MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Quiz 3 Solutions

Name: ______________________________ **Recitation Section:** _______________

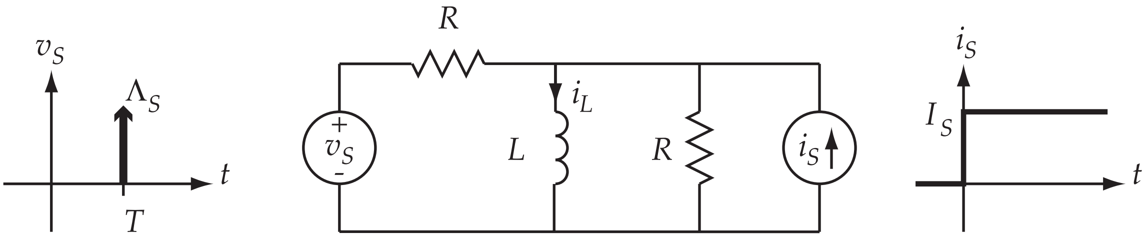
Recitation Instructor: _______________ **Teaching Assistant:** _______________

Make sure that your name is on all sheets. **Enter your answers directly in the spaces provided on the printed pages**. You may use the back of the previous page as a worksheet. Answers must be derived or explained, not just simply written down. The quiz is closed book, but **calculators are allowed**.

This quiz contains 8 pages including the cover sheet. Make sure that your quiz contains all 8 pages and that you hand in all 8 pages.

Problem	Points	Grade	Grader
1	30		
2	30		
3	40		
Total	100		

Problem 1: (*30 points*) This problem examines the transient response of the circuit shown below. In the circuit, $i_L = 0$ at $t = 0^-$.



(A) (*10 points*) Determine an expression for i_L due to $i_S = I_S u(t)$; that is, for $\Lambda_S = 0$.

Since $\Lambda_S = 0$, we replace the voltage source v_S with a short. Recognizing that this is a first order LR circuit, we need to find: 1. $i_L(0^+)$, 2. $i_L(t \to \infty)$ and 3. $\tau = L/R_{eq}$, where R_{eq} is the Thévenin equivalent resistance seen from the terminals of the inductor L.

1. To find $i_L(0^+)$, we recall that an inductor with $i_L(0^-) = 0$ acts as an open at $t = 0^+$. Thus, the current from the source goes through the two resistors and $i_L(0^+) = 0$.

2. To find $i_L(t \to \infty)$, we recall that an inductor acts as a short at $t \to \infty$. Thus, all the current from the source goes through the inductor and $i_L(t \to \infty) = I_S$.

3. To find $\tau = L/R_{eq}$, we turn off i_S by replacing it with an open. We then find that the Thévenin equivalent resistance seen from the terminals of the inductor L is the two resistors R in parallel, so that $R_{eq} = R/2$. Therefore, $\tau = 2L/R$.

We can construct the answer using the results 1, 2, and 3:

$$i_S(t) = \begin{cases} I_S(1 - e^{-t/\tau}); \quad \tau = 2L/R & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(B) (*10 points*) Determine an expression for i_L due to $v_S = \Lambda_S \delta(t-T)$; that is, for $I_S = 0$.

Since $I_S = 0$, we replace the current source i_S with an open. Since this is a linear system, the answer to an input shifted to $t = T$ is the answer to that same input at $t = 0$ shifted to $t = T$. Thus, we need to find the response to $v_S = \Lambda_S \delta(t)$ and shift the answer to $t = T$.

Again, recognizing that this is a first order LR circuit, we need to find: 1. $i_L(0^+)$ and 2. $i_L(t \to \infty)$. The time constant τ is the same independent of the inputs to the circuit; thus, it is as that found in part (A), $\tau = 2L/R$.

1. To find $i_L(0^+)$, we recall that an inductor with $i_L(0^-) = 0$ acts as an open at $t = 0^+$. The voltage flux Λ_S divides equally across the two resistors R and appears across the open terminals of the inductor as $\Lambda_S/2$. This flux generates a current instantly according to $\Lambda_L = Li_L$. Thus, $i_L(0^+) = \Lambda_L/L = \frac{\Lambda_S}{2L}$.

2. To find $i_L(t \to \infty)$, we recall that an inductor acts as a short at $t \to \infty$. Thus, the current through the inductor is $i_L(t \to \infty) = v_S(t \to \infty)/R$. Since v_S is an impulse, $v_S(t \to \infty) = 0$, and $i_L(t \to \infty) = v_S(t \to \infty)/R = 0$.

We can construct the answer combining the results 1 and 2, and shifting the response to $t = T$:

$$i_S(t) = \begin{cases} \frac{\Lambda_S}{2L} e^{-(t-T)/\tau}; \quad \tau = 2L/R & t > T \\ 0 & t < T \end{cases}$$

Alternatively, we can compute the step response and differentiate it to get the impulse response:

$$\left[\frac{\Lambda_S}{V_S}\frac{d}{dt}\right] V_S u(t-T) = \Lambda_S \delta(t-T)$$

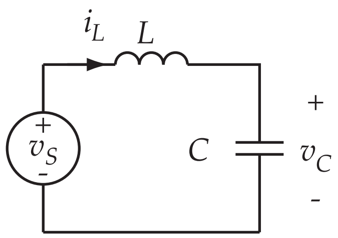
$$\left[\frac{\Lambda_S}{V_S}\frac{d}{dt}\right] \frac{V_S}{R}(1 - e^{-(t-T)/\tau}) = \frac{\Lambda_S}{R} e^{T/\tau} \frac{1}{\tau} e^{-t/\tau} = \frac{\Lambda_S}{2L} e^{-(t-T)/\tau}$$

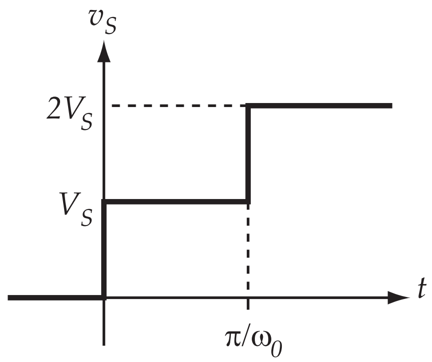
(C) (*10 points*) Determine an expression for i_L due to i_S and v_S together.

Applying superposition, the response is the sum of the responses found in parts (A) and (B):

$$i_S(t) = \begin{cases} I_S(1 - e^{-t/\tau}) + \frac{\Lambda_S}{2L} e^{-(t-T)/\tau}; \quad \tau = 2L/R & t > T \\ I_S(1 - e^{-t/\tau}); \quad \tau = 2L/R & 0 \le t < T \\ 0 & t < 0 \end{cases}$$

Problem 2: (*30 points*) In the circuit below, $\omega_0 \equiv 1/\sqrt{LC}$, $v_C(0^-) = 0$, and $i_L(0^-) = 0$.





(A) (*15 points*) Determine an expression for $v_C(t)$ for $t > 0$. If you determine $v_C(t)$ by inspection, state your reasoning CLEARLY.

The input is a sum of two steps, which can be written as $v_S(t) = V_S u(t) + V_S u(t - \pi/\omega_0)$. Using linearity, we can express the solution as $v_C(t) = v'_C(t) + v'_C(t - \pi/\omega_0)$, where $v'_C(t)$ is the response of the circuit to $v_S(t) = V_S u(t)$.

At $t = 0^+$, L is open and C is shorted. Thus, $i_L(0^+) = 0$, and $v'_C(0^+) = 0$. Also, $\dot{v}'_C(0^+) = i_L(0^+)/C = 0$. The particular solution is $v'_C(t) = V_S$. Since the system is undamped, the general solution takes the form

$$v'_C(t) = A\cos\omega_0 t + B\sin\omega_0 t + V_S$$

Applying the initial conditions,

$$v'_C(t) = V_S(1 - \cos\omega_0 t)$$

Thus, this is the solution for $0 \le t \le \pi/\omega_0$. For $t > \pi/\omega_0$, we need to add $v'_C(t - \pi/\omega_0)$ to our original $v'_C(t)$,

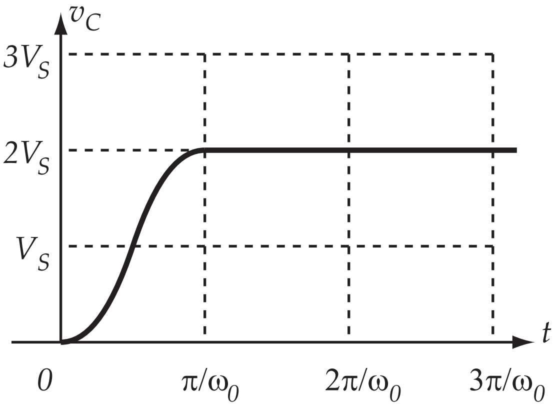
$$v'_C(t) + v'_C(t - \pi/\omega_0) = V_S(1 - \cos\omega_0 t) + V_S[1 - \cos\omega_0(t - \pi/\omega_0)]$$

$$= V_S(1 - \cos\omega_0 t) + V_S(1 + \cos\omega_0 t) = 2V_S$$

Combining these results,

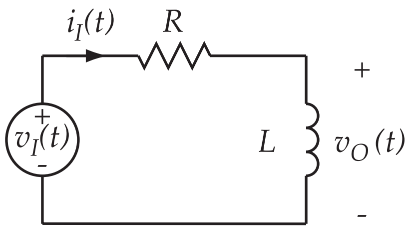
$$v_C(t) = \begin{cases} V_S(1 - \cos\omega_0 t) & 0 \le t \le \pi/\omega_0 \\ 2V_S & t > \pi/\omega_0 \end{cases}$$

(B) (*15 points*) Sketch and dimension $v_C(t)$ for $t > 0$.



Problem 3: (*40 points*) All circuits below operate in the sinusoidal steady state. For each circuit, determine the input impedance $Z(j\omega)$, the specified transfer function $H(j\omega)$, and the parameters V_o and ϕ which define $v_O(t) \equiv V_o \cos(\omega t + \phi)$, where V_o is a real, positive number.

(A) (*10 points*)



$v_I(t) = V_i \cos(\omega t)$

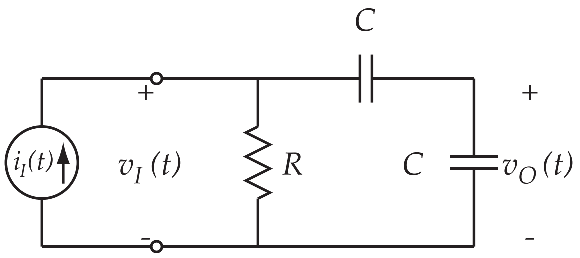
$$Z(j\omega) = \frac{V_i}{\hat{I}_i} = R + Lj\omega$$

$$H(j\omega) = \frac{\hat{V}_o}{V_i} = \frac{Lj\omega}{R+Lj\omega} = \frac{j\frac{L}{R}\omega}{1+j\frac{L}{R}\omega}$$

$$V_o = \frac{L\omega}{\sqrt{R^2+(L\omega)^2}}V_i$$

$$\phi = \pi/2 - \tan^{-1}\frac{L\omega}{R} = \tan^{-1}\frac{R}{L\omega}$$

(B) (*15 points*)



$i_I(t) = I_i \cos(\omega t)$

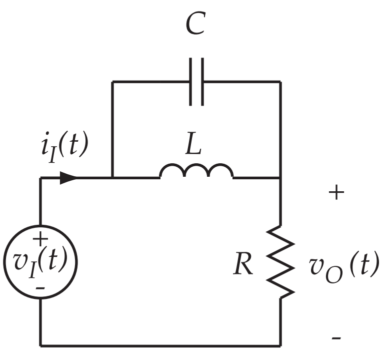
$$Z(j\omega) = \frac{\hat{V}_i}{I_i} = R||(\frac{1}{Cj\omega} + \frac{1}{Cj\omega}) = \frac{2R}{RCj\omega+2}$$

$$H(j\omega) = \frac{\hat{V}_o}{I_i} = \frac{\hat{V}_o}{\hat{V}_i}\frac{\hat{V}_i}{I_i} = \frac{\hat{V}_o}{\hat{V}_i}Z(j\omega) = \frac{\frac{1}{Cj\omega}}{\frac{1}{Cj\omega}+\frac{1}{Cj\omega}}Zj\omega = \frac{1}{2}Z(j\omega) = \frac{R}{RCj\omega+2}$$

$$V_o = \frac{R}{\sqrt{4+(RC\omega)^2}}I_i$$

$$\phi = -\tan^{-1}\frac{RC\omega}{2}$$

(C) (*15 points*)



$v_I(t) = V_i \cos(\omega t)$

$$Z(j\omega) = \frac{V_i}{\hat{I}_i} = R + Lj\omega || \frac{1}{Cj\omega} = R + \frac{Lj\omega}{1-LC\omega^2}$$
$$= R\left(1 + \frac{\frac{1}{RC}j\omega}{\frac{1}{LC}-\omega^2}\right) = R\left(\frac{\frac{1}{LC}-\omega^2+\frac{1}{RC}j\omega}{\frac{1}{LC}-\omega^2}\right)$$

$$H(j\omega) = \frac{\hat{V}_o}{V_i} = \frac{R}{R+Lj\omega||\frac{1}{Cj\omega}} = \frac{R-RLC\omega^2}{R-RLC\omega^2+Lj\omega} = \frac{\frac{1}{LC}-\omega^2}{\frac{1}{LC}-\omega^2+\frac{1}{RC}j\omega}$$

$$V_o = \frac{|R-RLC\omega^2|}{\sqrt{(R-RLC\omega^2)^2+(L\omega)^2}} V_i$$

$$\phi = -\tan^{-1}\frac{L\omega}{R-RLC\omega^2} + \tan^{-1}\left(\frac{0}{R-RLC\omega^2}\right)$$
$$= -\tan^{-1}\frac{L\omega}{R-RLC\omega^2} + \begin{cases} 0 & \omega < 1/\sqrt{LC} \\ \pi & \omega > 1/\sqrt{LC} \end{cases}$$

Note:

Since $\tan^{-1}$ is a multi-valued function, its range needs to be specified in order to define a unique solution within a factor of $2n\pi$. One way to do this is to explicitly spell out the numerator and denominator in the argument, a procedure that then defines the angle to be between 0 and 2π or $-\pi$ to $+\pi$. This is the approach taken in lecture and in Appendix C of the notes, and leads to the result given here. Another way is to specify that $\tan^{-1}$ lies between $-\pi/2$ and $\pi/2$ and add π when the angle moves out of this range. In any case, $\tan^{-1}$ must be clearly defined and that was worth 1 point for this part of the problem.