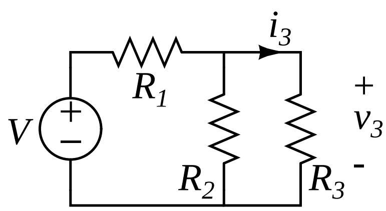
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

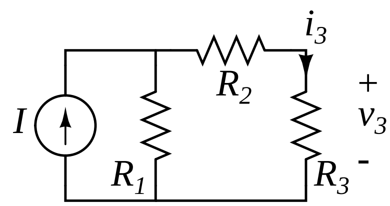
6.002 – Electronic Circuits
Spring 2002

Homework #2 Solutions

Exercise 2.1: For both networks shown below, determine the voltage v_3 across, and the current i_3 through, R_3.

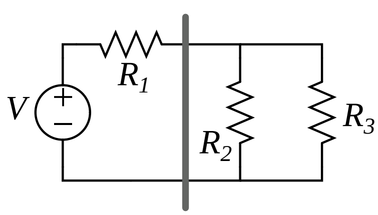


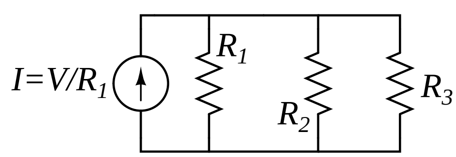
Network (a)



Network (b)

Answer: These networks show the usefulness of transforming between Thevenin and Norton equivalent circuits. In both networks, we can cut the circuit in half, so that the source and R_1 are on one side, while R_2 and R_3 are on the other. We then take the source and R_1 and convert them to their other form. The other side of the circuit (and this is the heart of Thevenin and Norton equivalents) will not know the difference.

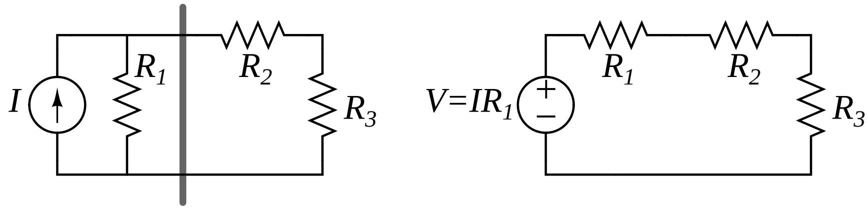




(A) We cut the circuit as shown above, left. Since we know the relation $V_{Th} = R_{eq} I_N$, we can compute the Norton equivalent circuit. Note that the resistance associated with the Norton equivalent equals that of the Thevenin resistance. The new circuit is shown above, right. The voltage across and current through R_3 is then

$$V_{R_3} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} I = \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} V$$

$$I_{R_3} = \frac{V_{R_3}}{R_3} = \frac{R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3} V$$



(B) Again, we cut the circuit as shown. This time, we convert to a Thevenin equivalent. The three resistors are now in series, as shown above, right. The voltage across and current through R_3 is then

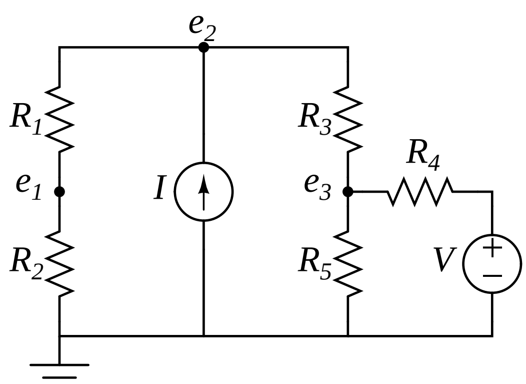
$$I_{R_3} = \frac{V}{R_1 + R_2 + R_3} = \frac{R_1 I}{R_1 + R_2 + R_3}$$

$$V_{R_3} = R_3 I_{R_3} = \frac{R_1 R_3 I}{R_1 + R_2 + R_3}$$

Exercise 2.2: Using the node method, develop a set of simultaneous equations for the network shown below that can be used to solve for the three unknown node voltages in the network. Express these equations in the form

$$G \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = S$$

where G is a 3×3 matrix of conductance terms and S is a 3×1 vector of terms involving the sources. You need not solve the set of equations for the node voltages.



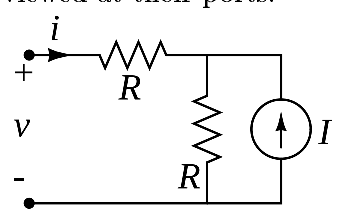
Answer: You could arrive at the conductance matrix by writing down the node equations and combining similar terms. However, as you get more familiar with this method, you should be able to "read" off the conductance matrix G and the source matrix S by simply looking at the circuit. The following outlines the procedure. Keep in mind that the following procedure has to be *modified* if there are *supernodes* in the circuit.

1. G(i, i) = summation of all conductances connected to node i.
2. G(i, j) = negative of the summation of all conductances connected between node i and j for i $\neq$ j. Note that G is a *symmetric* matrix.
3. S(i) = summation of: (1) current sources connected to node i, with current going in as positive and current going out as negative, and (2) terms in the form $V \cdot G$, where G is the conductance connecting the voltage source V and node i. Again, the sign is determined by following the sign of the current going from V to node i.
4. See your TA if you need further explanation.

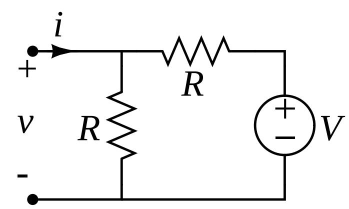
For this network:

$$\begin{bmatrix} G_1+G_2 & -G_1 & 0 \\ -G_1 & G_1+G_3 & -G_3 \\ 0 & -G_3 & G_5+G_4+G_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ I \\ VG_4 \end{bmatrix}$$

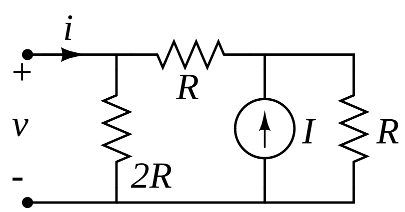
Problem 2.1: Find the Thevenin and Norton equivalents of the following networks, and graph their *i*-*v* relations as viewed at their ports.



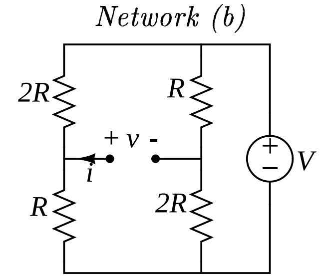
Network (a)



Network (b)

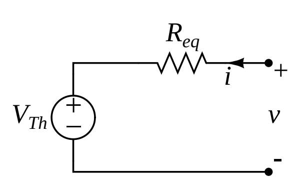


Network (c)

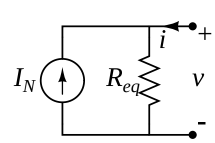


Network (d)

Answer:



Thevenin equivalent



Norton equivalent

For each circuit, we want V_{Th}, I_N, and R_{eq}, as represented in the circuits above. Note that the current at each port is defined to go *into* the positive terminal. This is the opposite direction from our definition of i_{sc}, which is the current we find from the positive terminal to the negative when we load the terminals with a short circuit. So, $V_{Th} = v_{oc}$ (the voltage at the port when it is open-circuit), $I_N = i_{sc}$ (the current through the port when it is short-circuit). This tells us further that the v-intercept of the i-v relation is V_{Th}, while the i-intercept is $-I_N$. Note that shutting off the sources and finding the equivalent resistance seen at the port determines the slope of the i-v relation. Note also that our v-i relations will be of the form

$$v = R_{eq} i + V_{Th}$$

while another way to relate them is with the inverse function

$$i = \frac{1}{R_{eq}} v - I_N$$

(A) In finding v_{oc}, there is no current through (and thus no voltage across) the resistor closest to the port. Thus v_{oc} is the voltage across the other resistor.

$$V_{Th} = v_{oc} = RI$$

When we short the terminals, the source current is split equally between the two resistors, since they are of equal value and are in parallel.

$$I_N = i_{sc} = \frac{R}{2R} I = \frac{I}{2}$$

The equivalent resistance follows.

$$R_{eq} = \frac{V_{Th}}{I_N} = 2R$$

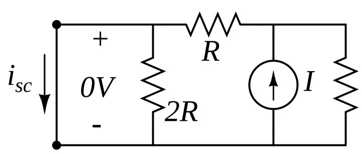
(B) This is the dual problem to (a).

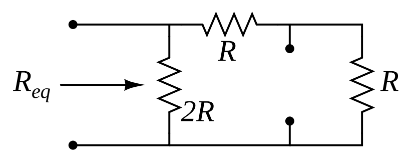
$$I_N = i_{sc} = \frac{V}{R}$$

$$V_{Th} = v_{oc} = \frac{R}{2R} V = \frac{V}{2}$$

$$R_{eq} = \frac{V_{Th}}{I_N} = \frac{R}{2}$$

(C) In this problem it is easiest to find i_{sc} and R_{eq}, and determine V_{Th} from those.





Shorting the port as in the figure above, we force the voltage across the $2R$ resistor to zero. Thus the current splits equally between the two paths (no current flows through the $2R$ resistor). So,

$$I_N = i_{sc} = \frac{I}{2}$$

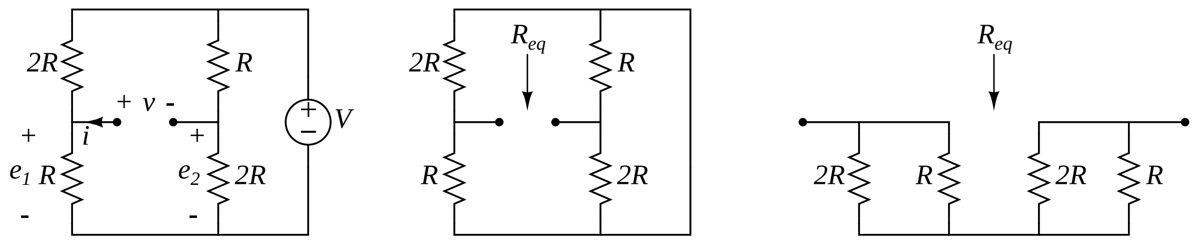
We now turn off the current source (turns into an open circuit) and compute the equivalent resistance, seen at the terminals. It is obvious from the figure above, right that the resistance is

$$R_{eq} = 2R \parallel (R + R) = R$$

Now, the Thevenin voltage is

$$V_{Th} = R_{eq} I_N = \frac{RI}{2}$$

(D) This is the most complicated network. It helps to define node voltages e_1 and e_2 as in the figure (left). Then $v_{oc} = e_1 - e_2$.



To find e_1 and e_2, we first notice that the left and right pairs of resistors are completely independent of one another. This is a result of the location of the voltage source, across both pairs. We then have two voltage dividers.

$$e_1 = \frac{R}{R + 2R} V = \frac{V}{3}$$

$$e_2 = \frac{2R}{R + 2R} V = \frac{2V}{3}$$

$$V_{Th} = v_{oc} = e_1 - e_2 = -\frac{V}{3}$$

Looking at the figure above, center, we see the circuit with the voltage source shut off. It is difficult to see how to analyze this circuit until it is redrawn as above, right. (If you are unsure the right drawing is the same as the middle one, label the nodes and trace each path.) We find that

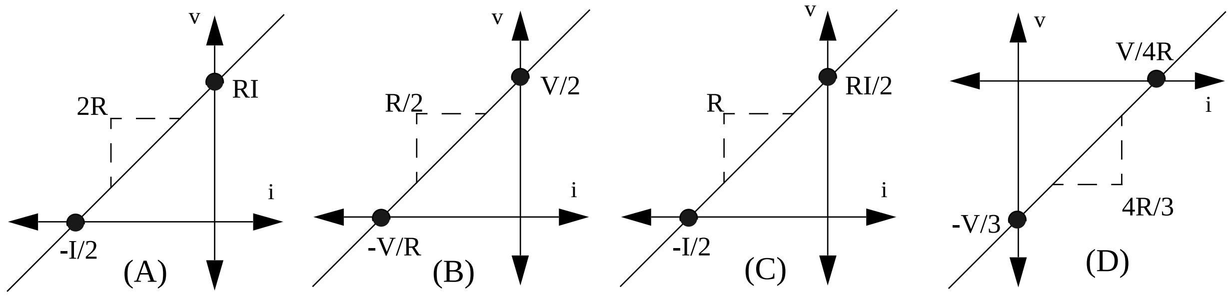
$$R_{eq} = (R \parallel 2R) + (2R \parallel R) = \frac{4}{3} R$$

The Norton current is then

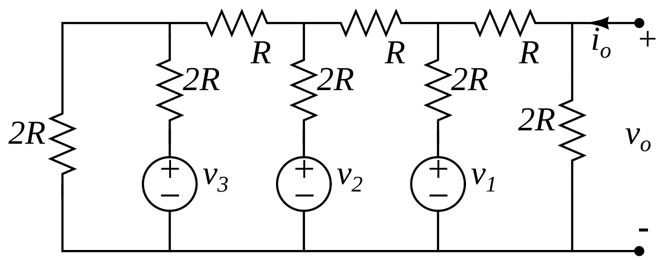
$$I_N = \frac{V_{Th}}{R_{eq}} = -\frac{V}{4R}$$

(E) Plots

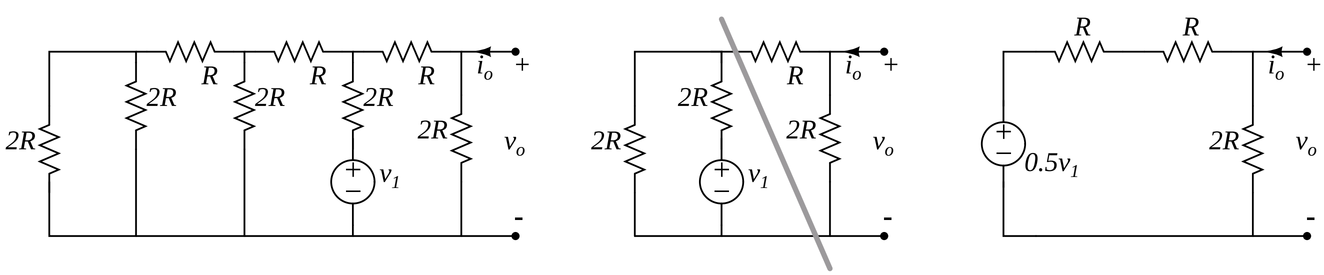
The v-i characteristics are as follows. Note that the axes are labelled, as well as the intercepts and slope.



Problem 2.2: Given the network shown below, find v_0 as a function of v_1, v_2 and v_3 assuming $i_0 = 0$. Hint: use superposition. Also, find the Thevenin equivalent of the network as viewed from its port. Finally, of what electronic circuit might the network be a part?

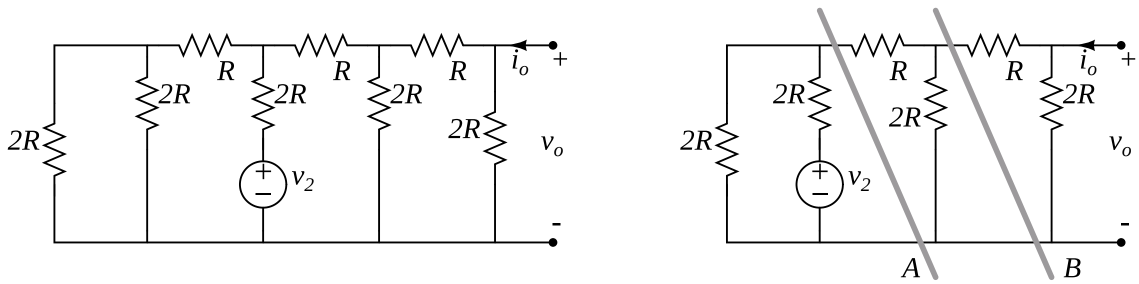


Answer: This problem is very difficult if approached the wrong way. Following the hints, we approach it from an easy direction. First, we shut off the sources v_2 and v_3. This leaves us with only v_1 as shown in the first circuit below.



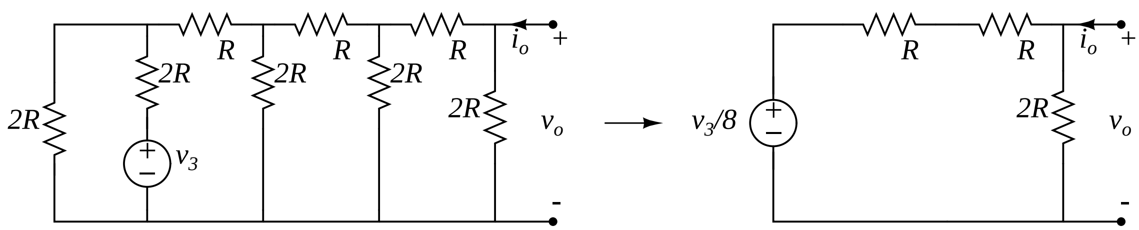
Looking at the leftmost circuit, we see we can combine many of the resistors to the left of the source v_1. Having done this, we cut the circuit as shown (middle circuit), and convert the source side to its Thevenin equivalent. On the right in the figure, we see the reconnected circuit. It is clear that the voltage v_0 is

$$v_0 = \frac{2R}{R + R + 2R} V_{Th} = \frac{v_1}{4}$$



Now we examine the contribution by v_2. Looking at the figure, we see the original circuit with v_1 and v_3 off. To the right, the resistors to the left of v_2 are combined, and two cuts are shown. It is obvious that the circuit to the left of cut A yields the same Thevenin equivalent as for the cut in the v_1 circuit. The circuit to the left of cut B has the same equivalent resistance, but the Thevenin voltage is cut by half. So,

$$v_0 = (\frac{1}{2})^2(\frac{v_2}{2}) = \frac{v_2}{8}$$



Looking at the figure above, left, we see the circuit when v_1 and v_2 are shut off. If we cut the circuit as shown, the portion on the left looks exactly like the v_2-only circuit. This means we can replace this by its Thevenin equivalent and reattach as shown above, right. We then further reduce this circuit by noticing the voltage divider halves the source voltage, or $v_{oc} = \frac{1}{2}V_s = \frac{1}{16}v_3$. By shutting off the source, the equivalent resistance is $R_{eq} = 2R \parallel (R + R) = R$. Note that this is the equivalent resistance of the entire circuit.

Assuming $i_0 = 0$, the relation between v_3 and v_0 is

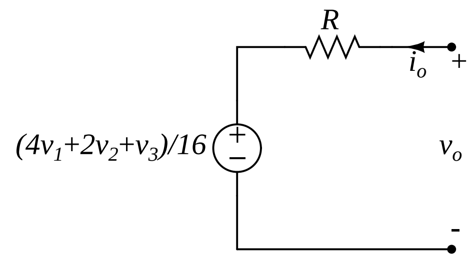
$$v_0 = (\frac{1}{2})^3(\frac{v_3}{2}) = \frac{v_3}{16}$$

Having the three components of v_0 when $i_0 = 0$, and having the equivalent resistance, we can draw the Thevenin equivalent circuit:

The last part of the problem asks for the behavior of v_0 with respect to the source voltages.

$$v_0 = \frac{1}{4}v_1 + \frac{1}{8}v_2 + \frac{1}{16}v_3$$

To understand the meaning of this equation when $v_1, v_2, v_3 \in \{0V, 3V\}$, suppose we define $b_i = \frac{16}{3}v_i, i = 1, 2, 3$. Then we find that



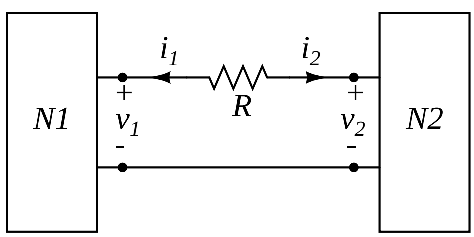
$$v_0 = \frac{3}{16}(2^2 b_1 + 2^1 b_2 + 2^0 b_3) = \frac{3}{16}(b_1 b_2 b_3)$$

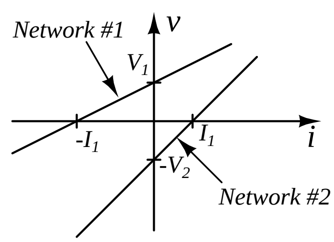
where $b_1 b_2 b_3$ is the binary representation of integers from 0 to 7. We have a digital-to-analog converter!

Problem 2.3: Two networks, N1 and N2, are described in terms of their i-v relations, and connected together through a single resistor, as shown below.

(A) Find the Thevenin and Norton equivalents of N1 and N2.

(B) Find the voltages v_1 and v_2 that result from the interconnection of N1 and N2 through a resistor as shown below.





Answer:

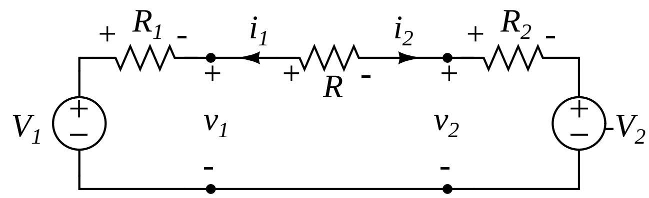
(A) At this point, we can simply read the Thevenin and Norton parameters right from the graph. For clarity, I will superscript the parameters by their network labels.

Network 1	Network 2
$V_{Th}^{n1} = V_1$	$V_{Th}^{n2} = -V_2$
$I_N^{n1} = I_1$	$I_N^{n2} = -I_1$
$R_{eq}^{n1} = \dfrac{V_1}{I_1}$	$R_{eq}^{n2} = \dfrac{-V_2}{-I_1} = \dfrac{V_2}{I_1}$

This is a good time to note that the equivalent resistance is positive (obvious but reassuring).

(B) One way to approach this problem is to use Thevenin equivalent circuits for $N1$ and $N2$ to find the current around the loop, and then use the device terminal relationships to find v_1 and

v_2. First we redraw the circuit substituting in the equivalent circuits and their element values:



Noting that $i_2 = -i_1$, writing KVL clockwise around the circuit gives the equationl

$$V_1 - i_2 R_{eq}^{n1} - i_2 R - i_2 R_{eq}^{n2} - (-V_2) = 0$$

We can combine like terms, and substitute in the values found in part (A) to solve for i_2:

$$V_1 - i_2 \left(\frac{V_1}{I_1} + R + \frac{V_2}{I_1} \right) + V_2 = 0$$

$$i_2 = \frac{V_1 + V_2}{\frac{V_1}{I_1} + R + \frac{V_2}{I_1}} = \frac{I_1(V_1 + V_2)}{V_1 + RI_1 + V_2}$$

The voltages v_1 and v_2 can be found by using the value for i_2 above, and the device laws for voltage sources and resistors.

$$v_1 = V_1 - i_2 R_1 = V_1 - \frac{I_1(V_1 + V_2)}{V_1 + RI_1 + V_2} * \frac{V_1}{I_1} = \frac{RI_1 V_1}{V_1 + RI_1 + V_2}$$

$$v_2 = -V_2 + i_2 R_2 = -V_2 + \frac{I_1(V_1 + V_2)}{V_1 + RI_1 + V_2} * \frac{V_2}{I_1} = -\frac{RI_1 V_2}{V_1 + RI_1 + V_2}$$

To check this answer, we can make sure that these node voltages produce i_2 through the resistor R. The current through R is $i_2 = \frac{V_1 + V_2}{\frac{V_1}{I_1} + R + \frac{V_2}{I_1}}$. The voltage across R is equal to $v_1 - v_2 = \frac{RV_1 I_1 + RV_2 I_1}{V_1 + RI_1 + V_2}$. This is just R times i_2, which is what we expected.

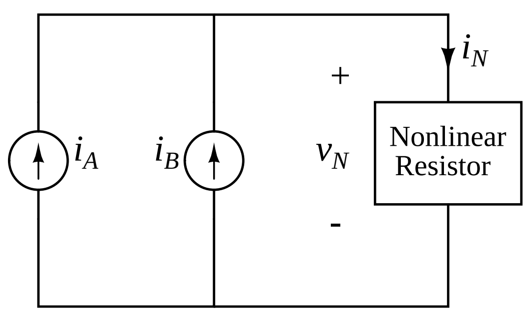
Problem 2.4: This problem studies the network shown below. The network contains two current sources and a nonlinear resistor. The nonlinear resistor has the terminal relation $i_{\mathrm{N}} = \alpha|v_{\mathrm{N}}|v_{\mathrm{N}}$, where α is a positive constant with units $\mathrm{A/V^2}$. One current source produces the current i_{A} while the second current source produces the current i_{B}.

(A) Determine v_N in terms of i_{A} and i_{B}.

(B) Let $i_{\mathrm{A}} = I_{\mathrm{A}}$, where I_{A} is a constant current, and let $i_{\mathrm{B}} = i_{\mathrm{b}}(t)$. Further, assume that $I_{\mathrm{A}} \gg |i_{\mathrm{b}}(t)| > 0$ so that i_{A} can be thought of as being only a large-signal bias current, and i_{B} can be thought of as being only a small-signal time-varying current. Using the result from Part (A), linearize v_{N} and express it in the form $v_{\mathrm{N}} = V_{\mathrm{N}} + v_{\mathrm{n}}(t)$ where V_{N} is a constant large-signal bias voltage and $v_{\mathrm{n}}(t)$ is small-signal time-varying voltage proportional to $i_{\mathrm{b}}(t)$.

(C) Using the result of Part (B), determine R such that $v_{\mathrm{n}}(t) = Ri_{\mathrm{b}}(t)$. Show that R is the incremental resistance of the nonlinear resistor, namely that $R = dv_{\mathrm{N}}/di_{\mathrm{N}}$ evaluated at the bias current I_{A}.

(D) From Part (C), it is apparent that the small-signal gain from the input current $i_{\mathrm{b}}(t)$ to the output voltage $v_{\mathrm{n}}(t)$ can be controlled by the bias current I_{A}. What problem can you see in using the circuit shown below as a controllable gain from the input current $i_{\mathrm{b}}(t)$ to the output voltage $v_{\mathrm{n}}(t)$?



Answer:

(A) We are given that $i_{\mathrm{N}} = \alpha|v_{\mathrm{N}}|v_{\mathrm{N}}$. Applying KCL to the top node gives $i_{\mathrm{N}} = i_{\mathrm{A}} + i_{\mathrm{B}}$. The sign of v_{N} matches the sign of i_{N} due to the absolute value sign in the device law for the nonlinear resistor. The substitution of KCL into that device law yields:

$$v_{\mathrm{N}} = \mathrm{sign}(i_{\mathrm{A}} + i_{\mathrm{B}})\sqrt{\frac{|i_{\mathrm{A}} + i_{\mathrm{B}}|}{\alpha}}$$

(B) Because both input currents have been defined to be positive, we can simplify the expression for v_{N} to $\sqrt{\frac{i_{\mathrm{A}}+i_{\mathrm{B}}}{\alpha}}$. To express v_{N} as a large-signal bias voltage and a small-signal time-varying voltage proportional to $i_{\mathrm{b}}(t)$, we need only take the first two terms of the Taylor series expansion of v_{N} in terms of i_{B}. To do so, substitute for i_{A} and i_{B}, and observe that

$$v_{\mathrm{N}} = \sqrt{\frac{I_{\mathrm{A}} + i_{\mathrm{b}}(t)}{\alpha}} = \sqrt{\frac{I_{\mathrm{A}}}{\alpha}}\sqrt{1 + \frac{i_{\mathrm{b}}(t)}{I_{\mathrm{A}}}}$$

Nest, use the Taylor Series expansion of $\sqrt{1+x} \approx 1 + \frac{x}{2}$ when $x \ll 1$ to get

$$v_{\mathrm{N}} \approx \sqrt{\frac{I_{\mathrm{A}}}{\alpha}} \left(1 + \frac{i_{\mathrm{b}}(t)}{2I_{\mathrm{A}}}\right)$$

where x has represented $i_{\mathrm{B}}/I_{\mathrm{A}}$. Rearranging this result then yields

$$v_{\mathrm{N}} \approx \sqrt{\frac{I_{\mathrm{A}}}{\alpha}} + \frac{i_{\mathrm{b}}(t)}{2}\sqrt{\frac{1}{\alpha I_{\mathrm{A}}}}$$

so that

$$V_{\mathrm{N}} = \sqrt{\frac{I_{\mathrm{A}}}{\alpha}}$$

$$v_{\mathrm{n}}(t) = \frac{i_{\mathrm{b}}(t)}{2}\sqrt{\frac{1}{\alpha I_{\mathrm{A}}}}$$

(C) We found that $v_{\mathrm{n}}(t) = \frac{i_{\mathrm{b}}(t)}{2}\sqrt{\frac{1}{\alpha I_{\mathrm{A}}}}$. So $R = \frac{1}{2}\sqrt{\frac{1}{\alpha I_{\mathrm{A}}}}$. The derivative of v_{N} with respect to i_{N} evaluated at I_{A} is:

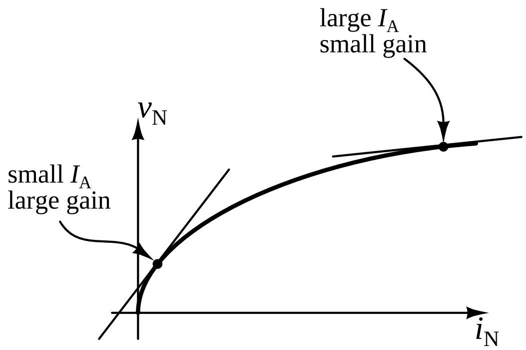
$$\frac{dv_{\mathrm{N}}}{di_{\mathrm{N}}}(I_{\mathrm{A}}) = \frac{d}{di_{\mathrm{N}}}\sqrt{\frac{i_{\mathrm{N}}}{\alpha}}\Bigg|_{I_{\mathrm{A}}}$$

$$\frac{dv_{\mathrm{N}}}{di_{\mathrm{N}}}(I_{\mathrm{A}}) = \frac{1}{2\alpha}\sqrt{\frac{\alpha}{i_{\mathrm{N}}}}\Bigg|_{I_{\mathrm{A}}}$$

$$\frac{dv_{\mathrm{N}}}{di_{\mathrm{N}}}(I_{\mathrm{A}}) = \frac{1}{2}\sqrt{\frac{1}{\alpha I_{\mathrm{A}}}} = R$$

(D) One problem with using this circuit as a controllable gain from $i_{\mathrm{b}}(t)$ to $v_{\mathrm{n}}(t)$ is that we must add the bias current I_{A} to $i_{\mathrm{B}}(t)$, which takes extra circuitry. Another problem is the presence of the bias current I_{A} wastes power.

Another more subtle problem becomes apparent if we look at the v-i graph of the output circuit, shown below. The gain is proportional to the slope of the transfer characteristic at the operating point.



One assumption we made above was that $\frac{i_{\mathrm{b}}(t)}{I_{\mathrm{A}}} \ll 1$. However, larger slopes (and therefore larger gains) are located at lower operating points. At lower operating points, $i_{\mathrm{b}}(T)$ begins to become comparable to I_{A}. This is bad, because the circuit no longer behaves according to the linearization, and we will see larger and larger distortions with higher and higher gains, that is as I_{A} is decreased.