}\Omega * 2\frac{\text{mA}}{\text{V}^2}\right)(V_A - V_T) \\
&= \left(5\text{k}\Omega * 2\frac{\text{mA}}{\text{V}^2}\right)(V_{IN} - V_T) * \left(5\text{k}\Omega * 2\frac{\text{mA}}{\text{V}^2}\right)(V_A - V_T) \\
&= 10 * 10 * 1.2 * .8 \\
&= 96
\end{aligned}$$

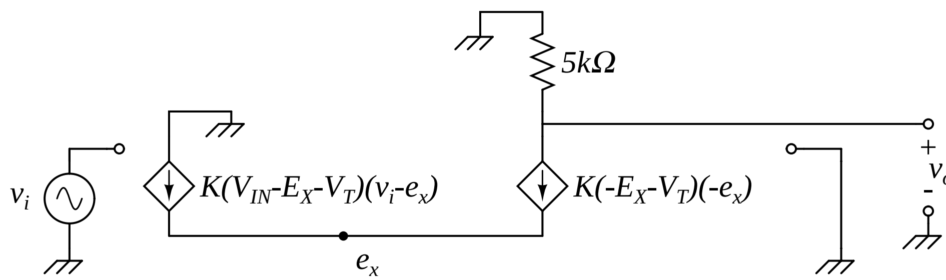
Which is identical to the answer from Part (E).

Problem 6.2 Answer:

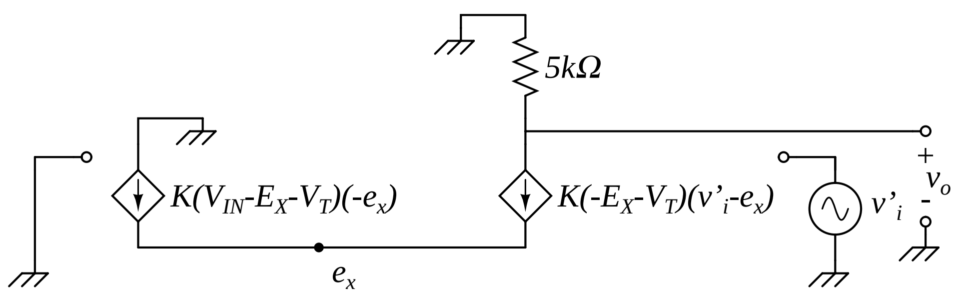
(A) The small signal model for the circuit is drawn below.



Notice that the small signal input voltage to M3 (the transistor on the bottom) is zero. That means the small signal drain current is also zero (M3's dependent source is an open circuit), and the model can be redrawn as follows:



(B) Modify the small-signal circuit as shown below.



The small signal currents into the node e_x must equal zero. So we know that

$$(2\text{mA/V})(v_i - e_x) + (2\text{mA/V})(-e_x) = 0$$

Solving this equation for e_x gives

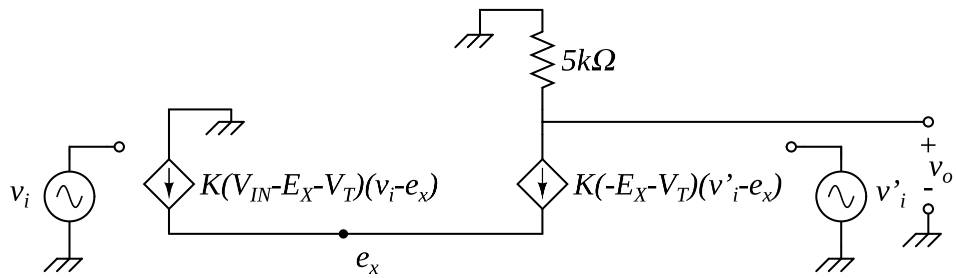
$$e_x = \frac{v_i}{2}$$

We can see that $v_o = -(10\text{V}^{-1})(-E_X - V_T)(-e_x)$. Substituting values in and dividing by v_i gives

$$\frac{v_o}{v_i} = 5$$

This is the same answer found in Problem 5.2.

(C) This is very similar to the derivation above. Modify the small-signal circuit as shown below.



The small signal currents into the e_x node must still equal zero, so

$$(2\text{mA/V})(-e_x) + (2\text{mA/V})(v_i' - e_x) = 0$$

Solving this for e_x gives

$$e_x = \frac{v_i'}{2}$$

This time $v_o = -(10\text{V}^{-1})(-E_X - V_T)(v_i' - e_x)$. Thus, the small signal gain is

$$\frac{v_o}{v_i'} = -5$$

(D) If both v_i and v_i' are applied simultaneously to the gates of M1 and M2, then:

$$(2\text{mA/V})(v_i - e_x) + (2\text{mA/V})(v_i' - e_x) = 0$$

We can see that

$$e_x = \frac{v_i + v_i'}{2}$$

Now $v_o = -(10\text{V}^{-1})(-E_X - V_T)(v_i' - \frac{v_i+v_i'}{2})$. This can be rewritten as:

$$v_o = 5(v_i - v_i')$$

This amplifier only amplifies the difference between the two input signals, hence the name "difference amplifier".

Problem 6.3 Answer:

(a) Recall that capacitances in parallel add, and capacitances in series have reciprocals that add (just like conductances). The two capacitors on the right form one capacitor with value $(C_3 + C_4)$. This is in series with C_2 to become $\frac{C_2(C_3+C_4)}{C_2+C_3+C_4}$. This is in parallel with C_1, making the total capacitance

$$C_{\text{total}} = C_1 + \frac{C_2(C_3 + C_4)}{C_2 + C_3 + C_4}$$

(b) Recall that inductances in series add, and inductors in parallel have reciprocals that add (just like resistors). The two inductors on the right form on inductors with value $(L_3 + L_4)$. This is in parallel with L_2 to become $\frac{L_2(L_3+L_4)}{L_2+L_3+L_4}$. This is in series with L_1, making the total inductance

$$L_{\text{total}} = L_1 + \frac{L_2(L_3 + L_4)}{L_2 + L_3 + L_4}$$

Problem 6.4 Answer:

(A) (a) Remember that the voltage across the capacitor is proportional to the amount of charge on the capacitor. In zero time (from $0^- \to 0^+$) no charge can accumulate on the capacitor (because there is not an infinite current in the circuit). Hence, the voltage across the capacitor at $t = 0+$ must be the same as it was at $t = 0^-$.

$$v(0^+) = 0$$

To find $v(\infty)$, consider the way the current i_S will divide into R_3 and R_2. At $t = 0$ there is no voltage across the capacitor. As time goes on the capacitor voltage increases, and more of i_S splits into R_2 than R_3. Eventually, all of i_S will flow through R_2, the voltage across the capacitor will equal the voltage across R_2, which is

$$v(\infty) = I_S R_2$$

Alternatively, consider the Thévenin equivalent circuit formed by i_S, R_1, and R_2. The Thévenin voltage is $I_S R_2$, and the Thévenin resistance is R_2. Once v reaches the value of the Thévenin voltage, no more current will flow in or out of the capacitor, and it's voltage will remain fixed at $I_S R_2$.

(b) Remember that the current through an inductor will be continuous unless there is an infinite voltage applied across it (the magnetic flux is proportional to the integral of the voltage applied across it over time). Because there is not an infinite voltage source in the circuit, the current through the inductor must be continuous, so

$$i(0^+) = i(0^-) = 0$$

At $t = \infty$ the inductor will act like a short circuit. Consider the Thévenin circuit formed by the voltage sources and resistors.

$$\begin{aligned} V_{Th} &= \frac{R_2}{R_1 + R_2} V_1 - \frac{R_1}{R_1 + R_2} V_2 \\ R_{Th} &= \frac{R_1 R_2}{R_1 + R_2} \end{aligned}$$

If L acts like a short at $t = \infty$, the current i will be

$$i(\infty) = \frac{V_{Th}}{R_{Th}} = \frac{V_1}{R_1} - \frac{V_2}{R_2}$$

(B) (a) The time constant is the capacitance times the equivalent resistance seen at the capacitor's terminals. This resistance is $R_2 + R_3$. The time constant is

$$\tau = C(R_2 + R_3)$$

(b) The time constant here is the inductance divided by the equivalent resistance seen at its terminals. This resistance is $R_1||R_2$. The time constant is

$$\tau = \frac{L(R_1 + R_2)}{R_1 R_2}$$

(C) We know that the solution will be of the form

$$F + (I - F)e^{-\frac{t}{\tau}}$$

where F is the final value of the circuit variable, I is the initial value, and τ is the time constant.

Using the above:

$$v(t > 0) = I_S R_2 \left(1 - e^{-\frac{t}{C(R_2+R_3)}}\right)$$

(b) Again, we can write the expression directly as above

$$i(t > 0) = \left(\frac{V_1}{R_1} - \frac{V_2}{R_2}\right)\left(1 - e^{-\frac{tR_1R_2}{L(R_1+R_2)}}\right)$$

(D) If $v_2(t) = V_2$ for all time, at $t = 0^-$ then, $i = -\frac{V_2}{R_2}$. The final value $i(\infty)$ is, as before:

$$i(\infty) = \frac{V_1}{R_1} - \frac{V_2}{R_2}$$

And the time constant is still the same. By inspection:

$$i(t > 0) = \frac{V_1}{R_1} - \frac{V_2}{R_2} - \frac{V_1}{R_1}e^{-\frac{tR_1R_2}{L(R_1+R_2)}}$$

(E) Consider the inductor to be a current source with value i as given above. We can use superposition to find i_{R_2}.

$$i_{R_2} = \frac{v_1}{R_1 + R_2} + \frac{v_2}{R_1 + R_2} - i\frac{R_1}{R_1 + R_2}$$

Using the expression for i we found in Part (D) above, we find for $t > 0$

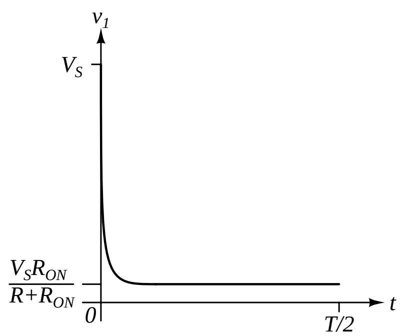
$$\begin{aligned} i_{R_2} &= \frac{V_1 + V_2}{R_1 + R_2} - \frac{R_1}{R_1 + R_2}\left(\frac{V_1}{R_1} - \frac{V_2}{R_2} - \frac{V_1}{R_1}e^{-\frac{tR_1R_2}{L(R_1+R_2)}}\right) \\ &= \frac{V_2 + \frac{R_1}{R_2}}{R_1 + R_2} + \frac{V_1}{R_1 + R_2}e^{-\frac{tR_1R_2}{L(R_1+R_2)}} \end{aligned}$$

Problem 6.5 Answer:

(A) At $t = 0$, v_{IN} switches to V_S, and the first MOSFET turns into a resistor with value R_{ON} in the SC model. The initial value of $v_1 = V_S$. The final value is $V_S \frac{R_{ON}}{R+R_{ON}}$. The time constant for the system is $\tau = C_{eq}\frac{R_{ON}R}{R_{ON}+R}$. Because $R_{ON} << R$, this expression reduces to $\tau = C_{eq}R_{ON}$. By inspection, as in Problem 4, we can write

$$v_1(0 < t < T/2) = V_S \frac{R_{ON}}{R + R_{ON}} + \left(V_S - V_S \frac{R_{ON}}{R + R_{ON}}\right) e^{-\frac{t}{C_{eq}R_{ON}}}$$

Note that $RC_{eq} << \frac{T}{2}$, so $R_{ON}C_{eq} << RC_{eq} << \frac{T}{2}$. This means that the capacitor will discharge *very* quickly, and the transition in v_1 is sharp.



(B) The Lo-Hi transition of the input happens at $t = 0$. We can find the time when v_1 falls below V_{OL} by setting the expression found in Part (A) equal to V_{OL} and solving for t. This gives

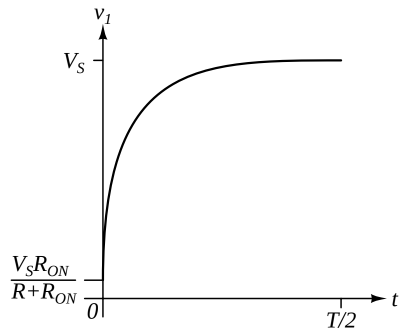
$$\Delta t = -C_{eq}R_{ON} \ln\left(\frac{V_{OL}(R_{ON} + R) - V_S R_{ON}}{V_S R}\right)$$

(C) This solution is similar to Part (A) and (B). At $t = \frac{T}{2}$ the first MOSFET turns off, and the capacitor begins charging up to V_S through the resistor R. The initial value of v_1 is $v_1(T/2) = V_S \frac{R_{ON}}{R_{ON}+R}$. The final value is $v_1(T) = V_S$, because $RC_{eq} << \frac{T}{2}$. The time constant is now $\tau = RC_{eq}$. By inspection:

$$v_1(T/2 < t < T) = V_S + \left(V_S \frac{R_{ON}}{R_{ON} + R} - V_S\right) e^{-\frac{t-\frac{T}{2}}{RC_{eq}}}$$

Notice the $t - \frac{T}{2}$ term in the exponent. The input transition occurs at $t = \frac{T}{2}$, so we must time-shift the output expression by $\frac{T}{2}$.

This is graphed below. Note that $R_{ON}C_{eq} << RC_{eq}$, so this transition happens slower than the one in Parts (A) and (B).



Again, set this expression equal to V_{OH} and solve to find

$$t = \frac{T}{2} - C_{eq} R \ln\left(\frac{(V_S - V_{OH})(R_{ON} + R)}{V_S}\right)$$

The time delay is the expression above minus $\frac{T}{2}$, which is

$$\Delta t = -C_{eq} R \ln\left(\frac{(V_S - V_{OH})(R_{ON} + R)}{V_S}\right)$$

(D) The switching speed of the buffer is reduced by lowering R, but two other important things are increased. The low output voltage is $V_S \frac{R_{ON}}{R+R_{ON}}$. If R is decreased, this is increased, decreasing the low-end noise margin. Also, decreasing R increases the power dissipated when the MOSFET is on.