Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Homework #9 Solutions

Problem 9.1 Answer:

(A) Notice that this is a series RLC circuit which is under-damped $\left(\frac{R}{2} < \sqrt{\frac{L}{C}}\right)$. The un-driven (homogeneous) solution is solved in the course notes, and will not be repeated here. Because $\frac{L}{R} << T$ we can assume that v_O will decay to a final constant state within the time T.

We know that before $t = 0$ the MOSFET is off so $v_O(0^+) = V_S$. When the MOSFET turns on v_O will ring down towards $V_S\frac{R_{ON}}{R_{ON}+R_{pu}}$. Let us make the assumption that $R_{ON} << R_{pu}$, this means the response for $0 < t < T$ will just be the un-driven response analyzed in the notes, with $i_L(0) = 0$ and $v_C(0) = V_S$.

For our purposes, let $\alpha = \frac{R}{2L}$ and $\omega_d = \sqrt{\frac{1}{LC} - \alpha^2}$. From the notes we can directly write

$$v_O(0 < t < T) = V_S\sqrt{1 + \left(\frac{\alpha}{\omega_d}\right)^2} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)$$

As we've seen in class, this is a decaying exponential oscillatory response. The R used to calculate α is the parallel combination of R_{ON} and R_{pu}.

For $T < t < 2T$ the answer is nearly identical. The difference is that the particular solutions is now V_S rather than 0. If we look closely at the problem, though, we can easily adjust the above solution without having to do too much work. In the notes, the constants A_1 and A_2 are found by evaluating $v_C(0)$ and $\frac{dv_C(0)}{dt}$ and setting them equal to their initial conditions. The derivative equation doesn't change, because the constant particular solution just disappears. Look carefully at the first equation though. From the notes the homogeneous response gives us

$$v_O(0) = A_1 + A_2 = V_S$$

In the driven case, though, we get

$$v_O(0) = A_1 + A_2 + V_S = 0$$

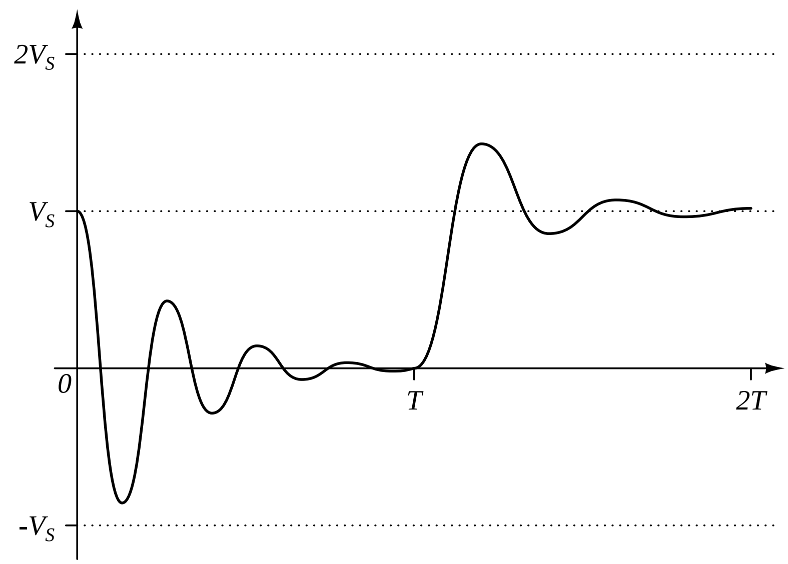
Notice the we can re-arrange the second equation to be

$$v_O(0) = A_1 + A_2 = -V_S$$

This isn't a big change from the homogeneous solution at all! What does this affect? Because $\frac{dv_O}{dt} = 0$, the only thing that this changes is the sign of A_1 and A_2 from the homogeneous solution. We can find $v_O(T < t < 2T)$ then be subtracting the homogeneous solution we found above from the particular solutions. This gives

$$v_O(T < t < 2T) = V_S\left(1 - \sqrt{1 + \left(\frac{\alpha}{\omega_d}\right)^2} e^{-\alpha t}\cos\left(\omega_d t - \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)\right)$$

Does this answer make sense? Yes! The derivative is still zero, meaning that the inductor current will be zero. The value of it at $t = 0$ is zero, which means the capacitor voltage is continuous, and it's a decaying exponential with a final value of V_S, which we expected. Note that this time the R used to calculate α is just R_{pu}, because the MOSFET is off. This means that α is bigger during the second half of a switching cycle, leading to a faster decay, and smaller ω_d (frequency) for the oscillations. Below is a graph of $v_O(t)$ for $0 < t < 2T$.



(B) We know that once the output of the circuit becomes a valid high output for a valid low input, it should not leave the valid high output region. This means that the lowest ringing trough when the output switches high must be $> 0.8V_S$.

To make the work easier, we can re-write the expression for v_O as

$$v_O(T < t < 2T) = V_S\left(1 - e^{-\alpha t}\left(\cos(\omega_d t) + \frac{\alpha}{\omega_d}\sin(\omega_d t)\right)\right)$$

The troughs (and peaks) occur when $\frac{dv_O(t)}{dt} = 0$. Let's evaluate the derivative:

$$\frac{dv_O(T < t < 2T)}{dt} = \alpha e^{-\alpha t}\cos(\omega_d t) + \omega_d e^{-\alpha t}\sin(\omega_d t) + \frac{\alpha^2}{\omega_d}e^{-\alpha t}\sin(\omega_d t) - \alpha e^{-\alpha t}\cos(\omega_d t)$$

The cos terms cancel each other, and we are left with

$$\frac{dv_O(T < t < 2T)}{dt} = e^{-\alpha t}\omega_d\left(\sin(\omega_d t) + \left(\frac{\alpha}{\omega_d}\right)^2 \sin(\omega_d t)\right)$$

The only time when this expression is equal to zero is when $\omega_d t$ is some integer multiple of π. We are interested in the point where $\omega_d t = 2\pi$, the first trough. Note that $\omega_d t = \pi$ corresponds to the first peak, and $\omega_d t = 0$ corresponds to the initial switch.

Using $\omega_d t = 2\pi$ in the modified expression for v_O above, the sin term is 0 and the cos term is 1, and we are left with

$$v_0\left(\frac{2\pi}{\omega_d}\right) = V_S(1 - e^{-\alpha t})$$

Setting this expression equal to $0.8V_S$, substituting in expressions for α and ω_d and solving for L gives

$$\begin{aligned}
0.8V_S &= V_S\left(1 - e^{-\frac{2R\pi}{2L\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}}}\right) \\
0.2 &= 0.e^{-\frac{2R\pi}{2L\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}}} \\
\ln(5) &= \frac{R\pi}{L\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}} \\
L\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2} &= \frac{R\pi}{ln(5)} \\
L^2\left(\frac{1}{LC}-\left(\frac{R}{2L}\right)^2\right) &= \frac{R^2\pi^2}{\ln^2(5)} \\
\frac{L}{C}-\left(\frac{R}{2}\right)^2 &= \frac{R^2\pi^2}{\ln^2(5)} \\
L &= CR^2\left(\frac{\pi^2}{\ln^2(5)}+\frac{1}{4}\right)
\end{aligned}$$

Evaluating this with the circuit parameters given yields

$$L \approx 40.602\text{nH}$$

Problem 9.2 Answer:

(A) Start from the right side of the circuit, and work back towards the source. The current that flows through R_2 (towards the right) is equal to the current through the resistor R_1 and the current into the capacitor C_1. Call this current i_2. It is

$$i_2 = \frac{v_{C_1}}{R_1} + C_1 \frac{dv_{C_1}}{dt}$$

Likewise, the source current is equal to the current through R_2 and C_2. This is

$$i_s = C_2 \frac{dv_{C_2}}{dt} + i_2$$

However, we have an expression for i_2 in terms of v_{C_1} above. We can also say

$$v_{C_2} = v_{C_1} + R_2 i_2$$

Using this information, we can write out the differential equation in terms of v_{C_1}.

$$\begin{aligned}
i_s &= C_2 \frac{dv_{C_2}}{dt} + i_2 \\
i_s &= C_2 \frac{d}{dt} \left[v_{C_1} + R_2 i_2 \right] + i_2 \\
i_s &= C_2 \frac{d}{dt} \left[v_{C_1} + R_2 \left(\frac{v_{C_1}}{R_1} + C_1 \frac{dv_{C_1}}{dt} \right) \right] + \frac{v_{C_1}}{R_1} + C_1 \frac{dv_{C_1}}{dt} \\
i_s &= C_2 \frac{d}{dt} \left[v_{C_1} \left(1 + \frac{R_2}{R_1} \right) + R_2 C_1 \frac{dv_{C_1}}{dt} \right] + \frac{v_{C_1}}{R_1} + C_1 \frac{dv_{C_1}}{dt} \\
i_s &= C_2 \left(1 + \frac{R_2}{R_1} \right) \frac{dv_{C_1}}{dt} + C_1 R_2 C_2 \frac{d^2 v_{C_1}}{dt^2} + \frac{v_{C_1}}{R_1} + C_1 \frac{dv_{C_1}}{dt} \\
i_s &= C_1 R_2 C_2 \frac{d^2 v_{C_1}}{dt^2} + \left(C_1 + C_2 + \frac{R_2 C_2}{R_1} \right) \frac{dv_{C_1}}{dt} + \frac{v_{C_1}}{R_1} \\
\frac{i_s}{C_1 R_2 C_2} &= \frac{d^2 v_{C_1}}{dt^2} + \left(\frac{1}{R_2 C_2} + \frac{1}{R_2 C_1} + \frac{1}{R_1 C_1} \right) \frac{dv_{C_1}}{dt} + \frac{v_{C_1}}{R_1 C_1 R_2 C_2}
\end{aligned}$$

(B) (i) Using the values given in the problem statement, the differential equation becomes

$$3 i_s = \frac{d^2 v_{C_1}}{dt^2} + 4 \frac{dv_{C_1}}{dt} + 3 v_{C_1}$$

The characteristic equation is

$$s^2 + 4s + 3 = (s+3)(s+1) = 0$$

It's roots are

$$s = -1 \quad \text{and} \quad s = -3$$

(ii) We know that the most general form of the homogeneous solution will be a linear combination of exponentials with natural frequencies from Part (i):

$$v_{C_1}(t) = A e^{-t} + B e^{-3t}$$

(iii) The input to our system is a constant, so we can suppose that the particular solution will be a constant, C, too. Let's substitute this into the differential equation, and we find

$$C = i_s$$

Note that C is in volts. Go back to the differential equation from Part (A) and note that if $v_{C_1} = C$ then

$$i_s = \frac{C}{R_1}$$

Because $R_1 = 1\Omega$ the particular solution is just i_sV.

(iv) The input is a step of current. From $t = 0^-$ to $t = 0^+$ no charge is injected into the circuit $\left(\int_{0^-}^{0^+} i_s dt = 0\right)$. This means that the capacitor voltage must be continuous at $t = 0$. The derivative of the capacitor voltage is proportional to the current flowing into it. At $t = 0^+$, though, C_2 will act like a short, and all of i_s will flow into it momentarily, so $\left.\frac{dv_{C_1}}{dt}\right|_{0^+} = 0$.

(v) The total answer is the particular solution from Part (iii) plus the homogeneous solution from part (ii). The constants A and B are used to satisfy the initial conditions from Part (iv). The general solution is

$$v_{C_1} = i_s + Ae^{-t} + Be^{-3t}$$

Evaluate this and it's derivative at zero in order to find A and B to satisfy the initial conditions. This gives

$$-i_s = A + B \quad \text{and} \quad 0 = A + 3B$$

Solve these this system of linear equations to find

$$A = -\frac{3i_s}{2} \quad \text{and} \quad B = \frac{i_s}{2}$$

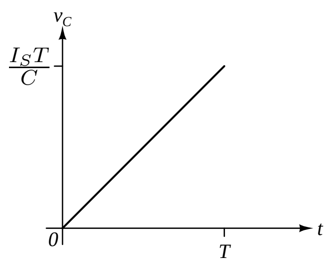
Putting all of this together we find

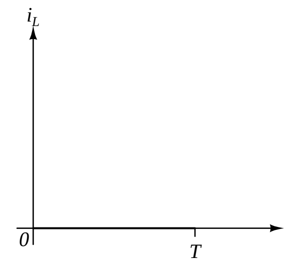
$$v_{C_1} = i_s \left(1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right)$$

Note that this is the solution for $t > 0$. If we were asked for the solution for all t, we would have multiplied the above answer by $u_{-1}(t)$ to reflect the fact that the circuit is in zero state for $t < 0$.

Problem 9.3 Answer:

(A) While the switch is in position B there is a short across the inductor, and all of I_S flows into C. The inductor current will remain at zero, and the capacitor voltage will increase linearly with time as $v_C = \frac{1}{C} I_S t$. This is sketched below.





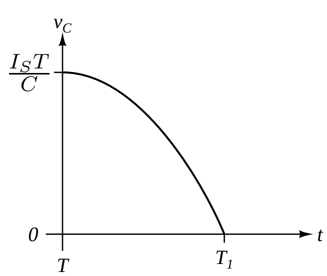
(B) We have already examined this problem in class. I_S has a short across it, and we're left with a parallel LC circuit with an initial state. We can directly write

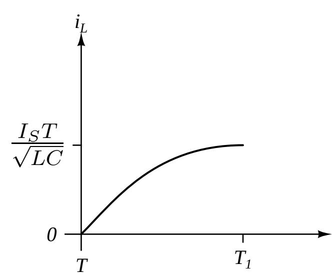
$$v_C(t) = \frac{I_S T}{C} \cos\left(\frac{t - T}{\sqrt{LC}}\right)$$

We know that $i_L = -i_C = -C\frac{dv_C}{dt}$.

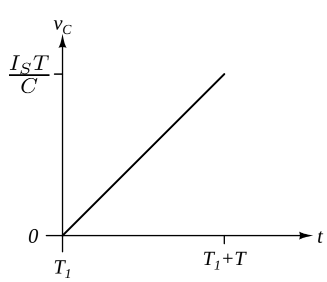
$$i_L(t) = \frac{I_S T}{\sqrt{LC}} \sin\left(\frac{t - T}{\sqrt{LC}}\right)$$

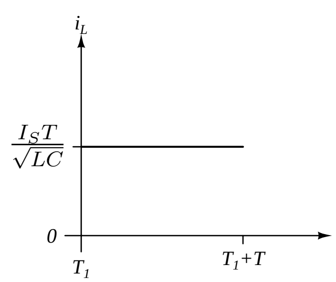
When the capacitor voltage is zero, $cos\left(\frac{T_1 - T}{\sqrt{LC}}\right) = 0$. This means $sin\left(\frac{T_1 - T}{\sqrt{LC}}\right) = 1$, so the inductor current is $i_L(T_1) = \frac{I_S T}{\sqrt{LC}}$. These waveforms are graphed below.





(C) Again, there is a short across the inductor. This means the voltage across the inductor is zero, so it's current won't change at all. It will remain at $i_L(T_1) = \frac{I_S T}{\sqrt{LC}}$. The capacitor voltage will charge linearly again, so $v_C(t) = \frac{1}{C} I_S (t - T_1)$.





(D) This time finding v_C and i_L is a little more complicated, because the initial conditions aren't quite as simple. However, we know the solution is going to be the sum of some sinusoids at the resonant frequency. We can guess

$$v_C(t) = A\cos\left(\frac{t - T_1 - T}{\sqrt{LC}}\right) + B\sin\left(\frac{t - T_1 - T}{\sqrt{LC}}\right)$$

We know that $v_C(T_1 + T) = \frac{I_S T}{C}$. The sin term above is zero at $t = T_1 + T$, so $A = \frac{I_S T}{C}$. Likewise, we know that the current out of the capacitor at $t = T + 1 + T$ must equal the inductor current, so $-C\frac{dv_C}{dt} = \frac{I_S T}{\sqrt{LC}}$. This means

$$-C\frac{dv_C}{dt} = \frac{I_S T}{\sqrt{LC}} = \frac{I_S T}{\sqrt{LC}}\sin\left(\frac{t - T_1 - T}{\sqrt{LC}}\right) + -C\frac{B}{\sqrt{LC}}\cos\left(\frac{t - T_1 - T}{\sqrt{LC}}\right)$$

Note that only the cos term is non-zero at $t = T_1 + T$. Solving for B gives $B = -\frac{I_S T}{C}$.

Now we have

$$v_C(t) = \frac{I_S T}{C}\left[\cos\left(\frac{t - T_1 - T}{\sqrt{LC}}\right) - \sin\left(\frac{t - T_1 - T}{\sqrt{LC}}\right)\right]$$

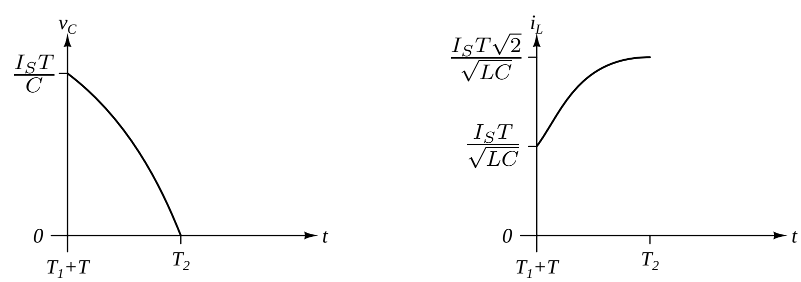
and we can find $i_L = -C\frac{dv_C}{dt}$

$$i_L(t) = \frac{I_S T}{\sqrt{LC}}\left[\sin\left(\frac{t - T_1 - T}{\sqrt{LC}}\right) + \cos\left(\frac{t - T_1 - T}{\sqrt{LC}}\right)\right]$$

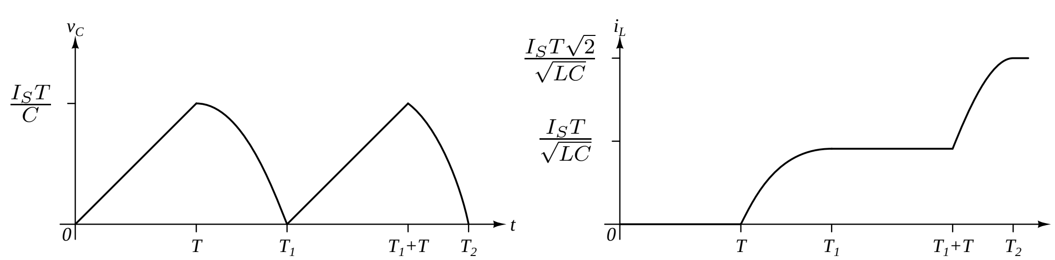
The capacitor voltage will be zero when

$$\cos\left(\frac{T_2 - T_1 - T}{\sqrt{LC}}\right) = \sin\left(\frac{T_2 - T_1 - T}{\sqrt{LC}}\right)$$

This means that $\frac{T_2 - T_1 - T}{\sqrt{LC}} = \frac{\pi}{4}$. Solving this for T_2 gives $T_2 = T + T_1 + \frac{\pi\sqrt{LC}}{4}$. We can find $i_L(T_2) = \frac{I_S T\sqrt{2}}{\sqrt{LC}}$. This is all graphed below.



(E) Here is a sketch of $v_C(t)$ and $i_L(t)$ for $0 < t < T_2$.



Problem 9.4 Answer:

(A) We already know that the voltage across the capacitor at $t = T$ is $v_C = \frac{I_S T}{C}$. The energy in the capacitor is

$$E_C = \frac{1}{2}\frac{I_S^2 T^2}{C}$$

(B) We know that $E_C = E_L$. So to find I_L:

$$\begin{aligned} E_L &= E_C \\ L i_L^2 &= \frac{I_S^2 T^2}{C} \\ i_L^2 &= \frac{I_S^2 T^2}{LC} \\ i_L &= \frac{I_S T}{\sqrt{LC}} \end{aligned}$$

Which is the same answer we got in 9.3(B).

(C) Again, the energy stored in the capacitor is just

$$E_C = \frac{1}{2}\frac{I_S^2 T^2}{C}$$

(D) This time, the final energy in the inductor needs to be equal to $\frac{I_S^2 T^2}{C}$, because $\frac{1}{2}\frac{I_S^2 T^2}{C}$ has been put into it twice.

$$\begin{aligned} E_L &= 2E_C \\ L i_L^2 &= \frac{I_S^2 T^2 2}{C} \\ i_L^2 &= \frac{2 I_S^2 T^2 2}{LC} \\ i_L &= \frac{I_S T \sqrt{2}}{\sqrt{LC}} \end{aligned}$$

which is the same answer we got in 9.3(D).

(E) After the nth cycle, we will have dumped $n * \frac{I_S^2 T^2}{2C}$ Joules into the inductor. We can find the inductor current as

$$\begin{aligned} E_L &= nE_C \\ L i_L^2 &= \frac{I_S^2 T^2 n}{C} \\ i_L^2 &= \frac{2 I_S^2 T^2 n}{LC} \\ i_L &= \frac{I_S T \sqrt{n}}{\sqrt{LC}} \end{aligned}$$