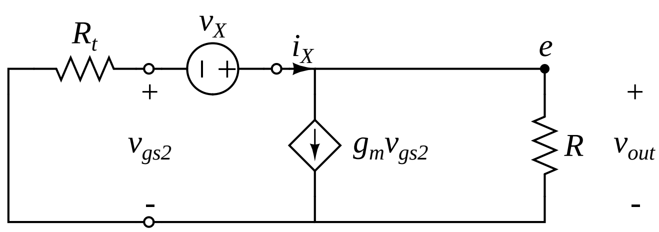
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Homework #7 Solutions

Problem 7.1 Answer:

(A) To find the Thévenin equivalent resistance seen by C_{GD}, replace the capacitor with a test voltage and current and set the independent sources to zero, as shown in the figure below.



We know that $v_{gs2} = -i_X R_t$. The voltage at the node labeled e is then $-i_X R_t + v_X$. We can write KCL at e:

$$i_X = g_m v_{gs2} + \frac{e}{R}$$

Substituting values in and solving for $\frac{v_X}{i_X}$ gives

$$\begin{aligned} i_X &= g_m(-i_X R_t) + \frac{-i_X R_t + v_X}{R} \\ i_X \left(1 + g_m R_t + \frac{R_t}{R}\right) &= \frac{v_X}{R} \\ \frac{v_X}{i_X} &= R + R_t(g_m R + 1) \end{aligned}$$

(B) The time constant associated with C_{GD} is the capacitance multiplied by the Thévenin resistance seen at it's terminals. This value is

$$\tau = C_{GD}(R + R_t(g_m R + 1))$$

(C) Because there are no infinite currents in the circuit, the voltage across C_{GD} must be continuous at $t = 0$. This means that $v_{gs2} = v_{out}$ at $t = 0^+$. We can write at $t = 0^+$:

$$\frac{v_t(0^+) - v_{out}(0^+)}{R_t} = \frac{v_{out}(0^+)}{R} + g_m v_{out}(0^+)$$

Solving this equation for v_{out} gives

$$v_{out}(0^+) = \frac{v_t R}{R_t + g_m R R_t + R}$$

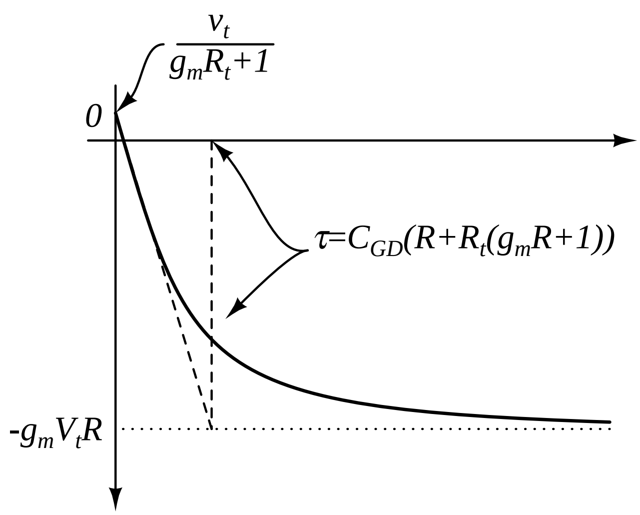
Note that from Part (B), $g_m R >> 1$, so the above expression reduces to

$$v_{out}(0^+) = \frac{v_t}{g_m R_t + 1}$$

Eventually C_{GD} will look like an open circuit, and v_{out} will settle to $v_{out} = -g_m v_t R$. By inspection

$$v_{out}(t) = -g_m v_t R + \left(\frac{v_t}{g_m R_t + 1} + g_m v_t R \right) e^{-\frac{t}{C_{GD}(R+R_t(g_m R+1))}}$$

This is graphed below.



Problem 7.2 Answer:

(A) We know the initial and final values of v (0 and V_0 respectively). The time constant is just RC. Consequently

$$v = V_0 \left(1 - e^{-\frac{t}{RC}}\right)$$

(B) The final energy is

$$E_C = \frac{1}{2} C V_0^2$$

The energy supplied by the voltage source is given by the following, where i is the current into the capacitor

$$E_{V_0} = \int_0^\infty (V_0 * i(t)) dt$$

Recognizing that $i = \frac{V_0 - v}{R}$, and using the answer from Part (A) we can rewrite this integral as

$$E_{V_0} = \frac{V_0^2}{R} \int_0^\infty (e^{-\frac{t}{RC}}) dt$$

This evaluates to

$$E_{V_0} = \frac{V_0^2}{R}(RC)\left(-e^{-\frac{\infty}{RC}} + e^{-\frac{0}{RC}}\right) = CV_O^2$$

There is a difference, only half of the energy supplied ended up in the capacitor. The other half was dissipated by the resistor during the charging process.

(C) In this circuit, the time constant has changed. The two capacitors in series become $\frac{CC_1}{C+C_1}$, so

$$\tau = R\frac{CC_1}{C+C_1}$$

When the switch closes, C_1 will begin to charge C through R. The voltage across C_1 can be expressed as $v_{C_1} = V_1 - \frac{1}{C_1}\int_0^t i(t)dt$, and the voltage across C can be expressed as $v_C = \frac{1}{C}\int_0^t i(t)dt$. We know that $i = \frac{v_{C_1}-v_C}{R}$, so we can write out an expression for i as follows:

$$i(t) = \frac{V_1 - \frac{1}{C_1}\int_0^t i(t)dt - \frac{1}{C}\int_0^t i(t)dt}{R}$$

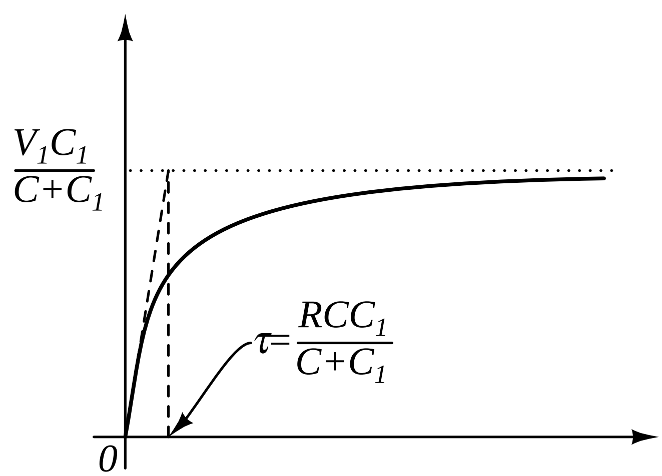
The resulting differential equation can be solved to find that

$$i(t) = \frac{V_1}{R}e^{-\frac{t}{\tau}}$$

We could have written this expression by inspection, as we knew the initial and final values of the current i. The final voltage on the capacitor C is then

$$v(\infty) = \frac{1}{C}\int_0^\infty i(t)dt = \frac{V_1\tau}{R} = V_1\frac{C_1}{C+C_1}$$

A sketch of $v(t)$ appears below.



If the same energy is to end up in C, $v(\infty)$ must be equal for both circuits. This means

$$V_1 = V_0(1 + \frac{C}{C_1})$$

(D) The energy dissipated by the resistor this time is equal to

$$\begin{aligned}
E_R &= R\int_0^\infty i^2(t)dt \\
&= \frac{V_1^2}{R}\int_0^\infty e^{-\frac{t}{\tau}}dt \\
&= \frac{V_1^2}{R}\tau\left[-e^{-\frac{\infty}{\tau}} + e^{-\frac{0}{\tau}}\right]dt \\
&= \frac{V_1^2}{R}\tau \\
&= V_O^2 C\left(1+\frac{C}{C_1}\right)
\end{aligned}$$

More energy is dissipate here than in Part (A).

(E) At $t = 0^+$ the current I_0 splits into R and L. However, we know that the current through L must be continuous unless there is a source of infinite voltage in the circuit, so initially all of the current flows into R, producing a voltage RI_0 across the inductor. This will cause current to begin to flow into L with time constant $\tau = \frac{L}{R}$ (remember that $v = L\frac{di}{dt}$ for an inductor). Eventually, the inductor will carry all of the current I_0. Given this information we can write

$$i(t) = I_0\left(1 - e^{-\frac{tR}{L}}\right)$$

Let the variable v correspond to the voltage across the inductor. We know that the energy supplied by the source will be equal to

$$E_S = I_0\int_0^\infty v(t)dt$$

The energy dissipated by the resistor is

$$E_R = \int_0^\infty (I_0 - i(t))v(t)dt$$

The energy stored by the inductor is

$$E_L = \int_0^\infty i(t)v(t)dt$$

We know that $v(t) = R(I_0 - i(t))$ from Ohm's law, and we have an expression for $i(t)$ above. Using this information we can evaluate these integrals and find

$$\begin{aligned}
E_S &= LI_0^2 \\
E_R &= \frac{1}{2}LI_0^2 \\
E_L &= \frac{1}{2}LI_0^2
\end{aligned}$$

Problem 7.3 Answer:

(A) (a) The general solution to the differential equation is

$$v(t) = \frac{1}{T} + Ae^{-\frac{t}{RC}}$$

The initial condition is that $v(0) = 0$. This means that the constant A in the above expression is $A = -\frac{1}{T}$.

(b) This time the general solution is

$$v(t) = Be^{-\frac{t-T}{RC}}$$

At $t = T$, this expression must have the same value as the one from part (i) above. This means that

$$B = \frac{1}{T}(1 - e^{-\frac{T}{RC}})$$

(c) We need to know $\lim_{T\to 0} B$. If we try to evaluate this directly we get $\frac{0}{0}$. We can use L'Hopital's rule to find

$$\begin{aligned} \lim_{T\to 0} B &= \lim_{T\to 0} \frac{\frac{d}{dT}\left(1 - e^{-\frac{T}{RC}}\right)}{\frac{d}{dT}T} \\ &= \lim_{T\to 0} \frac{1}{RC}e^{-\frac{T}{RC}} = \frac{1}{RC} \end{aligned}$$

The impulse response is then

$$v(t) = \frac{1}{RC}e^{-\frac{t}{RC}}$$

(B) Integrate the differential equation from $t = 0^-$ to $t = 0^+$.

$$\begin{aligned} RC\frac{dv(t)}{dt} + v(t) &= v_s(t) \\ \int_{0^-}^{0^+} \left[RC\frac{dv(t)}{dt} + v(t)\right] dt &= \int_{0^-}^{0^+} [v_s(t)]\, dt \\ \int_{0^-}^{0^+} RC\frac{dv(t)}{dt}dt + \int_{0^-}^{0^+} v(t)dt &= \int_{0^-}^{0^+} [v_s(t)]\, dt \\ [RCv(t)]|_{0^-}^{0^+} + 0 &= 1 \\ RCv(0^+) &= 1 \\ v(0^+) = \frac{1}{RC} \end{aligned}$$

Which is the same answer we got from Part (Aiii) above. The impulse response is then

$$v(t) = \frac{1}{RC}e^{-\frac{t}{RC}}$$

(C) We know that $v(0^+) = \frac{1}{C}\int_{0^-}^{0^+} i_c dt$. So

$$\begin{aligned} v(0^+) &= \frac{1}{C}\int_{0^-}^{0^+} i_c dt \\ &= \frac{1}{C}\int_{0^-}^{0^+} \frac{\delta(t)}{R} dt \\ &= \frac{1}{RC} \end{aligned}$$

Which is the same answer we got from Part (B).

(D) We've found the step response of a simple resistor-capacitor network many times. We know that it is

$$v(t) = 1 - e^{-\frac{t}{RC}}$$

To find the impulse response, differentiate this to find

$$v(t) = \frac{1}{RC} e^{-\frac{t}{RC}}$$

Which is the same answer we've been getting all along.

Problem 7.4 Answer:

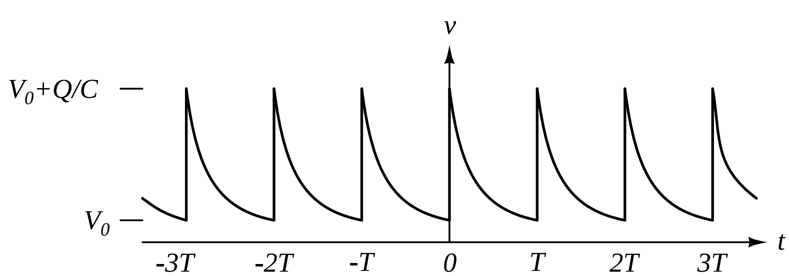
(A) The capacitor is initially charged to the unknown voltage V_0. At $t = 0$ an impulse of current with area Q Coulombs is applied to the capacitor. This means that at $t = 0^+$ the capacitor voltage will be $V_0 + \frac{Q}{C}$. You can verify this from the results of Problem 3 above, replacing the Thévenin voltage source with its Norton equivalent from this problem. Until $t = T$, this voltage will decay to 0 with time constant $\tau = RC$. For $0 < t < T$, then, $v(t)$ is

$$v(t) = \left(V_0 + \frac{Q}{C}\right) e^{-\frac{t}{RC}}$$

(B) Evaluate the expression from Part (A) at $t = T^-$, equate it to V_0, and solve:

$$\begin{aligned} V_0 &= \left(V_0 + \frac{Q}{C}\right) e^{-\frac{T}{RC}} \\ V_0\left(1 - e^{-\frac{T}{RC}}\right) &= \frac{Q}{C} e^{-\frac{T}{RC}} \\ V_O &= \frac{\frac{Q}{C} e^{-\frac{T}{RC}}}{1 - e^{-\frac{T}{RC}}} \end{aligned}$$

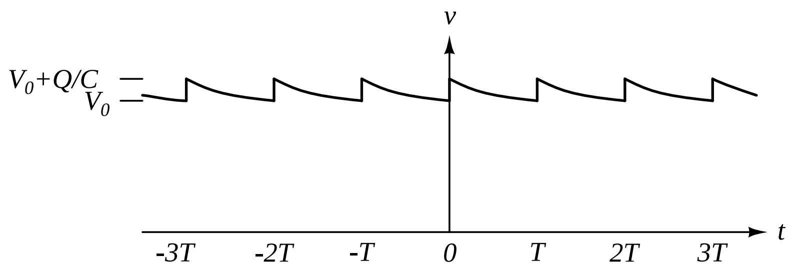
If $RC << T$ then $V_0 \approx 0$. The capacitor will discharge almost all the way between each impulse. The response will look like many impulse responses chained together, as shown below



If $RC >> T$, then $V_0 \approx \infty$. Take the Taylor series expansion of V_O and discard all but the first two terms. This is

$$V_O = \frac{\frac{Q}{C}\left(1 - \frac{T}{RC}\right)}{1 - \left(1 - \frac{T}{RC}\right)} = \frac{RQ}{T}$$

We can see that V_0 will approach ∞ as $\frac{1}{T}$. Physically, the capacitor will not discharge much at all between impulses, and the output will look fairly constant (at a high voltage), as shown below.



(C) Recall that the solution to a first-order linear differential equation is always a particular solution plus a homogeneous solution. We've already found the particular solution for the circuit in the parts above, it's just $v^*(t)$. All we need is the homogeneous solution, which we know is of the form

$$v_1(t) = Ae^{-\frac{t}{RC}}$$

All we need to find is the multiplying constant A to satisfy the initial conditions. Evaluating $v^*(t)$ at $t = 0-$ gives V_0. Remember that at $t = 0-$ the impulse has not been applied to the circuit yet, so the value of $v^*(t) = V_0$. We can write

$$\begin{aligned} 0 = v(0-) &= v^*(0^-) + v_1(0^-) \\ 0 &= V_0 + A \\ A &= -V_0 \end{aligned}$$

We have, finally:

$$v_1(t) = -V_0 e^{-\frac{t}{RC}}$$