Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
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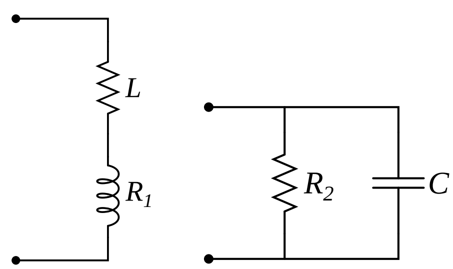
Homework #10 Solutions

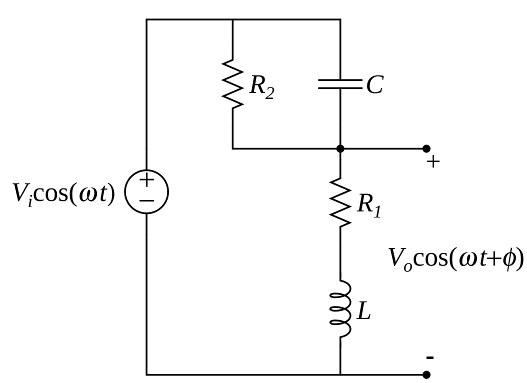
Exercise 10.1: Determine the impedance of each network shown below. Also, identify the asymptotic dependence of the impedances on frequency for very low frequencies and for very high frequencies, and explain the dependences physically.



Answer: For the first network, the two elements are in series, so their impedances combine like series resistors to be $Z_1 = j\omega L + R_1$. As $\omega \to 0$, the impedance of the inductor approaches zero, so the impedance is just R_1. The impedance of the inductor is much greater than that of the resistor when $\omega L \gg R_1$, so at high frequencies we can neglect the resistor, and the impedance is $j\omega L$.

In the second network, the two elements are in parallel to yield a total impedance of $Z_2 = \frac{R_2}{1+j\omega R_2 C}$. At low frequencies, the admittance of the capacitor is very low, so the total impedance approaches R_2. As frequency increases, the impedance of the capacitor becomes very small compared to that of the resistor, so the total impedance approaches $\frac{1}{j\omega C}$.

Exercise 10.2: Assume that the network shown below is operating in sinusoidal steady state. Determine the amplitude V_o and phase ϕ of the voltage across the series inductor and resistor. Hint: see Exercise 10.1.



Answer: The circuit is operating sinusoidal steady-state, so we can use complex amplitudes and impedances to determine V_o and ϕ. For the purposes of this problem, we will use "squiggle" notation for complex amplitudes. That is to say, $\tilde{V_i} = V_i$ and $\tilde{V_O} = V_o e^{j\omega}$.

From Exercise 10.1 above we can write

$$\begin{aligned}
\frac{\tilde{V_o}}{\tilde{V_i}} &= \frac{Z_1}{Z_1 + Z_2} = \frac{j\omega L + R_1}{j\omega L + R_1 + \frac{R_2}{1+j\omega R_2 C}} \\
&= \frac{j\omega L + R_1 - \omega^2 L R_2 C + j\omega R_1 R_2 C}{j\omega L + R_1 - \omega^2 L R_2 C + j\omega R_1 R_2 C + R_2} \\
\frac{\tilde{V_o}}{\tilde{V_i}} &= \frac{R_1 - \omega^2 L R_2 C + j\omega(L + R_1 R_2 C)}{R_1 + R_2 - \omega^2 L R_2 C + j\omega(L + R_1 R_2 C)} \\
&= \frac{\frac{R_1}{R_2} - \omega^2 LC + j\omega\left(\frac{L}{R_2} + R_1 C\right)}{1 + \frac{R_1}{R_2} - \omega^2 LC + j\omega\left(\frac{L}{R_2} + R_1 C\right)}
\end{aligned}$$

where all terms are unitless.

$$\begin{aligned}
V_o &= V_i \cdot \left|\frac{\tilde{V_o}}{\tilde{V_i}}\right| = \sqrt{\frac{(R_1 - \omega^2 L R_2 C)^2 + \omega^2 (L + R_1 R_2 C)^2}{(R_1 + R_2 - \omega^2 L R_2 C)^2 + \omega^2 (L + R_1 R_2 C)^2}} \cdot V_I \\
\phi &= \angle\frac{\tilde{V_o}}{\tilde{V_i}} = \tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_1 - \omega^2 L R_2 C}\right) - \tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_1 + R_2 - \omega^2 L R_2 C}\right)
\end{aligned}$$

Alternatively, we could rewrite ϕ as one tangent term by either using the identity

$$\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$$

or by removing the imaginary part of the denominator of $\frac{\tilde{V_o}}{\tilde{V_i}}$ by multiplying by the denominator's complex conjugate. This gives:

$$\phi = \tan^{-1}\left(\frac{\omega R_2 (L + R_1 R_2 C)}{(R_1 - \omega^2 L R_2 C)(R_1 + R_2 - \omega^2 L R_2 C) + \omega^2 (L + R_1 R_2 C)^2}\right)$$

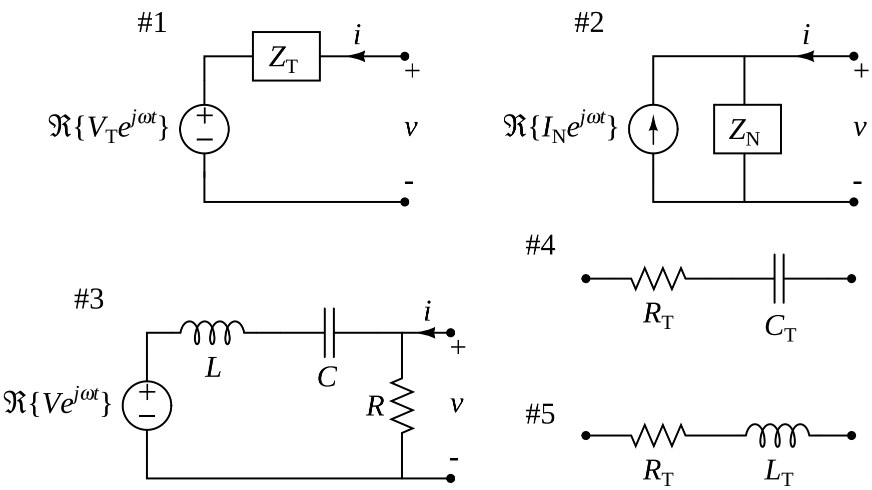
Problem 10.1: This problem explores the Thevenin and Norton equivalence of networks operating in sinusoidal steady state. All networks considered here are comprised of linear resistors, capacitors and inductors, and voltage and current sources all operating at the same frequency ω. Therefore, all branch currents and voltages operate at the frequency ω.

(A) Determine the relations between V_T, Z_T, I_N and Z_N which must exist for the *i*-*v* relations at the terminals of Networks #1 and #2 to be identical when operating in sinusoidal steady state.

(B) Review the arguments for networks involving only resistors and sources, and then briefly explain why Networks #1 and #2 may serve as the Thevenin and Norton equivalents, respectively, of an arbitrary network operating in sinusoidal steady state.

(C) Determine $V_{\rm T}$ and $Z_{\rm T}$ in the Thevenin equivalent of Network #3.

(D) Suppose $Z_{\rm T}$ in Part C is implemented with Networks #4 and #5. Determine $R_{\rm T}$ and $C_{\rm T}$ in Network #4, and $R_{\rm T}$ and $L_{\rm T}$ in Network #5, in terms of R, L, C and ω. Under what circumstances is Network #4 preferred over Network #5, and vice versa?



Answer: Since we know that $Z_{\rm T}$ and $Z_{\rm N}$ are complex, we can rewrite them in polar form:

$$\begin{aligned} Z_{\rm T} &= |Z_{\rm T}|e^{j\phi_{Z_{\rm T}}} \\ Z_{\rm N} &= |Z_{\rm N}|e^{j\phi_{Z_{\rm N}}} \end{aligned}$$

(A) We can determine the relations between $V_{\rm T}$, $Z_{\rm T}$, $I_{\rm N}$ and $Z_{\rm N}$ by applying a complex test voltage, $v(t) = \tilde{v}e^{j\omega t}$, into the ports of both #1 and #2, and determining the complex current $i(t) = \tilde{i}e^{j\omega t}$ flowing into the positive terminal. For #1 we get:

$$\tilde{i}e^{j\omega t} = \frac{\tilde{v}e^{j\omega t} - V_{\rm T}e^{j\omega t}}{|Z_{\rm T}|e^{j\phi_{Z_{\rm T}}}}$$

Dividing both sides by $e^{j\omega t}$, we get:

$$\tilde{i} = \frac{\tilde{v}}{|Z_{\rm T}|e^{j\phi_{Z_{\rm T}}}} - \frac{V_{\rm T}}{|Z_{\rm T}|e^{j\phi_{Z_{\rm T}}}}$$

For #2 we get:

$$\tilde{i}e^{j\omega t} = \frac{\tilde{v}e^{j\omega t}}{|Z_{\rm N}|e^{j\phi_{Z_{\rm N}}}} - I_{\rm N}e^{j\omega t}$$

which gives us:

$$\tilde{i} = \frac{\tilde{v}}{|Z_{\rm N}|e^{j\phi_{Z_{\rm N}}}} - I_{\rm N}$$

By observing that the coefficient in front of both $\tilde{v}$ terms must be equal, we obtain a relationship between $Z_{\rm N}$ and $Z_{\rm T}$:

$$Z_{\rm T} = |Z_{\rm T}|e^{j\phi_{Z_{\rm T}}} = |Z_{\rm N}|e^{j\phi_{Z_{\rm N}}} = Z_{\rm N}$$

or simply

$$Z_{\rm T} = Z_{\rm N}$$

since $|Z_{\mathrm{T}}| = |Z_{\mathrm{N}}|$ and $\phi_{Z_{\mathrm{T}}} = \phi_{Z_{\mathrm{N}}}$. Also, the right-hand most terms must be equivalent for the i-v relationship to be the same:

$$I_{\mathrm{N}} = \frac{V_{\mathrm{T}}}{|Z_{\mathrm{T}}|e^{j\phi_{Z_{\mathrm{T}}}}} = \frac{V_{\mathrm{T}}}{Z_{\mathrm{T}}}$$

This result could be shown to hold for real sinusoidal inputs by taking the real part of the resulting equations, which would result in the same analysis. We can note that Thevenin and Norton equivalences can be used when dealing with impedances.

(B) We note from out above analysis that the two circuits can be used as Thevenin and Norton equivalents when the inputs are sinusoidal because we see that for any arbitrary sinusoidal voltage of characteristic frequency ω and phase ϕ_{V}, there is a direct mapping to a specific sinusoidal current of the same frequency with a specific phase. If the same voltage were applied to both output terminals of the two circuits, then the same current (including phase and magnitude) would be observed. Thus, if either circuit were hooked up to an arbitrary network, they would function exactly the same and thus be electronically interchangeable (i.e. equivalent).

(C) Using our impedance transformations, we can determine a Thevenin equivalent circuit for #3. First, the Thevenin Impedance, Z_{T}, can be found by shorting out (setting to 0) the voltage source on the left-hand side and finding the total equivalent resistance as seen from the two ports:

$$Z_{\mathrm{T}} = R || \left(j\omega L + \frac{1}{j\omega C} \right) = \frac{R - \omega^2 RLC}{1 - \omega^2 LC + j\omega RC}$$

The Thevenin voltage can be found by assuming no current flows into the positive terminal ($i = 0$) and measuring the terminal voltage. This becomes a voltage divider:

$$V_{\mathrm{T}} = V \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

Note that, in particular, V_{T} is complex, and therefore has a phase shift.

(D) First, we put Z_{T} into its complex cartesian form by realifying the denominator:

$$\begin{aligned} Z_{\mathrm{T}} &= \frac{R - \omega^2 RLC}{1 - \omega^2 LC + j\omega RC} \cdot \left(\frac{1 - \omega^2 LC - j\omega RC}{1 - \omega^2 LC - j\omega RC} \right) \\ Z_{\mathrm{T}} &= \frac{R(1 - \omega^2 LC)^2}{(1 - \omega^2 LC)^2 + (\omega RC)^2} - j \frac{\omega RC(R - \omega^2 RLC)}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \end{aligned}$$

If we wish for #4 to be equivalent, then its impedance must equal that of Z_{T} for #3:

$$R_{\mathrm{T}} + \frac{1}{j\omega C_{\mathrm{T}}} = R_{\mathrm{T}} - j\frac{1}{\omega C_{\mathrm{T}}} = Z_{\mathrm{T}}$$

The real parts must be equivalent, as well as the imaginary parts.

$$\begin{aligned} R_{\mathrm{T}} &= \frac{R(1 - \omega^2 LC)^2}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \\ C_{\mathrm{T}} &= \frac{(1 - \omega^2 LC)^2 + (\omega RC)^2}{\omega^2 R^2 C(\omega^2 LC - 1)} \end{aligned}$$

The same goes with #5:

$$R_{\mathrm{T}} + j\omega L = Z_{\mathrm{T}}$$

Again,

$$\begin{aligned} R_{\mathrm{T}} &= \frac{R(1-\omega^2 LC)^2}{(1-\omega^2 LC)^2 + (\omega RC)^2} \\ L_{\mathrm{T}} &= \frac{R^2 C(1-\omega^2 LC)}{(1-\omega^2 LC)^2 + (\omega RC)^2} \end{aligned}$$

Depending on whether the value ω is less than for greater than $\frac{1}{\sqrt{LC}}$, L_{T} or C_{T} will be a positive value. Since capacitors and inductors always have positive, real values for C and L, we would implement the Network that would give us positive values. So, for $\omega < \frac{1}{\sqrt{LC}}$, we would implement Network #5. For $\omega > \frac{1}{\sqrt{LC}}$, we would implement Network #4.

Problem 10.2: This problem examines the very simple tuner for an AM radio shown below. Here, the tuner is the parallel inductor and capacitor. The injection of radio signals into the tuner by the antenna is modeled by a current source, while the Norton resistance of the antenna in parallel with the remainder of the radio is modeled by a resistor. (You will learn about antenna modeling in 6.014.) The AM radio band extends from 540 kHz through 1600 kHz. The information transmitted by each radio station is constrained to be within ±5 kHz of its center frequency. (You will learn about AM radio transmission in 6.003.) To prevent frequency overlap of neighboring stations, the center frequency of each station is constrained to be a multiple of 10 kHz. Therefore, the purpose of the tuner is to pass all frequencies within 5 kHz of the center frequency of the selected station, while attenuating all other frequencies.

(A) Assume that $I(t) = I\cos(\omega t)$. Find $v(t)$ where $v(t) = V\cos(\omega t + \phi)$, and both V and ϕ are functions of ω. Note that $v(t)$ is the output of the tuner, namely the signal that is passed on to the remainder of the radio.

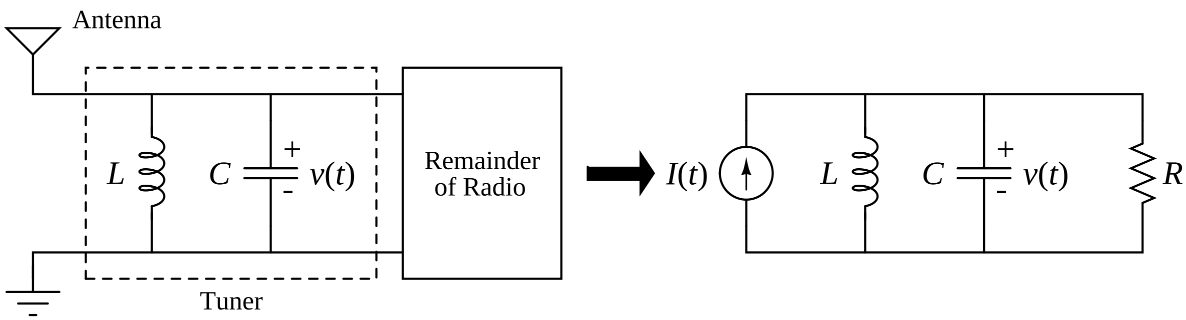
(B) For a given combination of I, C, L and R, at what frequency is V maximized?

(C) Assume that $L = 365\ \mu$H. Over what range of capacitance must C vary so that the frequency of maximum V/I may be tuned over the entire AM band. Note that tuning the frequency of maximum V/I to the center frequency of a particular station tunes in that station.

(D) As a compromise between passing all frequencies within 5 kHz of a center frequency and rejecting all frequencies outside that band, let the design of R be such that $V(1\text{ MHz} \pm 5\text{ kHz})/V(1\text{ MHz}) \approx 0.25$ when the tuner is tuned to 1 MHz. Given this design criterion, determine R.

(E) Given your design for R, determine $V(1\text{ MHz} \pm 10\text{kHz})/V(1\text{ MHz})$. Also, determine Q for the tuner and its load resistor when the tuner is tuned to 1 MHz.

(F) Suppose the tuner is tuned to another station and then quickly tuned to the station broadcasting at 1 MHz. Approximately how long will it take for $v(t)$ to depend primarily on the signal from the station broadcasting at 1 MHz. Assume that both stations broadcast signals of equal strength. Hint: consider the time-domain interpretation of Q.



Answer:

(A) Using the impedance transformations, we can write

$$\tilde{V} = \tilde{I} \cdot Z_{\text{eq}} = \tilde{I} \cdot \frac{1}{Y_{\text{eq}}}$$

Noting that admittances in parallel add, we find

$$\begin{aligned}
\tilde{V} &= \frac{\tilde{I}}{\frac{1}{j\omega L} + j\omega C + \frac{1}{R}} \\
&= \frac{j\omega L}{j^2\omega^2 LC + j\omega\frac{L}{R} + 1} \cdot I \\
&= \frac{j\omega L}{1 - \omega^2 LC + j\omega\frac{L}{R}} \cdot I = Ve^{j\phi}
\end{aligned}$$

Taking the magnitude and the phase of this complex expression for $\tilde{V}$, we find

$$\begin{aligned}
V &= I \cdot \frac{\omega L}{\sqrt{(1-\omega^2 LC)^2 + \left(\omega\frac{L}{R}\right)^2}} \\
\phi &= \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega\frac{L}{R}}{1-\omega^2 LC}\right) = \tan^{-1}\left(\frac{1-\omega^2 LC}{\omega\frac{L}{R}}\right) \\
v(t) &= V\cos(\omega t + \phi)
\end{aligned}$$

(B) For the purposes of this section, let

$$X = (1-\omega^2 LC)^2 + \left(\frac{\omega L}{R}\right)^2$$

To find ω when V is maximized, we have to find ω such that $\frac{dV}{d\omega} = 0$. Looking at the numerator of the messy derivative, we see that

$$\begin{aligned}
0 &= IL\sqrt{X} - \frac{I\omega L}{2\sqrt{X}}\left[2(1-\omega^2 LC)^2(-2\omega LC) + 2\omega\left(\frac{L}{R}\right)^2\right] \\
0 &= X - \omega\left[-2\omega LC(1-\omega^2 LC) + \omega\left(\frac{L}{R}\right)^2\right] \\
0 &= (1-\omega^2 LC)^2 + \omega^2\frac{L^2}{R} + 2\omega^2 LC - 2\omega^4 L^2C^2 - \omega^2\frac{L^2}{R}
\end{aligned}$$

Solving this polynomial for ω we find

$$\omega_{\max(\mathrm{V})} = \omega_{\mathrm{N}} = \frac{1}{\sqrt{LC}}$$

(C) To operate over the entire AM band, 540 kHz $\leq \frac{\omega_{\mathrm{N}}}{2\pi} \leq$ 1600 kHz, we can write

$$540 \times 10^3 \leq \frac{1}{2\pi\sqrt{LC}} \leq 1600 \times 10^3$$

Rearranging terms to find C yields

$$\frac{1}{L(2\pi(540 \times 10^3))^2} \geq C \geq \frac{1}{L(2\pi(1600 \times 10^3))^2}$$

Substituting in $L = 365\ \mu$H gives

$$238 \text{ pF} \geq C \geq 27.1 \text{ pF}$$

(D) First, find the amplitude at the center frequency 1 MHz using Parts (A) and (B).

$$@\omega = \omega_{\mathrm{N}} = \frac{1}{\sqrt{LC}}, \quad |V| = IR$$

The frequency 5 kHz away can be represented as a fraction γ of the center frequency.

$$\begin{aligned}
\omega &= \gamma\omega_{\mathrm{N}} = \frac{\gamma}{\sqrt{LC}} \\
V(\gamma\omega_{\mathrm{N}}) &= \frac{I\gamma\frac{L}{\sqrt{LC}}}{\sqrt{(1-\gamma^2)^2 + \frac{\gamma^2 L^2}{LCR^2}}} = \frac{IL}{\sqrt{LC\left(\frac{1}{\gamma}-\gamma\right)^2 + \left(\frac{L}{R}\right)^2}} \\
\frac{V(\gamma\omega_{\mathrm{N}})}{V(\omega_{\mathrm{N}})} &= \frac{L}{\sqrt{R^2LC\left(\frac{1}{\gamma}-\gamma\right)^2 + L^2}} \equiv \beta \\
\frac{L^2}{\beta^2} &= R^2LC\left(\frac{1}{\gamma}-\gamma\right)^2 + L^2 \\
\frac{L(1-\beta^2)}{\beta^2} &= R^2C\left(\frac{1}{\gamma}-\gamma\right)^2 \\
\Rightarrow R &= \sqrt{\frac{L(1-\beta^2)}{\beta^2 C\left(\frac{1}{\gamma}-\gamma\right)^2}}
\end{aligned}$$

Substituting $L = 365\ \mu$H, $\beta = .25$, $C = \frac{1}{L(2\pi\times 10^6)} = 69.4$ pF, and $\gamma = .995$ gives

$$R = 886 \text{ k}\Omega$$

using $\gamma = 1.005$ gives

$$R = 890 \text{ k}\Omega$$

We will choose the case where $\frac{V(\gamma\omega_{\mathrm{N}})}{V(\omega_{\mathrm{N}})} > .25$. Since $\omega = .995$ MHz is a tighter constraint on R:

$$R = 886 \text{ k}\Omega$$

(E) $V(1\text{ MHz} \pm 10\text{ kHz})$ is given by

$$\frac{V(\gamma\omega_N)}{V(\omega_N)} = \frac{L}{\sqrt{R^2 LC\left(\frac{1}{\gamma} - \gamma\right)^2 + L^2}} \quad \text{where} \quad \gamma = .99$$

Therefore

$$\frac{V(.99\omega_N)}{V(\omega_N)} = .128$$

Finding Q is quick:

$$Q\frac{\omega_N}{2\alpha} = \frac{RC}{\sqrt{LC}} = R\sqrt{\frac{C}{L}} = 386$$

(F) Our circuit will have a transient that will decay away as $e^{-\alpha t}$ as soon as we switch stations. The time constant is $\frac{1}{\alpha} = 2RC = 123\ \mu$s.

There are many reasonable engineering approximations for when a transient has mostly died away. Note that after three time constants, the transient has decayed to $e^{-3} \approx .05 = 5\%$ of its initial value. $3 \cdot 2RC = 369\ \mu$s. You could also have used a different number of time constants as your reference.

Problem 10.3: In this problem, a low-voltage sinusoidal source is coupled to a resistive load through an inductor-capacitor network as shown below. The role of the network is to boost the voltage at the load.

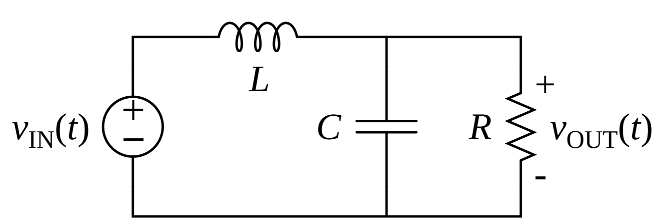
(A) Derive the differential equation that relates $v_{OUT}(t)$ to $v_{IN}(t)$.

(B) Assume that the circuit operates in sinusoidal steady state with $v_{IN}(t) = V_I\cos(\omega t)$. Let $v_{OUT}(t)$ take the form $v_{OUT} = V_O\cos(\omega t + \phi)$. Determine $N \equiv V_O/V_I$ and ϕ.

(C) For a given L and ω, determine the value of C which maximizes N, and for this value of C, determine N.

(D) Suppose that $v_{IN}(t)$ is abruptly set to zero in an attempt to remove the voltage at the load. In this case, the amplitude of the load voltage will decay in proportion to $e^{-t/\tau}$. Assuming that C is chosen to maximize N following the result from Part (C), determine τ in terms of N and ω.

(E) In view of the results of Parts (C) and (D), what is the disadvantage of using an inductor-capacitor network to boost the voltage which excites the load?



Answer:

(A) We begin by writing KVL around the loop to yield

$$v_{\mathrm{IN}}(t) = v_L(t) + v_{\mathrm{OUT}}(t)$$

Substituting the inductor's constituent relationship, gives us

$$v_{\mathrm{IN}}(t) = L\frac{di_{\mathrm{L}}(t)}{dt} + v_{\mathrm{OUT}}(t).$$

Using KCL, Ohm's law and the constituent relationship for the capacitor we can write,

$$i_{\mathrm{L}}(t) = i_{\mathrm{C}}(t) + i_{\mathrm{R}}(t) = C\frac{dv_{\mathrm{OUT}}(t)}{dt} + \frac{v_{\mathrm{OUT}}(t)}{R}.$$

Substituting this result into (2) gives the following:

$$v_{\mathrm{IN}}(t) = L\frac{d}{dt}\left[C\frac{dv_{\mathrm{OUT}}(t)}{dt} + \frac{v_{\mathrm{OUT}}(t)}{R}\right] + v_{\mathrm{OUT}}(t)$$

Finally, if we distribute the derivative and normalize the highest order derivative we have our answer:

$$\frac{v_{\mathrm{IN}}(t)}{LC} = \frac{d^2v_{\mathrm{OUT}}(t)}{dt^2} + \frac{1}{RC}\frac{dv_{\mathrm{OUT}}(t)}{dt} + \frac{v_{\mathrm{OUT}}(t)}{LC}.$$

(B) This problem is readily handled using impedance techniques, however since we already have the differential equation it is faster to assume complex exponential inputs and solve from there. Let the input and output be complex exponentials of the following form:

$$\tilde{V_{\mathrm{I}}}e^{j\omega t} \quad \text{and} \quad \tilde{V_{\mathrm{O}}}e^{j\omega t}.$$

Please note the usage of the tilde, this denotes that the variable (amplitude) is complex in general. Furthermore we recognize that

$$v_{\mathrm{IN}}(t) = \Re\{\tilde{V_{\mathrm{I}}}e^{j\omega t}\} \quad \text{and} \quad v_{\mathrm{OUT}}(t) = \Re\{\tilde{V_{\mathrm{O}}}e^{j\omega t}\}$$

since we are interested in a cosinusoidal input. Now we are ready to begin by substituting the complex inputs into the differential equation, this gives the following:

$$\frac{\tilde{V_{\mathrm{I}}}e^{jwt}}{LC} = -w^2\tilde{V_{\mathrm{O}}}e^{jwt} + \frac{jw\tilde{V_{\mathrm{O}}}e^{jwt}}{RC} + \frac{\tilde{V_{\mathrm{O}}}e^{jwt}}{LC}.$$

We can simplify this expression by canceling out all of the complex exponentials and rewriting the equation as the ratio of $\tilde{V_{\mathrm{O}}}/\tilde{V_{\mathrm{I}}}$ as follows,

$$\frac{\tilde{V_{\mathrm{O}}}}{\tilde{V_{\mathrm{I}}}} = \frac{1}{-w^2LC + \frac{jwL}{R} + 1}.$$

When the question asks for N, what they mean is $|\tilde{V_{\mathrm{O}}}/\tilde{V_{\mathrm{I}}}|$ or the magnitude of the complex ratio (transfer function) and when they ask for ϕ they simply mean $\angle(\tilde{V_{\mathrm{O}}}/\tilde{V_{\mathrm{I}}})$ or the angle of the complex ratio (transfer function). The magnitude is just the $|Num|/|Den|$ hence we can write,

$$\frac{V_{\mathrm{O}}}{V_{\mathrm{I}}} = \left|\frac{\tilde{V_{\mathrm{O}}}}{\tilde{V_{\mathrm{I}}}}\right| = \frac{1}{\sqrt{(1-w^2LC)^2 + (\frac{wL}{R})^2}}$$

While the phase is just $\angle(Num) - \angle(Den)$ which leads us to the following,

$$\phi = -\tan^{-1}\left(\frac{wL}{R - w^2RLC}\right)$$

(C) Looking at the expression for the output magnitude we found in Part (B), we realize we can maximize the magnitude by minimizing the denominator. Only one term depends on C and it's squared. Thus the expression is maximized with respect to C when the quantity $(1 - \omega^2 LC)$ is equal to 0. This occurs when

$$C = \frac{1}{\omega^2 L}.$$

For this value of C, we find that the output-to-input amplitude ratio is

$$N = \left.\frac{V_O}{V_I}\right|_{C=1/(\omega^2 L)} = \frac{R}{\omega L}.$$

(D) We know that the system is second order and will have a homogeneous response that is essentially a decaying sinusoid. We also know that the rate of decay goes as $e^{-\alpha t}$, where α is given by the characteristic equation $((j\omega)^2 + 2\alpha j\omega + \omega_o^2)$ of (5) and is

$$\alpha = \frac{1}{\tau} = \frac{1}{2RC}$$

Solving for τ and rewriting in terms of N yields,

$$\tau = \frac{2N}{\omega}$$

(E) From our expression for τ we see that for a given ω increasing N and hence the level of "boosting" will result in a larger time constant, τ. As a result if we wish to boost the voltage using this method, and decide to remove the voltage at the load by turning off the input, we must wait considerably longer while the transient dies away. In the limit that $R \to \infty$ we find that $N \to \infty$ but $\tau \to 0$, thus we get infinite boosting, but the circuit will ring forever and will never decay to zero even after we set the source to zero. Essentially we have an energized LC circuit with no resistance to dissipate the energy.