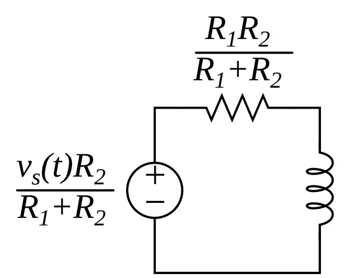
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Homework #8 Solutions

Problem 8.1 Answer: Before we begin, let's construct the Thévenin equivalent circuit connected to the inductor. It is shown in the figure below.



(a) This input is just a unit step. The current through an inductor must be continuous unless an infinite voltage is applied across it, so we know the voltage step must appear across the inductor, not the resistor. Given this:

$$\begin{aligned} \tau &= \frac{L(R_1+R_2)}{R_1R_2} \\ v(0) &= \frac{R_2}{R_1+R_2}\text{V} \\ v(\infty) &= 0\text{V} \end{aligned}$$

Now we can write

$$v(t) = \frac{R_2}{R_1+R_2} e^{-\frac{tR_1R_2}{L(R_1+R_2)}} u(t)$$

The $u(t)$ was added in to show that for $t < 0$ the network has zero state.

(b) This input is the unit ramp. The unit ramp is the integral of the unit step scaled by $\frac{1}{T}$, so to find the ramp response, we can just integrate the step response and scale it by $\frac{1}{T}$ to find:

$$\begin{aligned} v(t) &= \frac{1}{T}\int_0^t \frac{R_2}{R_1+R_2} e^{-\frac{kR_1R_2}{L(R_1+R_2)}} dk \\ &= -\frac{L}{TR_1} e^{-\frac{kR_1R_2}{L(R_1+R_2)}} \bigg|_0^t \\ &= \frac{L}{TR_1}\left(1 - e^{-\frac{tR_1R_2}{L(R_1+R_2)}}\right) u(t) \end{aligned}$$

(c) This input is the unit ramp minus the unit ramp shifted to the right by T. We can just sum the responses to these inputs to find

$$v(t) = \frac{L}{TR_1}\left(1 - e^{-\frac{tR_1R_2}{L(R_1+R_2)}}\right)u(t) - \frac{L}{TR_1}\left(1 - e^{-\frac{(t-T)R_1R_2}{L(R_1+R_2)}}\right)u(t-T)$$

(d) This input is the one from Part (c), plus the unit step. The response is

$$v(t) = \left(\frac{L}{TR_1} + \left(\frac{R_2}{R_1+R_2} - \frac{L}{TR_1}\right)e^{-\frac{tR_1R_2}{L(R_1+R_2)}}\right)u(t) - \frac{L}{TR_1}\left(1 - e^{-\frac{(t-T)R_1R_2}{L(R_1+R_2)}}\right)u(t-T)$$

(e) This input is the one from Part (d), minus the one from Part (c), shifted to the right by $2T$, minus the step response shifted to the right by $3T$.

$$\begin{aligned} v(t) &= \left[\left(\frac{L}{TR_1} + \left(\frac{R_2}{R_1+R_2} - \frac{L}{TR_1}\right)e^{-\frac{tR_1R_2}{L(R_1+R_2)}}\right)u(t) - \frac{L}{TR_1}\left(1 - e^{-\frac{(t-T)R_1R_2}{L(R_1+R_2)}}\right)u(t-T)\right] \\ &\quad - \left[\frac{L}{TR_1}\left(1 - e^{-\frac{(t-2T)R_1R_2}{L(R_1+R_2)}}\right)u(t-2T) - \frac{L}{TR_1}\left(1 - e^{-\frac{(t-3T)R_1R_2}{L(R_1+R_2)}}\right)u(t-3T)\right] \\ &\quad - \frac{R_2}{R_1+R_2}e^{-\frac{(t-3T)R_1R_2}{L(R_1+R_2)}}u(t-3T) \end{aligned}$$

Problem 8.2 Answer:

(A) During the first half-period MOSFET M1 is turned on. Energy is being dissipated in both the pullup resistor, and the MOSFET as the capacitor C_{eq} discharges. The best way to find the energy dissipated here is to find the energy provided by V_S, and the energy stored in C_{eq} that must be lost in M1 . The energy lost by C_{eq} is easy. The initial capacitor voltage is V_S, and the final voltage is $V_S\frac{R_{ON}}{R_{ON}+R}$. The energy lost by the capacitor is then

$$\begin{aligned} E_C &= \frac{1}{2}C_{eq}\left[V_S^2 - \left(V_S\frac{R_{ON}}{R_{ON}+R}\right)^2\right] \\ E_C &= \frac{1}{2}C_{eq}V_S^2\left[1 - \left(\frac{R_{ON}}{R_{ON}+R}\right)^2\right] \end{aligned}$$

The energy provided by the power supply (all of which is dissipated by M1 and R), assuming that $T >> \tau_1$ can be found as

$$\begin{aligned} E_{V_S} &= \int_0^{\frac{T}{2}} (V_S * i_{V_S})dt \\ &= V_S\int_0^{\frac{T}{2}} \left(\frac{V_S - v_1(t)}{R}\right)dt \\ &= \frac{V_S}{R}\int_0^{\frac{T}{2}} V_S\frac{R}{R_{ON}+R}\left(1 - e^{-t/\tau_1}\right)dt \\ &= \frac{V_S^2}{R_{ON}+R}\int_0^{\frac{T}{2}}\left(1 - e^{-t/\tau_1}\right)dt \\ &= \frac{V_S^2}{R_{ON}+R}\left(t + \tau_1 e^{-t/\tau_1}\right)\Big|_0^{\frac{T}{2}} \\ &= \frac{V_S^2}{R_{ON}+R}\left(\frac{T}{2} - \tau_1\right) \end{aligned}$$

The total energy dissipated is $E_{V_S} + E_C$. Using the fact that $\tau_1 = C_{eq}(R||R_{ON})$ we find

$$\begin{aligned} E &= E_{V_S} + E_C \\ &= \frac{V_S^2}{R_{ON}+R}\left(\frac{T}{2} - C_{eq}\frac{R_{ON}R}{R_{ON}+R}\right) + \frac{1}{2}C_{eq}V_S^2\left(1 - \left(\frac{R_{ON}}{R_{ON}+R}\right)^2\right) \end{aligned}$$

(B) During the second half-period, M1 is off, and the only element dissipating power is the resistor R. The power supply is also providing energy which will be stored in the capacitor, and dissipated by M1 during the next half-period. The easiest way to find the energy dissipated by the resistor is to find the energy supplied by the power supply, and subtract the energy stored by the capacitor. We know that

$$E_C = \frac{1}{2}C_{eq}V_S^2\left(1 - \left(\frac{R_{ON}}{R_{ON}+R}\right)^2\right)$$

The energy supplied by the power supply is

$$\begin{aligned} E_{V_S} &= \frac{V_S^2}{R_{ON}+R}\int_{\frac{T}{2}}^{T} e^{-(t-T/2)/\tau_2}dt \\ &= \frac{V_S^2}{R_{ON}+R}\left[-\tau_2 e^{-(t-T/2)/\tau_2}\right]\Big|_{\frac{T}{2}}^{T} \\ &= \frac{V_S^2}{R_{ON}+R} * \tau_2 \\ &= \frac{V_S^2}{R_{ON}+R} * RC_{eq} \end{aligned}$$

So the total energy dissipated over the second half-period is

$$E = \frac{V_S^2}{R_{ON}+R} * RC_{eq} - \frac{1}{2}C_{eq}V_S^2\left(1 - \left(\frac{R_{ON}}{R_{ON}+R}\right)^2\right)$$

(C) The total energy dissipated is the sum of the answers from Parts (A) and (B):

$$E = \frac{V_S^2}{R_{ON}+R}\left(\frac{T}{2} - C_{eq}\frac{R_{ON}R}{R_{ON}+R} + RC_{eq}\right)$$

Take this answer and divide by the period T to find the average power

$$\overline{P} = \underbrace{\frac{V_S^2}{2(R_{ON}+R)}}_{\text{Static Power}} + \underbrace{\frac{V_S^2}{T(R_{ON}+R)}\left(RC_{eq} - C_{eq}\frac{R_{ON}R}{R_{ON}+R}\right)}_{\text{Dynamic Power}}$$

Note that if we take the limit of the above expression at $R_{ON} << R$, we find

$$\overline{P} = \frac{V_S^2}{2R} + V_S^2 C_{eq} f$$

Which is the answer derived in lecture.

(D) The power dissipated is

$$\overline{P} \approx .1136\text{mW} + .02066\text{mW} = .1343\text{mW}$$

From the above, we see that the static power dominates.

Problem 8.3 Answer:

(a) $\overline{AB + C}$

A	B	C	Z
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

(b) $\overline{AB + C\overline{D}}$

A	B	C	D	Z
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

(c) $\overline{(A + \overline{D})(B + C)}$

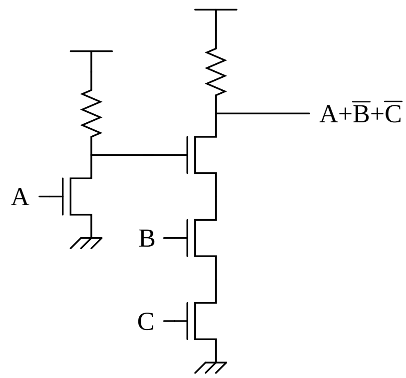
A	B	C	D	Z
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

(d) $(A + \overline{D})(B + C)$

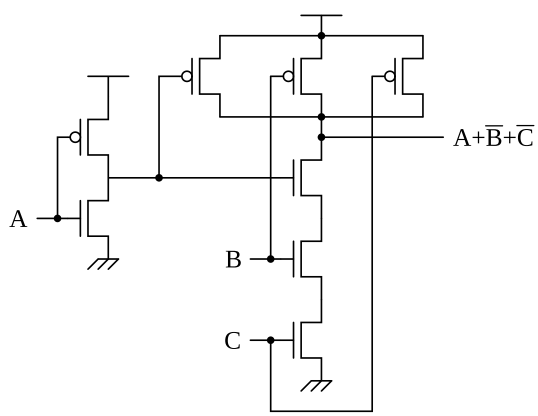
A	B	C	D	Z
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Problem 8.4 Answer:

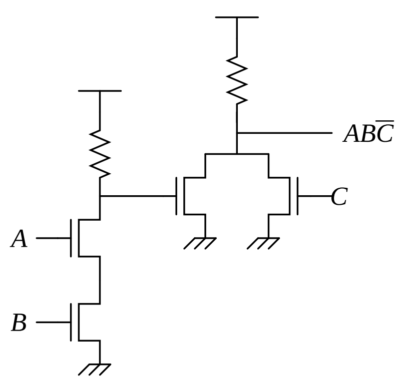
(A) Recall that our solution from Problem 4.2 looked like the following.



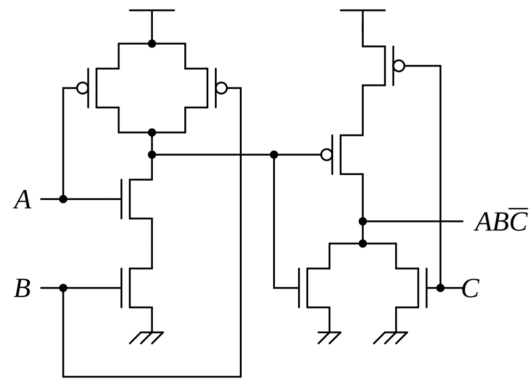
In the inverter just replace the pullup resistor with a PFET. The NFETs in the second stage are all in series. The corresponding PFETs get placed in parallel in place of the pullup resistor, as shown below.



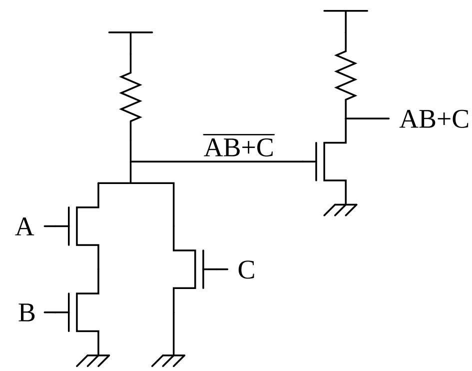
(B) Recall that our solution from Problem 4.2 looked like the following.



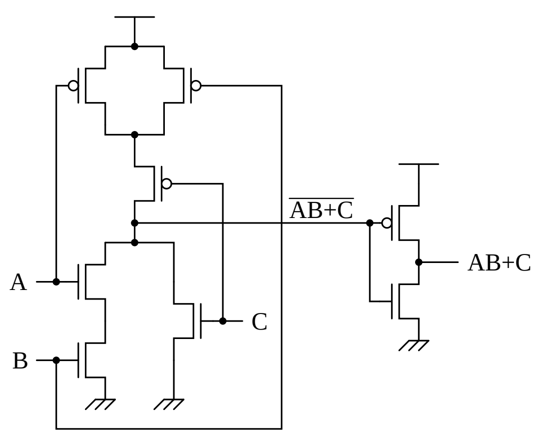
In the NAND gate the NFETs are in series, so the corresponding PFETs get placed in parallel. Conversely, in the NOR gate, the NFETs are in parallel, so the PFETs are in series, as shown below.



(C) Recall that our solution from Problem 4.2 looked like the following.



Again, in the inverter we just replace the pullup resistor with a PFET. In the first gate's NFETs, A and B are in series, in parallel with C. The PFETs will be A and B in parallel, in series with C, as shown below.



Problem 8.5 Answer: To help us find the average power dissipation first consider the following table showing the inputs, and the logic levels at M4's gate and drain.

A	B	C	M4's Gate	OUT
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	1	0
1	1	1	0	1

When the output is low, static power is dissipated by the pullup resistor at the output (remember that $R_{ON} << R$ so we can approximate $R_{ON} + R \approx R$). If the output is high, the same static power is dissipated by the first gate's pullup resistor. Over $8T$ then the static power dissipates $8T\frac{V_S^2}{R}$ of energy.

Looking at the table above we find that M4's gate capacitor C_{GS} goes through three charge/discharge cycles. Each of these dissipates CV_S^2 of energy (half while charging it, and half while discharging). This is a total of $3CV_S^2$ over $8T$.

The time average power is then

$$\overline{P} = \frac{V_S^2}{R} + \frac{3}{8T}CV_S^2$$