Non-linear circuits.

- We're going to consider the following circuit with a non-linear 2-terminal device and

\[
\begin{align*}
I_0 &= 50 \text{ mA} \\
R_1 &= 12.5 \Omega \\
R_2 &= 3 \Omega \\
R_3 &= 2.5 \Omega
\end{align*}
\]

- We are told: \( i_N = I_S e^{\frac{UN}{V_T}} \) where \( I_S = 10 \text{ mA} \) \( V_T = 25 \text{ mV} \)

- We want to:
  1. find the Chebyshev equivalent circuit seen from the terminals of the non-linear device.
  2. Use this to plot the load line for the circuit.
  3. Find the operating point \((i_N, v_N)\) for the non-linear device for the given numbers.
  4. Find a small-signal model for the non-linear device for small perturbations in \( i_N \) around \( I_0 \).

(a) To find the Chebyshev circuit

(i) \( R_N \) is found by setting \( I_0 \to 0 \Rightarrow \) open circuit

\[
\begin{align*}
R_1 &\text{ (in parallel)} \\
R_2 &\text{ (in parallel)} \\
R_3 &\text{ (in series)}
\end{align*}
\]

\( R_{th} = R_2 + R_3 \).

(ii) \( V_{th} \) is the open circuit voltage at the device terminals.
\[ V_{th} = V_{oc} = Vr_2 = I_0 R_2. \]

So we can redraw the circuit with the Chevénin equivalent rows:

\[ V_{th} = I_0 R_2 \]

\[ I_{sc} = \frac{V_{th}}{R_{th}} = \frac{150\text{mV}}{500\text{mV}} = 30\text{mA} \]

where \( R_{th} = R_2 + R_3 \).

(b) Plotting the load line:

We can also write the equation of the load line:

\[ i_n = 0.03 - 0.2 u_n. \]

(c) Superimposing the load line on a plot of \( i_n = f(u_n) = I_s e^{u_n/V_t} \), we can find the operating point \((u_n, i_n)\).
So we are drawing a tangent to the curve at \( V_N = V_N \) and finding the slope:

\[
\text{Slope} = \left. \frac{\partial f}{\partial V_N} \right|_{V_N = V_N} = \frac{V_n}{I_N}
\]

For our example:

\[
i_N = I_s \frac{e^{V_n/V_T}}{V_T}
\]

\[
\Rightarrow i_N \approx \left. \frac{\partial}{\partial V_N} \left[ I_s \frac{e^{V_n/V_T}}{V_T} \right] \right|_{V_N = V_N} \cdot V_N
\]

\[
\approx \left. \frac{I_s}{V_T} \frac{e^{V_n/V_T}}{V_T} \right|_{V_N = V_N} \cdot V_N
\]

\[
i_N \approx \left. \frac{10 \text{mA}}{25 \text{mV}} \right| e^{25 \text{mV}/25 \text{mV}} \cdot V_N
\]

\[
i_N \approx 1.087 \ V_N \quad \text{... Linear function.}
\]
By inspection (since it's difficult to solve analytically), we can say

\[ V_N \approx 25 \text{ mV} \]
\[ I_N \approx 25 \text{ mA} \]

(d) Small Signal Model.

Now that we know where \( i_N \) will "sit", we can see how it will change with small variations in \( V_N \).

- Remember:
  \[ i_N = I_N + i_n \]
  \[ V_N = V_n + V_n \]

DC/Bias \quad \text{SIGNAL}

- From the Taylor Series:
  \[ i_N = f(V_N) \quad \text{non-linear function} \]
  \[ \Rightarrow i_N = f(V_N) + \frac{df}{du} \bigg|_{V_N} (V_N - V_N) + \ldots \]
  \[ = I_N + \frac{df}{du} \bigg|_{V_N} (V_N - V_N) \]

\[ i_N = I_N + \frac{df}{du} \bigg|_{V_N} (V_N - V_N) \]

\[ i_N - I_N = \left[ \frac{df}{du} \bigg|_{V_N} \right] V_N \]

\[ i_N - I_N = \left[ \frac{df}{du} \bigg|_{V_N} \right] V_N \]
First Order Circuit:

1. We have 3 elemental building blocks:

\[ V_R = i_R R \]
\[ V_L = L \frac{di_L}{dt} \]
\[ i_C = C \frac{dv_C}{dt} \]

2. In the absence of impulses in the system:

\[ i_L(0^-) = i_L(0^+) \quad \text{... } i_L \text{ is continuous} \]
\[ v_C(0^-) = v_C(0^+) \quad \text{... } v_C = 0 \]

3. In steady state conditions:

- \( L \rightarrow \) Short circuit
- \( C \rightarrow \) Open circuit

Our plan of attack is the following:

(a) Find a differential equation in terms of the state variable for the circuit.

- For circuits with \( C \), this is \( v_C \)
- For circuits with \( L \), this is \( i_L \)

(b) Read off \( x \) from the O.D.E.:

\[ C \frac{dx}{dt} + x = A i(t) \quad \text{where } x = x(t) = \text{state variable} \]

Note: \( x \) can also be found from inspection where

\[ x = \begin{cases} \frac{L}{R_{eff}} & \text{for inductor circuits} \\ \frac{R_{eff} C}{C} & \text{capacitor} \end{cases} \]

\( R_{eff} \) is the Thevenin / Norton equivalent resistance seen from the terminals of the \( L \) or \( C \).
(c) Write \( x(t) = P + \frac{V_{0}e^{-t/\tau}}{\tau} \) for the state variable.

Particular solution

(a) Solve \( P \) and \( \tau \) from initial and final conditions.

Example 1:

\[ U_{C}(t) = V_{1}u(t) \quad \text{(Step function)} \]

and \( U_{C}(t=0^-) = 0 \), find \( U_{C}(t) \) and \( i(t) \) for \( t \to 0 \).

(a) \( U_{1}(t) = iR + U_{C} \)

\[ V_{1} = RC \frac{dU_{C}}{dt} + U_{C} \quad \text{... for} \quad t \to 0. \]

(b) \( \Rightarrow \tau = RC \) ... could also have read off from \( \tau = \text{RafC} \)

(c) \( U_{1}(t) = P + V_{0}e^{-t/\tau} \) where \( \tau = RC \).

(d)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( U_{C}(t) )</th>
<th>( i(t) )</th>
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<tbody>
<tr>
<td>( t = 0^- )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( t = 0^+ )</td>
<td>0</td>
<td>( V_{i}/R )</td>
</tr>
<tr>
<td>( t \to \infty )</td>
<td>( V_{1} )</td>
<td>0</td>
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\[ U_{C}(t) = V_{1}(1 - e^{-t/\tau}) \]

\[ i(t) = \frac{V_{i}}{R} e^{-t/\tau} \]
Example 2:

\[ i_{L}(t) = I_{0}, \quad u(t) = 0 \quad \text{and} \quad i_{L}(0^{-}) = I_{0} \]

Find \( i_{L}(t) \)

(a) KCL: \[ i_{L}(t) = i_{R} + i_{L} \]
[\[ i_{L} = \frac{V}{R} + i_{L} \]
[\[ i_{L} = \frac{L}{R} \frac{di_{L}}{dt} + i_{L} \]

(b) \[ \Rightarrow \xi = \frac{L}{R} \]

(c) \[ i_{L}(t) = P + Ke^{-t/\xi} \]

(d)

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<td>( -I_{0}R )</td>
</tr>
<tr>
<td>( t=0^+ )</td>
<td>( I_{0} )</td>
<td>( (I_{1}-I_{0})R )</td>
</tr>
<tr>
<td>( t=\infty )</td>
<td>( I_{1} )</td>
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Continuous, "short circuit"

Solving for \( P, K \):

\[ i_{L}(t=0^+) = I_{0} = P + K \]
\[ i_{L}(t=\infty) = I_{1} = P \]

\[ \Rightarrow K = I_{0} - P = I_{0} - I_{1} \]

\[ i_{L}(t) = I_{1} + (I_{0} - I_{1}) e^{-t/\xi} \]

where \( \xi = \frac{L}{R} \).
We can plot $v(t)$ from intuition (or if not, just remember $V = V_L = L \frac{di_L}{dt}$)

\[ v(t) \]

\[ (I_0 - I_i)R \]

\[ -I_0R \]

\[ t \]

\[ z \]

\[ u(t) = -I_0R. \]

... $v(t)$ is discontinuous.
Example 3.

Consider the circuit on the left. For $t < 0$, the switch is open so $i_1(t < 0) = 0$. At $t = 0$, the switch closes. Plot $i_1(t)$ and $u_L(t)$ for $t \geq 0$.

(a) Let's redraw the circuit after the switch closes:

$bV$: (for loop with $L$):

\[
V_S = \frac{1}{R_1} \frac{d}{dt} i_1, \]  
\[
\frac{V_S}{R_1} = i_1 + \frac{L}{R_1} \frac{d}{dt} i_1, \]

$\Rightarrow \frac{V}{R_1} = \frac{1}{R_1}$

(b) $i_1(t) = P + V e^{-t/R_1}$

(c)

$\begin{array}{c|c|c}
 t = 0^- & i_1(t) & V_S(t) \\
 t = 0^+ & 0 & 0 \\
 t \rightarrow \infty & V_S/R_1 & 0 \\
\end{array}$

Voltage bias to appear across $L$ since $u_{R_1} = 0$

$L$ is a short circuit

$\begin{array}{cc}
i_1(t) = \frac{V_S}{R_1} \left(1-e^{-t/R_1}\right) & U_L(t) = V_S e^{-t/R_1}
\end{array}$
at some time $t = T$, the switch opens. Consider $T < 4\tau_1$ so that the system is not in steady state yet. Plot $i_1(t)$, $v_2(t)$ and $v_{sw}(t)$ for all time.

(a) Reduce the circuit when the switch opens:

**KCL at bottom node:**

$$i_1 + i_2 = 0$$

$$\Rightarrow i_1 = -i_2$$

**KVL for right-hand loop:**

$$i_1R_1 + L \frac{di_1}{dt} - i_2R = 0$$

$$i_1R_1 + i_1R_2 + L \frac{di_1}{dt} = 0$$

$$i_1(R_1 + R_2) + L \frac{di_1}{dt} = 0$$

$$i_1 + \frac{L}{R_1 + R_2} \frac{di_1}{dt} = 0$$

$$\Rightarrow \zeta_2 = \frac{L}{R_1 + R_2}$$

(b) $i_1(t) = \rho + ke^{-t/\zeta_2}$

$$\begin{array}{c|c|c|c}
 t = T^- & i_1(t) & v_2(t) & v_{sw}(t) \\
 I_1 & V_1 & 0 & \text{Switch is closed} \\
 t = T^+ & I_1 & -I_1(R_1 + R_2) & V_s + I_1R_2 \\
 t \to \infty & 0 & 0 & V_s \\
\end{array}$$

$$i_1(t = T^-) = \frac{V_s}{R_1} (1 - e^{-T/\tau_1}) \triangleq I_1$$

$$\Rightarrow i(t = T^+) = I_1 \quad \text{... continuous current}$$
Similarly \( v_L(t = T^-) = V_S e^{-t/\tau_2} \triangleq V_1 \)

For \( v_W(t = T^+) = V_S - i_2(t = T^+) R_2 \)
\[ = V_S - i_1(t = T^+) R_2 \]
\[ = V_S + I_1 R_2. \]

And \( v_L(t = T^+) = V_S - i_1(t = T^+) R_1 - v_W(t = T^+) \)
\[ = V_S - I_1 R_1 - (V_S + I_1 R_2) \]
\[ = -I_1 (R_1 + R_2) \]

Now for steady state:
\( i_L(t \to \infty) = 0 \) ... energy stored in \( L \) at \( t = T \)

\( u_L(t \to \infty) = 0 \) ... inductor is a short

\( u_W(t \to \infty) = V_S \)