Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2003

Quiz 2

- Please write your name on each page of the exam in the space provided, and circle the name of your recitation instructor and the time of your recitation at the bottom of this page.

- Please verify that there are 16 pages in your exam.

- To the extent possible, do all of your work on the pages contained within this exam. In particular, try to do your work for each question within the boundaries of the question, or on the back side of the page preceding the question. Extra pages are also available at the end of your exam.

- You may use one double-sided page of notes and a calculator while taking this exam.

- Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
<th>Grader</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Name: Solutions

Instructor: Perreault  Antoniadis  Chaniotakis  Umans  Kolodziejski
Time: 10 11 11 11 12 12 1 2 3
Problem 1  -  25 Points

Yikes Inc. has manufactured a large batch of identical MOSFETs which unfortunately do not display square-law characteristics. Instead, they are characterized by the following equation when operated in saturation:

\[ i_{DS} = K(v_{GS} - V_T) \]

The MOSFETs operate in saturation when \( v_{GS} \geq V_T \) and \( v_{DS} \geq v_{GS} - V_T \). Yikes begins to experiment with these devices and builds the following circuit containing a single MOSFET M1. The supply voltage is \( V_{SU} \). Assume for this problem that both \( V_{SU} \) and \( K \) are greater than 0.

**Diagram:**

\( V_{SU} \)
\[ R \]
\[ D \]
\[ i_G = 0 \]
\[ M1 \]
\[ G \]
\[ S \]
\[ V_S \]
\[ I_0 \]

(A) Calculate \( v_S \), the node voltage at the source, assuming saturation. Note that the gate terminal \( G \) of the MOSFET is connected to ground. (8 points)

\[ i_{DS} = I_0 = K \left( V_T - V_S - V_T \right) = -K V_S - K V_T \]

\[ \Rightarrow K V_S = -I_0 - K V_T \]

\[ v_S = -\frac{I_0}{K} - V_T \]
(B) What is the maximum value of $R$ for which $M1$ satisfies the saturation discipline? (5 points)

\[ V_{DS} = 0 - V_S = \frac{I_o}{R} + V_T \geq V_T \]

\[ V_{DS} \geq V_T \]

\[ \Rightarrow (V_{SS} - I_o R) - V_S \geq V_T - V_S - V_T \]

\[ \Rightarrow I_o R \leq V_{SS} + V_T \]

\[ \Rightarrow R \leq \frac{V_{SS} + V_T}{I_o} \]

Maximum value of $R = \frac{V_{SS} + V_T}{I_0}$

(C) For a given value of $R$, what are the constraints on $I_o$ for which $M1$ satisfies the saturation discipline? (5 points)

1. $I_o$ must satisfy $V_{DS} \geq V_T$, need $I_o > 0 A$.
2. $V_{SS} - I_o R \geq -V_T$

\[ \Rightarrow I_o R \leq V_{SS} + V_T \]

\[ \Rightarrow I_o \leq \frac{V_{SS} + V_T}{R} \]

Minimum value of $I_o = 0 A$  

Maximum value of $I_o = \frac{V_{SS} + V_T}{R}$
(D) Draw a small signal model for the two-terminal device constructed by connecting the gate and drain of the Yikes MOSFET M1 as shown in the figure below. Derive an expression for the relationship between the small signal current $i_{ds}$ and the small signal voltage $v_{ds}$. (7 points)

\[ i_{ds} = \frac{\partial i_{ds}}{\partial v_{ds}} \left| \begin{array}{c} v_{ds} \\ \end{array} \right. = K v_{ds} \]

**Small signal model:**

$\frac{1}{v_{ds}} \xrightarrow{1/K} \frac{1}{i_{ds}}$
Problem 2 – 25 Points

After years of research, Yehaa Inc. has invented a new MOSFET which is shown below along with its small signal model under the saturation discipline.

This new MOSFET has an extremely high transconductance $g_m$ for reasonable bias values. The only catch is that $g_m$ is highly sensitive to temperature. As a result the traditional amplifiers designed using the MOSFET have unpredictable gains. Yehaa researchers work on this problem, and after a few more months announce they have discovered the breakthrough amplifier circuit shown below. They claim that under certain conditions the small signal gain of the new amplifier is insensitive to $g_m$. In this problem you will verify their claim.

Assume for this problem that the MOSFET always operates under the saturation discipline, and that it is biased such that its transconductance is $g_m$.

(A) Draw the small signal equivalent circuit for the amplifier. Clearly mark the small signal input $v_i$ and the small signal output $v_o$ in your circuit. (4 points)
(B) Determine the small signal transfer function $v_o/v_i$ for the circuit. (8 points)

\[
v_{gs} = v_i - R_s j_m v_{gs} \Rightarrow v_{gs}(1 + j_m R_s) = v_i
\]

\[
\Rightarrow v_o = 0 - j_m R_i v_{gs}
\]

\[
= -\frac{j_m R_i v_i}{1 + j_m R_s}
\]

\[
\frac{v_o}{v_i} = -\frac{j_m R_i}{1 + j_m R_s}
\]
(C) Determine one or more constraints that are required in order for the claims made by the Yehaa researchers to be true. In other words, what are the constraints under which the small signal gain is independent of $g_m$. Express these constraints in terms of one or more of the following parameters: $g_m$, $R_L$, and $R_S$. (5 points)

\[
\begin{align*}
\text{In order to have claim true, we want} \\
\quad g_m R_S & \gg 1
\end{align*}
\]

\[
\Rightarrow \quad \frac{V_o}{V_i} = -\frac{g_m R_L}{g_m R_S} = -\frac{R_L}{R_S}
\]

(D) Emboldened by their recent success, the Yehaa researchers make a second claim that the small signal output resistance of the amplifier is independent of $R_S$. If true, this property would make the new amplifiers even more appealing to existing customers using the older amplifier model (for which $R_S = 0$).

Determine an expression for the small signal output resistance of the amplifier and use it to confirm or disprove the second claim. (Recall that the small signal output resistance is the Thevenin resistance seen between the output node and ground). (8 points)

\[
\text{Put } V_{\text{test}} \text{ at output, find } i_{\text{test}} \Rightarrow R_o = \frac{V_{\text{test}}}{i_{\text{test}}} \quad \text{Turn off}
\]

Since independent

\[
\Rightarrow \quad \text{KCL at output: } \quad 0 = -\frac{V_{\text{test}}}{R_L} + i_{\text{test}} = -g_m V_S
\]

\[
\Rightarrow \quad 0 = (1 + g_m R_S) \left( -\frac{V_{\text{test}}}{R_L} + i_{\text{test}} \right)
\]

\[
\text{But } 1 + g_m R_S \neq 0 \Rightarrow \quad \frac{V_{\text{test}}}{i_{\text{test}}} = R_L
\]

Small signal output resistance $= R_L \quad \Rightarrow \text{claim is true.}$
Problem 3 – 25 Points

(A) Sketch by inspection \( v(t) \) in the figure below, for \( t > 0 \) given \( i(0) = I_0 \). What is the time constant \( \tau \)?

Give expressions for initial and final values in your sketch. (8 points)

\[ \tau = \frac{L}{R_{eq}} \quad R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \]

[Diagram of a circuit with resistors in parallel and a voltage source.]

\( I(0) = -I_0 R_{eq} \)

\[ V(\infty) = \frac{I_0}{R_{eq}} \quad t = \infty \]
(B) Sketch by inspection $v(t)$ in the figure below, for $t > 0$ given that the switch is moved from 1 to 2 at $t = 0$. What is the time constant $\tau$? Give expressions for initial and final values in your sketch. (8 points)

\[ R \text{ and } C \text{ in parallel} \]

\[ C_0 = C_1 + C_2 \quad R_0 = \frac{R_1 R_2}{R_1 + R_2} \]

\[ v(0) = V_0 \]

\[ t = \infty, \text{ capacitors are open} \Rightarrow v(t) = \frac{R_1}{R_1 + R_2} V_0 \]

\[ v(t) = V_0 e^{-\frac{R_1}{R_1 + R_2} t} \]

\[ \tau = R e \frac{C_0}{C_0} = (C_1 + C_2) \left( \frac{R_1 R_2}{R_1 + R_2} \right) \]
(C) Sketch by inspection $v(t)$ in the figure below, for $t > 0$ given that $i(0) = I_0$ and $v(0) = 0$. What is the value of $\omega_0$? Give an expression for the maximum value of $v$ in your sketch. (9 points)

Undamped 2nd order system $\Rightarrow \omega = \frac{1}{\sqrt{LC}}$

By energy conservation, $V_{\text{max}}$ when $i = 0$ (i.e., inductor has given all energy to capacitor)

$\Rightarrow E_i = \frac{1}{2} C (i_0)^2 + \frac{1}{2} L i_0^2 = \frac{1}{2} L i_0^2$

$\Rightarrow \frac{V_{\text{max}}^2}{C} = \frac{1}{2} L i_0^2$

$\Rightarrow V_{\text{max}} = \pm i_0 \sqrt{\frac{L}{C}}$ (choose positive to get "max")

Maximum value of $v = I_0 \sqrt{\frac{L}{C}}$
Problem 4 — 25 Points

Minnie Delay is interested in designing a light in her dorm room that turns brighter gradually when switched on, and dims gradually when switched off. Having just learned RC circuits in 6.002, she comes up with the following design.

The bulb can be modeled as a resistance of value $R$. Similarly, when turned ON, the switch can be modeled as a resistor of value $R_{ON}$. Assume that $V_S$ is a positive DC voltage. The following questions concern $v_B$, the node voltage at $B$ (note that $v_B$ is NOT the voltage across the bulb).

(A) When the switch is in the ON state, what is the steady state value of the node voltage $v_B$? (5 points)

$$v_B = \frac{R_{ON}}{R_{ON} + R} V_S$$

Steady state value of $v_B = \frac{R_{ON}}{R_{ON} + R} V_S$
(B) After being ON for a long time, the switch is turned OFF at time $T_0$. Sketch the form of $v_B$ starting at $T_0$. Clearly indicate the initial values, final values (in the steady state), and the time constant $\tau_0$. (6 points)

\[ T_0 = R C \]
(C) After being OFF for a long time, the switch is turned ON at time $T_1$. Sketch the form of $v_B$ starting at $T_1$. Clearly indicate the initial value, final value (in the steady state), and the time constant $\tau_1$. Also, write an expression for $v_B$ as a function of $t$, starting at time $T_1$. Remember that the switch has a resistance $R_{ON}$ when turned ON. (9 points)

When switch on $\Rightarrow v_s$

Circuit can only have 1 $\tau$, so consider case when

$V_s = 0$. Then $\tau = C \frac{R_{ON}}{R + R_{ON}}.$

$V_B(T_1) = V_s$, $V_B(\infty) = \frac{R_{ON}}{R_{ON} + R} V_s$

Remember that time starts from $t = T_1$

$\Rightarrow V_B(t) = \frac{V_s}{R_{ON} + R} \left[ \frac{R_{ON}}{R_{ON} + R} e^{-\left(\frac{t - T_1}{\tau_1}\right)} \right]$

![Diagram showing the voltage $v_B$ over time $t$ with $T_1$ and $V_s$]
(D) For the situation in Part (C), how long does it take the voltage $v_D$ to reach half its initial value (i.e., its value at $T_1$) from the instant the switch is turned ON. Assume that $R = 2R_{ON}$ for this part. You may express your answer in terms of $T_1$. (5 points)

\[
\text{Wanted: } v_D \text{ to reach } \frac{V_S}{2}, \text{ given } R = 2R_{ON}
\]

\[
\Rightarrow \frac{V_S}{2} = \frac{V_S}{3R_{ON}} \left[ R_{ON} + 2R_{ON} e^{-\frac{(t-T_1)}{T_1}} \right]
\]

\[
\Rightarrow \frac{3}{2} = 1 + 2 e^{-\frac{t}{T_1}}
\]

\[
\Rightarrow \frac{1}{4} = e^{-\frac{t}{T_1}}
\]

\[
\Rightarrow \Delta t = -T_1 \ln \left( \frac{1}{4} \right)
\]

Time for $v_D$ to reach half its initial value $= \Delta t = -T_1 \ln \left( \frac{1}{4} \right)$