6.002 Circuits and Systems

Final Review

CIRCUITS AND SYSTEMS

FINAL REVOLUTIONS

Jason Kim

2003
6.002 Circuits and Systems: Outline of Topics

- **Resistor Networks**
  - **Concepts:** Node Method, KCL, KVL, Superposition, Thevenin and Norton equivalent circuit models, Power.

- **1st Order Circuits**
  - **Concepts:** Constitutive Laws, RC networks, RL networks, Homogeneous and Particular solution, Time constant $\tau$, response to an impulse, power, Low/High/Band/Notch pass filters, sinusoidal steady-state, transients, Impedance Model, Bode Plots (Magnitude and Phase).

- **2nd Order Circuits**
  - **Concepts:** LC networks, LRC networks, damping coefficient $\alpha$, natural frequency $\omega_n$, Underdamped/Critical/Overdamped Systems, Resonance, $Q$ factor, Half-power point, Sinusoidal steady-state, transients, Power (real/reactive), Impedance Model, ELI ICE, Bode Plots (Magnitude and Phase).

- **Digital Abstraction**
  - **Concepts:** Boolean Logic (truth table, formula, gates, and transistor level), primitive laws, DeMorgan’s Law (formula and gate equivalent), MSP (minimum sum of products), MPS (minimum product of sums).

- **MOSFET Transistors**
  - **Concepts:** Large Signal Model (S, SR, SCS, SVR), Small Signal Model, Saturation/Linear Region, $R_{on}$, Loadline, Operating Point, Input/Output Resistance, Current Gain, Power Gain, Noise Margin.

- **Op-Amp Circuits**
  - **Concepts:** Single/Multiple input op-amp circuits, differentiator, integrator, adder, subtractor, op-amp with R, L, C, negative feedback, positive feedback, Bode Plots (Magnitude and Phase), Input/Output resistance, Schmitt Trigger, Hysteresis, Cascaded stages.

- **Diodes**
  - **Concepts:** $i-v$ characteristics, model, diode w/ R, L, C, and op-amp, Peak detection, Clipper circuits, Incremental Analysis.

- **Energy and Power**
  - **Concepts:** Energy and Power relationships, nMOS and CMOS logic inverter power dissipation, static and dynamic power, power reduction techniques.
Resistor:

- **time domain:** \( V = IR \)
- **freq. domain:** \( Z = R \)

**Series:** \( R_1 + R_2 \)
**Parallel:** \( \frac{R_1 R_2}{R_1 + R_2} \)

\[ V_o = \frac{R_2}{R_1 + R_2} V_i \]

Voltage Divider
\[ I_2 = \frac{R_1}{R_1 + R_2} I_1 \]

Current Divider

**Node Analysis (KCL, KVL):**
- *KCL:* Sum of all currents entering and leaving a node is zero.
- *KVL:* Sum of all voltages around a closed loop path is zero. (be careful with polarities)
  1) LABEL all current directions and voltage polarities. (Remember, currents flow from + to -)
  2) write KCL & KVL Equation.
  3) substitute and solve.

**Simplification (by inspection):**

- **Thevenin Equivalent Circuit Model**
  - Three variables: \( V_{th} = V_{oc} \), \( R_{th} = R_N \), and \( I_{N} = I_{sc} \). They are related by \( V_{oc} = I_{sc} R_{th} \).
  - \( V_{th} = V_{oc} \): Leave the port open (thus no current flow at the port) and solve for \( V_{oc} \).
    For a resistive network, this gives you a point on the V-axis of the i-v plot.
  - \( I_{N} = I_{sc} \): Short the port (thus no voltage across the port) and solve for \( I_{sc} \) flowing out of + and into -. For a resistive network, this gives you a point on the I-axis of the i-v plot.
  - \( R_{th} \): Set all sources to zero, except dependent sources. Solve the resistive network.

When setting sources to zero, V source becomes short and I source becomes open. R_{th} may also be found by attaching I_{test} and V_{test} and setting \( R_{th} = \frac{V_{test}}{I_{test}} \).

If dependent sources are present, set only the independent sources to zero and attach I_{test} and V_{test}. Use KCL and KVL to find the expression \( \frac{V_{test}}{I_{test}} = R_{th} \).

\( R_{th} = \frac{V_{test}}{I_{test}} \)

\* For both Thevenin & Norton E.C.M.’s, they have the same power consumption at the port as the original circuit, but not for its individual components. Power dissipated at \( R_{th} \) does not equal power dissipated at the resistors of the original circuit. Same holds for the power delivered by the sources in the Thevenin model and the original circuit. Thevenin and Norton E.C.M.’s are for terminal i-v characteristics only.

- **Power = Real Power = IV = I^2R = V^2/R** (energy dissipated as heat)
- \( \langle Power \rangle = \frac{1}{T} \int_0^T I(t)V(t)dt \)
1. Primitive Elements

- Superposition

"In a linear network with a number of independent sources, the response can be found by summing the response to each independent sources acting alone, with all other independent sources set to zero."

\[
V_R = V_{R_1}|_{V_{2-n} = 0} + V_{R_2}|_{V_{1-n} = 0} + \cdots + V_{R_n}|_{V_{1-n} - 0}
\]

1) Leave one source on and turn off all other sources. 
   (Voltage source "off" = short & Current source "off" = open)
2) Find the effect from the "on" source.
3) Repeat for each sources.
4) Sum the effect from each sources to obtain the total effect.

For cases where a linear dependent source is present along with multiple independent sources, DO NOT turn off the dependent source. Leave the dependent source on and carry it in your expressions. Tackle the dependent source term last by solving linear equations. Remember that the variable which the dependent source is depended on is affected by the individual independent sources that you are turning on and off.

**Capacitor:**
- **time domain:** \( I = \frac{dV}{dt} \)
- **freq. domain:** \( Z = \frac{1}{i\omega C} \)

High Frequency = Short & Low Frequency = Open

Series: \( \frac{C_1C_2}{C_1 + C_2} \) Parallel: \( C_1 + C_2 \)

\( V_C(0^-) = V_C(0^+) \) except when the input source is an impulse.
\( V_C(t) \) is continuous while \( I_C(t) \) may be discontinuous.

"I C E" : Current(I) LEADS Voltage(EMF) by \( 90^\circ \) (mnemonic: ELI ICE)

Energy Conservation: \( \int i dt = Q = CV \).

Energy Stored: \( E = \frac{1}{2} CV^2 \). (no energy dissipation)

Note: Two identical capacitors, \( C_1 \) and \( C_2 \), in series may act like an open circuit after a long time where the total voltage across the capacitors are split between the two cap’s. For this case, \( V_C \) does not discharge to zero but \( V_{C1} = V_{C2} \).

\[ + \quad R \quad - \]
\[ + \quad C_1 \quad C_2 \quad + \]
\[ + \quad V_{C1} \quad + \quad V_{C2} \]
\[ - \quad V_{C1}(0^-) \neq V_{C2}(0^-) \]

**Inductor:**
- **time domain:** \( V = \frac{di}{dt} \)
- **freq. domain:** \( Z = L_s = j\omega L \)

High Frequency = Open & Low Frequency = Short

Series: \( L_1 + L_2 \) Parallel: \( \frac{L_1L_2}{L_1 + L_2} \)

\( I_L(0^-) = I_L(0^+) \) except when the input source is an impulse.
\( I_L(t) \) is continuous while \( V_L(t) \) may be discontinuous.
1. Primitive Elements

"E L I" : Voltage(EMF) LEADS Current(I) by 90° (mnemonic: ELI ICE)

Energy Conservation: \[ \int V \, dt = \lambda = LI . \]

Energy Stored: \[ E = \frac{1}{2} L I^2 . \] (no energy dissipation)

Note: Two identical inductors, \( L_1 \) and \( L_2 \), in parallel may act like a short circuit after a long time and have current flow through this loop indefinitely. For this case, \( i_L \) is not zero but \( i_{L1} = -i_{L2} \).

\[
\begin{align*}
V_{\text{td}} &= \int \lambda \Rightarrow L_1 = \frac{E}{2} \\
i_{L1}(0^-) &= i_{L2}(0^-)
\end{align*}
\]

LC:

Series@ \( \omega_o \) = Short & Parallel@ \( \omega_o \) = Open

Energy transfers between inductors and capacitors sinusodially (180° out of phase).
Also, \( V_c \) peaks when \( I_L = 0 \) and \( I_L \) peaks when \( V_c = 0 \).
2. First Order Circuits

First Order RC & RL Circuits

**RC Network**

\[ V_1 \quad V_C \quad \frac{R}{C} \quad i_C \quad + \quad V_c(0^+)=0 \]

**RL Network**

\[ i_L \quad \frac{L}{R} \quad V_L \quad + \quad i_L(0^+)=0 \]

**KVL:** \( V_i = I_c R + V_c \)

**KCL:** \( I_c = C \frac{dV_c}{dt} \)

**Differential Eq:** \( \frac{dV_c}{dt} + \frac{V_c}{RC} = \frac{V_i}{RC} \)

**Time Constant:** \( \tau = RC \)

*One time constant \( \tau \) charges/discharges 63% of the maximum value. Small time constant shows the effect of the exponential curve sooner, while big time constant shows a linear line for a long time before the exponential effect becomes visible.*

**Homogeneous:** \( Ae^{\frac{t}{\tau}} \)

**Particular:** **Step Input**

**Total:**

*Notice the *Duality* between \( v_c \) & \( i_L \) and \( i_c \) & \( v_L \).*
2. First Order Circuits

Approach to First Order Circuits

1) Set up differential equations, using KCL and KVL. Use the constitutive laws for C and L.
\[ \frac{dV_c}{dt} + \frac{V_c}{RC} = \frac{V_i}{RC} \]

2) Find Homogeneous solution by setting input to zero and substituting \( V_{h}=Ae^{st} \) as a solution.
\[ Ae^{st} + \frac{1}{RC}Ae^{st} = 0 \]
\[ \therefore s = -\frac{1}{RC} \]

3) Find Particular solution. Remember the output follows the form of the input.
\[ V_{p} = V_{i} : t > 0 \]

4) Combine Homogeneous and Particular solutions, and use initial conditions to find missing variables.
   (Be careful when input is an impulse!)
\[ V_{o} = V_{h} + V_{p} = Ae^{\frac{-t}{RC}} + V_{i} \]
\[ V_{o} = A + V_{i} = 0 \quad \text{at } t=0^+ \]
\[ \therefore A = -V_{i} \]
\[ V_{o} = V_{i}\left(1 - e^{\frac{-t}{RC}}\right) \quad \text{at } t=0^+, \quad V_{o}=0 \quad \text{and} \quad t=\infty, \quad V_{o}=V_{i}. \]

*After a long time, transients(homogeneous solution) die away, leaving only the particular solution.
Thus, the output may look different from the input in the beginning but slowly catches up to follow the input.

Impulse as Input

ex. RC circuit: \( V_{i} = Q\delta t \) and \( V_{o}(0^{\text{minus}}) = 0 \) and \( \frac{dV_{o}}{dt} + \frac{V_{o}}{RC} = \frac{V_{i}}{RC}. \)
\[ dV_{o} + \frac{1}{RC}V_{o}dt = \frac{Q}{RC}\delta t dt \]
\[ \int dV_{o} + \frac{1}{RC} \int V_{o}dt = \frac{Q}{RC} \int \delta t dt \rightarrow V_{o}(0^{\text{plus}}) - V_{o}(0^{\text{minus}}) = \frac{Q}{RC} \]
\[ V_{o}(0^{\text{plus}}) - 0 = \frac{Q}{RC} \rightarrow \therefore V_{o}(0^{\text{plus}}) = \frac{Q}{RC} \]
3. Second Order Circuits

Second Order LRC Circuits

#1

\[ \begin{align*}
V_o &= \frac{j\omega LR}{(j\omega)^2 LCR + j\omega L + R} \\
\frac{V_o}{i_i} &= \frac{R}{(j\omega)^2 LRC + j\omega L + R} \\
\omega_o &= \frac{1}{\sqrt{LC}} \\
\end{align*} \]

Diff. Eq.: \[ \frac{2}{\tau_1} \frac{\partial^2 V_o}{\partial t^2} + \frac{1}{\LC} \frac{\partial V_o}{\partial t} + \frac{V_o}{LC} = \frac{1}{\LC} \frac{\partial i_i}{\partial t} \]

Time Constant: \( \tau = \frac{1}{\sqrt{LC}} \)

"Band Pass Filter"

#2

\[ \begin{align*}
V_o &= \frac{R}{(j\omega)^2 LRC + j\omega L + R} \\
\omega_o &= \frac{1}{\sqrt{LC}} \\
\end{align*} \]

Diff. Eq.: \[ \frac{2}{\tau_1} \frac{\partial^2 V_o}{\partial t^2} + \frac{1}{\LC} \frac{\partial V_o}{\partial t} + \frac{V_o}{LC} = \frac{V_i}{LC} \]

Time Constant: \( \tau = \frac{1}{\sqrt{LC}} \)

"Low Pass Filter"

#3

\[ \begin{align*}
i_i &= \frac{j\omega C}{(j\omega)^2 LC + j\omega RC + 1} \\
\frac{i_i}{V_i} &= \frac{1}{(j\omega)^2 LC + j\omega RC + 1} \\
\end{align*} \]

Diff. Eq.: \[ \frac{2}{\tau_1} \frac{\partial^2 i_i}{\partial t^2} + \frac{R}{\LC} \frac{\partial i_i}{\partial t} + \frac{i_i}{LC} = \frac{1}{\LC} \frac{dV_i}{dt} \]

Time Constant: \( \tau = \frac{1}{\sqrt{LC}} \)
3. Second Order Circuits

- "peakiness" $Q = \frac{\omega_n}{2\alpha}$ & Half-Power Point = $\omega_o \pm \alpha$

![Band Pass Filter]

**Characteristic Equation** (Homogeneous Solution)

Obtain a characteristic equation by setting input to zero and replacing $\frac{d}{dt}$ with $s$ to get into the form:

$$s^2 + 2\alpha s + \omega_n^2 = 0$$

where $\omega_n$ is the "resonant" or "natural" frequency and $\alpha$ is the damping coefficient.

solving $s$ with quadratic formula,

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} = -\alpha \pm j\sqrt{\omega_n^2 - \alpha^2} = -\alpha \pm j\omega_d; \quad \omega_d$$ is the damped natural frequency

1) "Overdamped" $\alpha > \omega_n$ $s_1 \neq s_2$ $s_{1,2}$ are real

2) "Critically Damped" $\alpha = \omega_n$ $s_1 = s_2$ $s_{1,2}$ are real

3) "Underdamped" $\alpha < \omega_n$ $s_1 \neq s_2$ $s_{1,2}$ are complex

**Approach to Second Order Circuits (Transients)**

1) Set up 2nd order differential equations.

$$\frac{\partial^2 i_L}{\partial t^2} + \frac{R}{L} \frac{\partial i_L}{\partial t} + \frac{i_L}{LC} = \frac{1}{L} \frac{dV_i}{dt}$$ (ex. circuit #3)

2) Obtain characteristic equation and identify $\omega_n$ and $\alpha$.

$$s^2 + 2\left(\frac{R}{2L}\right)s + \left(\frac{1}{\sqrt{LC}}\right)^2 = 0; \quad \therefore \omega_n = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \therefore \alpha = \frac{R}{2L}$$
3. Second Order Circuits

3) Find Homogeneous solution (for the appropriate system type)

Overdamped:
\[ V_h = A e^{-(\alpha - \sqrt{\alpha^2 - \omega_0^2})t} + B e^{-(\alpha + \sqrt{\alpha^2 - \omega_0^2})t} \]

Critically Damped:
\[ V_h = A e^{-\alpha t} + B e^{-\alpha t} \]

Underdamped:
\[ V_h = A e^{-\alpha t} \cos \omega_d t + B e^{-\alpha t} \sin \omega_d t \]

4) Find Particular solution. Remember the output follows the form of the input.

5) Combine Homogeneous and Particular solutions, and use initial conditions to find missing variables. (Be careful when input is an impulse!)

\[ V_o = V_p + V_h = V_f + A e^{-\alpha t} \cos \omega_d t + B e^{-\alpha t} \sin \omega_d t \]
\[ i_L = C e^{-\alpha t} (-A \omega_d \sin \omega_d t + B \omega_d \cos \omega_d t) \]
\[ V_o(0) = V_f + A = 0 \quad \therefore A = -V_f \]
\[ i_L(0) = CB \omega_d = 0 \quad \therefore B = 0 \]

Thus, \[ V_o = V_f (1 - e^{-\alpha t} \cos \omega_d t) \] and \[ i_L = V_f C \omega_d e^{-\alpha t} \sin \omega_d t \]

Approach to Second Order Circuits (Sinusoidal Steady-State)

1) Replace the primitive elements with their impedance models and solve the network like a resistor network.

\[ i_L = I \cos(\omega t) \quad V_o = V \sin(\omega t + \phi) \]

2) Put the equation into the form of a real amplitude and a complex phase.

\[ \tilde{V}_o = \frac{\omega L}{\sqrt{(1 - \omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2}} i_L \]

\[ j = \begin{bmatrix} \text{imaginary part} \\ \text{real part} \end{bmatrix} \]

3) Match the coefficient V and phase shift \( \phi \)

\[ V = \frac{\omega L}{\sqrt{(1 - \omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2}} I \quad \text{and} \quad \phi = -\arctan \left( \frac{\frac{\omega L}{R}}{2 (1 - \omega^2 LC)} \right) \]

*notice that at the resonant frequency \( \omega_o = \frac{1}{\sqrt{LC}} \), the magnitude of the output is just R, and the LC circuit just behaves as if nothing is there (=open circuit). See circuit #1.

If the input was a sine wave instead of a cosine wave, then we would add an additional phase shift of 90 degrees, since \( \sin(\omega t) = \cos(\omega t - \frac{\pi}{2}) \). Thus, our new phase shift would be \( \phi' = \frac{\pi}{2} - \phi \).
4. Magnitude & Phase Plots

Poles & Zeros

obtain a transfer function of the system by replacing \( \frac{d}{dt} \) with \( s \), which is also equal to \( j\omega \),

\[
H(j\omega) = \frac{(10 + 100\tau)}{s(10 + s)} = \frac{10 + j(100\omega RC)}{j\omega(10 + j\omega RC)}
\]

roots of the numerator are "zeros"
roots of the denominator are "poles"

Break Point Frequency: \( \omega_{BP} \)

Poles and Zeros are break point frequencies. Equate real and imaginary parts for each term in the numerator and denominator (i.e., factor and find the roots) to get the break point frequencies:

numerator: \( \omega_z = \frac{1}{10RC} \)
denominator: \( \omega_p = \frac{10}{RC} \)

Magnitude Plot

short cut: "zeros" give you a +1 slope. (constant before \( \omega_z \), +1 slope after \( \omega_z \); in log scale)
"poles" give you a -1 slope. (constant before \( \omega_p \), -1 slope after \( \omega_p \); in log scale)

Thus,

\[
\begin{array}{ccc}
Z1: & 0 & +1 \\
P1: & 0 & 0 \\
P2: & -1 & -1 \\
\end{array}
\]

\[
\begin{array}{ccc}
\omega_p^1 & \omega_z & \omega_p^2 \\
-1 & 0 & -1 \\
\end{array}
\]

\[
\sum: \omega_p^1 + \omega_z + \omega_p^2 = 0
\]

Since \( \omega_p^1 = 0 \), which is the left most end of the frequency axis (in log scale), the magnitude plot starts off at -1 slope.

*both axes are in log scale.

Approach to Plotting Magnitude

1) Find all ZEROs and POLEs. (i.e., \( \omega_{BP} \)'s)
2) Draw and Label Axes. Denote break point frequencies.
3) Beginning at the far end of the \( \omega \)-axis, plot for \( \omega \) very small
   
   if \( \text{ZERO at } \omega = 0 \), start with +1 slope, \(+2 \text{ slope if double ZERO } \omega = 0, \text{ and } +3 \text{ for triple, ...}\)
   \( \text{POLE at } \omega = 0 \), start with -1 slope, \((-2 \text{ slope if double POLE } \omega = 0, \text{ and } -3 \text{ for triple, ...})\)
   otherwise, start with a constant. (label!)
4. Magnitude & Phase Plots

4) As you come up on the frequency axis, for each POLE you hit, go down by -1 slope, for each ZERO you hit, go up by +1 slope.

5) At $\omega_{BP}$, your actual curve will be off from the sharp asymptote corner by 3dB.

Phase Plot

short cut: "zeros" give you a $+\frac{\pi}{2}$ phase over ±decade.

"poles" give you a $-\frac{\pi}{2}$ phase over ±decade.

example,

Thus,

<table>
<thead>
<tr>
<th>Z1:</th>
<th>0</th>
<th>0</th>
<th>$+\frac{\pi}{4}$</th>
<th>$+\frac{\pi}{2}$</th>
<th>$+\frac{\pi}{2}$</th>
<th>$+\frac{\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1:</td>
<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{2}$</td>
<td></td>
</tr>
<tr>
<td>P2:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{\pi}{4}$</td>
<td>$-\frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

Approach to Plotting Phase

1) Find all ZEROs and POLEs. (i.e., $\omega_{BP}$’s)

2) Draw and label all axes and break point frequencies, including their +/- decades.

3) Beginning at the far left of the $\omega$-axis, plot for $\omega$ very small

   if ZERO at $\omega=0$, start with $+\frac{\pi}{2}$ phase, (+$\pi$ slope if double Zero@ $\omega=0$, and $+\frac{3\pi}{2}$ for triple, ...)

   POLE at $\omega=0$, start with $-\frac{\pi}{2}$ phase, (-$\pi$ slope if double Zero@ $\omega=0$, and $-\frac{3\pi}{2}$ for triple, ...)

   otherwise, start at 0 phase.

4) As you come up the $\omega$-axis, for each POLE you hit, make a change of $-\frac{\pi}{2}$ phase over a ±decade.
4. Magnitude & Phase Plots

\[ \frac{a_p}{10} \sim a_p \sim 10a_p \]. For each ZERO, make a change of \( \pm \frac{\pi}{2} \) phase over a \( \pm \)decade.

5) Be careful when the regions overlap, the effect from each pole and zero may cancel each other out.

6) At \( \omega_{BP} \), the actual curve is off by \( 6^\circ \).

**Bode Plot Building Blocks**

\[
\begin{array}{|c|c|c|}
\hline
H(j\omega) & S & \frac{1}{S} \\
\hline
|H(j\omega)| & +1 & +1 \\
\hline
\angle H(j\omega) & 0 & 0 \\
\hline
S + 100 & +1 & +1 \\
\frac{1}{S + 100} & +1 & +1 \\
\hline
\end{array}
\]
5. Digital Abstraction

Boolean Logic

*become comfortable expressing in different forms: truth table ↔ formula ↔ gates ↔ transistor level.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>OUT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \text{OUT} = (A \cdot B) + C \]

*Note that in the transistor-level implementation, transistors in series form an AND expression and those in parallel form an OR expression. The final output is always inverted in the pull-up implementation, equivalent to adding a NOT(inverter) to the output.

Primitive Rules

\[
\begin{align*}
A + (B + C) & = (A + B) + C \\
A + B & = B + A \\
A + (B \cdot C) & = (A + B) \cdot (A + C) \\
A + 1 & = 1 \\
A + 0 & = A \\
A + \bar{A} & = 1 \\
A + A & = A \\
A + (A \cdot B) & = A \\
A + (\bar{A} \cdot B) & = A + B
\end{align*}
\]

\[
\begin{align*}
A \cdot (B \cdot C) & = (A \cdot B) \cdot C \\
A \cdot B & = B \cdot A \\
A \cdot (B + C) & = (A \cdot B) + (A \cdot C) \\
A \cdot 0 & = 0 \\
A \cdot 1 & = A \\
A \cdot \bar{A} & = 0 \\
A \cdot A & = A \\
A \cdot (A + B) & = A \\
A \cdot (\bar{A} + B) & = A \cdot B
\end{align*}
\]

(associative)  (commutative)  (distributive)  (complement)  (idempotence)  (absorption)  (absorption)

DeMorgan’s Laws

1) \[ \overline{A \cdot B} = \overline{A} + \overline{B} \] Gate Equivalent:

2) \[ \overline{A + B} = \overline{A} \cdot \overline{B} \] Gate Equivalent:

Minimum Sum of Products(MSP): (1)first level AND’s, (2) combine with OR’s.  ex) \( (A \cdot B) + (B \cdot C) \)

Minimum Product of Sums(MPS): (1)first level OR’s, (2) combine with AND’s.  ex) \( (A+B) \cdot (B+C) \)

Common Gates

\[
\begin{array}{c|c|c}
A & B & C \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
A & B & C \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
A & B & C \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
A & B & C \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
A & B & C \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{array}
\]
**6. MOSFET Transistor (n-channel)**

**Large Signal Model**

**S Model:**

When \( V_{DS} < V_{GS} - V_{T} \),

**SCS Model (Saturation):**

\[ V_{GS} < V_{T} \]

**SR Model:**

When \( V_{DS} \geq V_{GS} - V_{T} \),

**SVR Model (Linear):**

\[ V_{GS} \geq V_{T} \]

\[ i_{DS} = \frac{K(V_{GS} - V_{T})^2}{2} \]

\[ R_{on} = \frac{1}{K(V_{GS} - V_{T})} \]

MOSFET characteristics

Load Line superimposed on MOS characteristics for operating point analysis of MOSFET Amplifier

**MOSFET Amplifier**

Small signal gain: \( \frac{v_o}{v_i} = -K(V_{T} - v_{T})R_L = -g_mR_L \) *Note that the output is a function of both \( V_I(DC) \) and \( v_i(AC) \).

Large Signal Model (SCS) is used to obtain the operating point, and Small Signal Model is used to obtain the output using Linear Approximation as shown on the right above. The Linear Approximation line is different for each \( V_I \).
6. MOSFET Transistor (n-channel)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (for small signal model)</th>
<th>Parameter</th>
<th>Value (for small signal model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input Resistance</td>
<td>$\infty$</td>
<td>Current Gain</td>
<td>$K(V_I - v_T)(R_L \parallel R_O) \frac{R_I}{R_O}$ ; $\propto$ if $R_i = \infty$</td>
</tr>
<tr>
<td>Output Resistance</td>
<td>$R_L$</td>
<td>Power Gain</td>
<td>$[K(V_I - v_T)(R_L \parallel R_O)]^2 \frac{R_I}{R_O}$ ; $\propto$ if $R_i = \infty$</td>
</tr>
</tbody>
</table>

Noise Margins

To improve noise margins: (1) increase $V_{IL}$ and lower $V_{IH}$, and/or (2) increase $V_{OH}$ and lower $V_{OL}$.

1: increase $v_T$ of the MOSFET. This would make the transistor to turn on at a higher threshold voltage, increasing $V_{IL}$ and lowering $V_{IH}$.

2: increase load resistance $R_L$ or (W/L) ratio of the MOSFET since $V_O \propto \frac{R_{on}}{R_{on} + R_L}$ and $R_{on} \propto \frac{L}{W}$. 

Inverter characteristic curve
7. Op-Amp Circuits

**Model**

- **Operations**
  * The Difference \( V^+ - V^- \) is amplified by \( A(10^6) \) as \( V_{out} \). The + and - terminal inputs have infinite impedance, so no current flows into the terminals. Most Op-Amp circuits make use of negative feedback where output tries to follow the input. A typical op-amp has output limits imposed by the power supply of +/- 15V.

- **Approach**
  1) Set \( V^+ = V^- \). (ideal op-amp w/ negative feedback)
  2) Do KCL at + and - terminal. (no current flows into the + and - terminals)
  3) Solve for the expression \( V_o/V_i \).

**Negative Feedback**

- Inverting Op Amp:

  ![op-amp abstraction](attachment:op-amp.png)

  ![op-amp model](attachment:op-amp.png)

  ![more accurate op-amp model](attachment:op-amp.png)

  * No current flows into the + and - terminals.
  * \( V_{out} = A(V^+ - V^-) \Rightarrow V^+ - V^- = V_{out}/A \sim 0 \). Thus, \( V^+ = V^- \).
  * KCL @ \( V^- \), \( V_{out} = -\frac{R_f}{R_i}V_{in} \). In General, \( V_{out} = \frac{Z_f}{Z_i}V_{in} \) thus, \( \frac{V_{out}}{V_{in}} = H(j\omega) = \frac{Z_f}{Z_i} \)
  * Input Resistance: \( R_i = R_f || (R_f + r_i) || \left( \frac{R_f + r_i}{A} \right) \approx \frac{R_f + r_i}{A} \); if \( r_i = \infty \), \( r_i \) term goes away
  * Output Resistance: \( R_o = \frac{r_t}{1 + A \frac{R_i}{R_f + R_f} + \frac{A R_f}{R_i + R_f}} \approx \frac{r_t}{A \frac{R_f}{R_i + R_f}} \)

- Non-inverting Op Amp:

  ![op-amp abstraction](attachment:op-amp.png)

  ![op-amp model](attachment:op-amp.png)

  ![more accurate op-amp model](attachment:op-amp.png)

  * \( V_{out} = \frac{R_1 + R_2}{R_2} V_{in} \)
  * Input Resistance: \( R_i \approx r_i \left[ \frac{AR_2}{R_1 + R_f + R_2} \right] \)
  * Output Resistance: \( R_o \approx \frac{r_t}{A \frac{R_2}{R_1 + R_2}} \)
7. Op-Amp Circuits

Important Circuits

Adder

\[ V_o = \frac{R_3}{R_1} V_1 + \frac{R_3}{R_2} V_2 \]

Subtractor

\[ V_o = \frac{R_2}{R_1} (V_2 - V_1) \]

Integrator

\[ V_o = \frac{1}{RC} \int V_i dt \]

Differentiator

\[ V_o = RC \frac{dV_i}{dt} \]

Double Integrator

\[ V_o = \frac{1}{LC} \int \int V_i dt \]

Integrator

\[ V_o = \frac{R}{L} \int V_i dt \]

Differentiator

\[ V_o = \frac{L}{R} \frac{dV_i}{dt} \]

Double Differentiator

\[ V_o = \frac{1}{LC} \frac{d^2 V_i}{dt^2} \]

Several of these op-amp circuits may be cascaded in stages. For such cases, you may find the $V_o/V_i$ relationship for each stage using impedances and multiply them together to obtain the total $V_o/V_i$.

example:

\[ v_a = \frac{R}{L} \int V_o dt \]
\[ v_b = -V_i - RC \frac{dV_o}{dt} \]
\[ v_o = \frac{v_a + v_b}{2} \]

Thus,
\[ \frac{dV_i}{dt} = -RC \frac{d^2 V_o}{dt^2} - 2 \frac{dV_o}{dt} \frac{RV_o}{L} \]
**8. Energy and Power**

**nMOS Inverter**

Static: \(V_{in} [0, T/2] = \text{high}\)

Dynamic:

\[
P = \frac{V_{sup}^2}{2(R_L + R_{on})^2} \cdot f
\]

Since it’s for \([0, T/2]\) which is half the period

\[
V_{delta} = V_{sup} \left(\frac{R_L}{R_L + R_{on}}\right)
\]

Combining static and dynamic average power,

\[
\bar{P} = \frac{V_{sup}^2}{2(R_L + R_{on})} + C \cdot V_{sup}^2 \cdot f \cdot \frac{R_L^2}{(R_L + R_{on})^2}
\]

When \(R_L >> R_{on}\),

\[
\bar{P} = \frac{V_{sup}^2}{2R_L} + C \cdot V_{sup}^2 \cdot f
\]

\(\bar{P}_{\text{static}}\) independent of frequency. \(\bar{P}_{\text{dynamic}}\) related to switching capacitor MOSFET on half the time.

**CMOS Inverter**

Dynamic: \(V_{in} [0, T/2] = \text{high}\)

Dynamic: \(V_{in} [T/2, T] = \text{low}\)

\[
P = \frac{C \cdot V_{sup}^2 \cdot f}{2}
\]

\[
P = \frac{C \cdot V_{supply} \cdot f}{2}
\]

*No static power dissipation (in ideal CMOS). This is done by implementing nMOS logic function and pMOS logic function to be opposites, using DeMorgan’s Law. This ensures that only one function (either nMOS OR pMOS) is true, shorting the output either to ground OR supply voltage.

Ex. from top of Sec. 5 Digital Abstraction,

nMOS function: \((A\bar{B})+C \Rightarrow \text{take the the opposite function: } (A\bar{B})\bar{C}\)

\(\Rightarrow \text{Use DeMorgan’s Law: } (A\bar{B})\bar{C} = (A + \bar{B}) \bar{C} \); the result is the pMOS function

(note, transistors in series constitutes an AND, and in parallel constitutes an OR)

**Ways to reduce power...**

- Reduce switching speed: reduce \(f\)
- Reduce supply voltage: reduce \(V_{supply}\)
- Reduce circuit elements; efficient circuit implementation: reduce \(C\)
- Turn off clock when not in use: reduce \(f\) for part of the chip
9. Non-Linear Circuits

Diode

• Model

DC Analysis: For the large signal model, the diode acts like a short with a voltage drop of 0.6V when the diode is on (Vd>0.6V), and like an open when the diode is off (Vd<0.6V).

\[ i_D = I_s \left( e^{\frac{qV_D}{kT}} - 1 \right) \]
where \( I_s \sim 10^{-12} \) Amps for Silicon diodes.

AC Analysis: For the incremental model, the diode acts like a current dependent resistor. The resistor value is set by the nominal(DC) operating current \( I_d \):

\[ r_d = \frac{K T}{q I_d} = \frac{26mV}{I_d} \]

• Half-wave Rectifier: Clips the negative portion of the input wave and passes only the positive portion of the input wave with a diode drop of 0.6V.

\[-V_i + i_D R + V_D = 0\]

• Peak Detector

\[ *\text{using ideal diode} \]

Good Luck!

12. 10. 2003. JK.

www.mit.edu/~jkim/6.002

comments to jkim@mit.edu