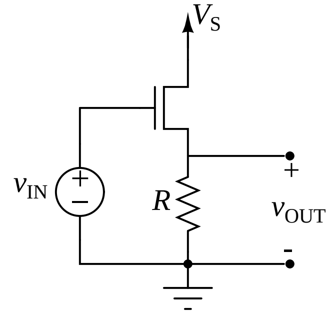
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

**6.002 – Electronic Circuits**
**Spring 2002**

# Homework #5 Solutions

**Exercise 5.1:** This problem studies the MOSFET amplifier shown below. A saturation-region model for the MOSFET is also given below. Assuming that the MOSFET operates in its saturation region, determine $v_{\text{OUT}}$ as a function of $v_{\text{IN}}$. Also, determine the range of $v_{\text{IN}}$ and the corresponding range of $v_{\text{OUT}}$ over which the MOSFET operates in its saturation region.




Saturation:

$0 \leq (v_{\text{GS}} - V_{\text{T}}) \leq v_{\text{DS}}$
$i_{\text{D}} = \frac{K}{2}(v_{\text{GS}} - V_{\text{T}})^2$

**Answer:** The voltage drop across $R$ is $v_{\text{OUT}}$ which is $Ri_{\text{D}}$. Recognizing that $v_{\text{GS}} = v_{\text{IN}} - v_{\text{OUT}}$, and assuming that the MOSFET is in saturation

$$v_{\text{OUT}} = \frac{RK}{2}(v_{\text{IN}} - v_{\text{OUT}} - V_{\text{T}})^2$$

This yields the quadratic equation

$$v_{\text{OUT}}^2 - 2(v_{\text{IN}} - V_{\text{T}})v_{\text{OUT}} - \frac{2}{RK}v_{\text{OUT}} + (v_{\text{IN}} - V_{\text{T}})^2 = 0$$

Using the quadratic formula gives

$$v_{\text{OUT}} = v_{\text{IN}} - V_{\text{T}} + \frac{1}{RK} - \sqrt{\frac{1}{R^2K^2} + \frac{2}{RK}(v_{\text{IN}} - V_{\text{T}})}$$

The square root term is subtracted because $v_{\text{OUT}} \leq v_{\text{IN}} - V_{\text{T}}$; otherwise the MOSFET would be in the cutoff region. To find the ranges for $v_{\text{IN}}$ and $v_{\text{OUT}}$ that correspond to the saturation region of operation, replace $v_{\text{GS}}$ in the saturation condition with $v_{\text{IN}} - v_{\text{OUT}}$, replace $v_{\text{DS}}$ with $V_{\text{S}} - v_{\text{OUT}}$, and rearrange a few terms to get

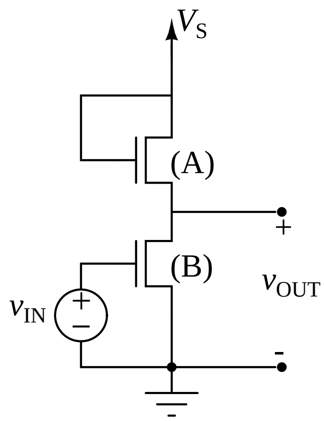
$$v_{\text{OUT}} \leq v_{\text{IN}} - V_{\text{T}} \leq V_{\text{S}}$$

The first condition is satisfied for all $v_{\text{IN}} - V_{\text{T}} \geq 0$. The second is always satisfied if $v_{\text{IN}}$ never exceeds $V_{\text{S}} + V_{\text{T}}$. Plugging these two range values into the expression for $v_{\text{OUT}}$ above, we find

$$V_{\text{T}} \leq v_{\text{IN}} \leq V_{\text{S}} + V_{\text{T}}$$

$$0 \leq v_{\mathrm{OUT}} \leq V_{\mathrm{S}} + \frac{1}{RK} - \sqrt{\frac{1}{R^2K^2} + \frac{2}{RK}V_{\mathrm{S}}}$$

**Exercise 5.2:** A "linear" MOSFET amplifier may be constructed using two MOSFETs as shown below. Note that the transconductances $K_{\mathrm{A}}$ and $K_{\mathrm{B}}$, and the threshold voltages $V_{\mathrm{TA}}$ and $V_{\mathrm{TB}}$, of the two MOSFETs are different. Assuming that both MOSFETs operate in their saturation regions, determine $v_{\mathrm{OUT}}$ as a function of $v_{\mathrm{IN}}$. Also, determine the range of $v_{\mathrm{IN}}$ and the corresponding range of $v_{\mathrm{OUT}}$ over which both MOSFETs operate in their saturation region.




Saturation:

$$0 \leq (v_{\mathrm{GSA}} - V_{\mathrm{TA}}) \leq v_{\mathrm{DSA}}$$
$$i_{\mathrm{DA}} = \tfrac{K_{\mathrm{A}}}{2}(v_{\mathrm{GSA}} - V_{\mathrm{TA}})^2$$
$$0 \leq (v_{\mathrm{GSB}} - V_{\mathrm{TB}}) \leq v_{\mathrm{DSB}}$$
$$i_{\mathrm{DB}} = \tfrac{K_{\mathrm{B}}}{2}(v_{\mathrm{GSB}} - V_{\mathrm{TB}})^2$$

**Answer:**

To find $v_{\mathrm{OUT}}$ as a function of $v_{\mathrm{IN}}$, write KCL at the node connected to $v_{\mathrm{OUT}}$ under the assumption that no current exits the output terminals. This yields:

$$i_{\mathrm{DA}} - i_{\mathrm{DB}} = 0$$

Using the saturation region equations for the drain currents in the above equations gives:

$$\frac{K_{\mathrm{A}}}{2}(v_{\mathrm{GSA}} - V_{\mathrm{TA}})^2 = \frac{K_{\mathrm{B}}}{2}(v_{\mathrm{GSB}} - V_{\mathrm{TB}})^2$$

Looking at the figure above, $v_{\mathrm{GSB}} = v_{\mathrm{IN}}$ and $v_{\mathrm{GSA}} = V_{\mathrm{S}} - v_{\mathrm{OUT}}$. Substituting these into the equation above gives:

$$K_{\mathrm{A}}(V_{\mathrm{S}} - v_{\mathrm{OUT}} - V_{\mathrm{TA}})^2 = K_{\mathrm{B}}(v_{\mathrm{IN}} - V_{\mathrm{TB}})^2$$

Taking the square root of both sides and rearranging a few terms provides an expression for $v_{\mathrm{OUT}}$ in terms of $v_{\mathrm{IN}}$.

$$v_{\mathrm{OUT}} = V_{\mathrm{S}} - V_{\mathrm{TA}} - \sqrt{\frac{K_{\mathrm{B}}}{K_{\mathrm{A}}}}(v_{\mathrm{IN}} - V_{\mathrm{TB}})$$

To find the ranges for $v_{\mathrm{IN}}$ and $v_{\mathrm{OUT}}$ that both MOSFETS are in saturation, examine the two inequalities

$$0 \leq (v_{\mathrm{GSA}} - V_{\mathrm{TA}}) \leq v_{\mathrm{DSA}}$$
$$0 \leq (v_{\mathrm{GSB}} - V_{\mathrm{TB}}) \leq v_{\mathrm{DSB}}$$

Let's consider MOSFET B first. We already know that $v_{\mathrm{GSB}} = v_{\mathrm{IN}}$. The first requirement on $v_{\mathrm{IN}}$ to keep MOSFET B in the saturation regions is then:

$$V_{\mathrm{TB}} \leq v_{\mathrm{IN}}$$

Using the fact that $v_{\mathrm{DSB}} = v_{\mathrm{OUT}}$, the other requirement on $v_{\mathrm{IN}}$ for MOSFET B is:

$$(v_{\mathrm{IN}} - V_{\mathrm{TB}}) \leq v_{\mathrm{OUT}}$$

Substituting using the equation we found for $v_{\mathrm{OUT}}$ above yields:

$$(v_{\mathrm{IN}} - V_{\mathrm{TB}}) \leq V_{\mathrm{S}} - V_{\mathrm{TA}} - \sqrt{\frac{K_{\mathrm{B}}}{K_{\mathrm{A}}}}(v_{\mathrm{IN}} - V_{\mathrm{TB}})$$

A little algebra gives the second restriction on $v_{\mathrm{IN}}$ for MOSFET B:

$$v_{\mathrm{IN}} \leq V_{\mathrm{TB}} + \frac{(V_{\mathrm{S}} - V_{\mathrm{TA}})}{1 + \sqrt{\frac{K_{\mathrm{B}}}{K_{\mathrm{A}}}}}$$

Now let's consider MOSFET A. The first requirement is

$$0 \leq (V_{\mathrm{S}} - v_{\mathrm{OUT}} - V_{\mathrm{TA}})$$

which can be rewritten $v_{\mathrm{OUT}} \leq V_{\mathrm{S}} - V_{\mathrm{TA}}$. Using the expression for $v_{\mathrm{OUT}}$ in terms of $v_{\mathrm{IN}}$ gives

$$V_{\mathrm{S}} - V_{\mathrm{TA}} - \sqrt{\frac{K_{\mathrm{B}}}{K_{\mathrm{A}}}}(v_{\mathrm{IN}} - V_{\mathrm{TB}}) \leq V_{\mathrm{S}} - V_{\mathrm{TA}}$$

This inequality is always true for $v_{\mathrm{IN}} - V_{\mathrm{TB}} \geq 0$. The second requirement for MOSFET A is

$$v_{\mathrm{GSA}} - V_{\mathrm{TA}} \leq v_{\mathrm{DSA}}$$

Substituting in for $v_{\mathrm{GSA}}$ and $v_{\mathrm{DSA}}$ yields

$$V_{\mathrm{S}} - v_{\mathrm{OUT}} - V_{\mathrm{TA}} \leq V_{\mathrm{S}} - v_{\mathrm{OUT}}$$

This inequality is always true. Combining these results gives the total restrictions on $v_{\mathrm{IN}}$ to be

$$V_{\mathrm{TB}} \leq v_{\mathrm{IN}} \leq V_{\mathrm{TB}} + \frac{(V_{\mathrm{S}} - V_{\mathrm{TA}})}{1 + \sqrt{\frac{K_{\mathrm{B}}}{K_{\mathrm{A}}}}}$$

The corresponding range of $v_{\mathrm{OUT}}$ is then

$$V_{\mathrm{S}} - V_{\mathrm{TA}} \geq v_{\mathrm{OUT}} \geq (V_{\mathrm{S}} - V_{\mathrm{TA}})\left(1 - \frac{1}{\sqrt{\frac{K_{\mathrm{A}}}{K_{\mathrm{B}}}} + 1}\right)$$

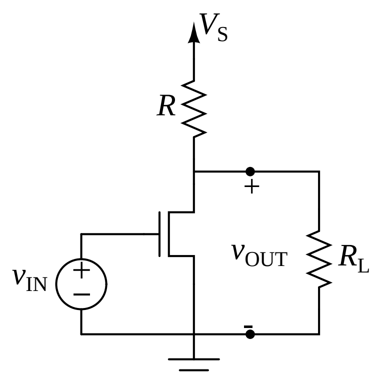
**Problem 5.1:** So far we have studied MOSFET amplifiers that have no load. That is, the current circulating through the output port of each amplifier was zero. For example, in Problem 4.3 the current out of the fist amplifier and into the second amplifier was zero because $i_{\mathrm{G}} = 0$ for the second MOSFET. In this problem, which studies the amplifier shown below, the output current is no longer zero. The load below is a resistor that does draw current from the amplifier. Hint, in analyzing this amplifier, consider the use of both Thevenin equivalence and load line analysis to simplify the problem. Also, review your solution to Problem 4.3.

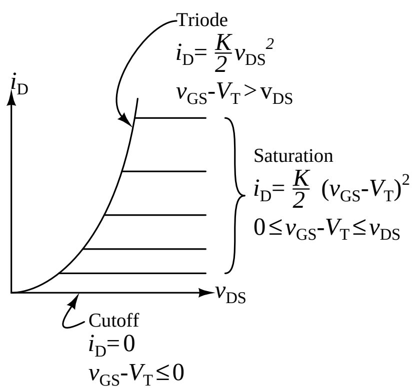
Once again, use a simplified model for the MOSFET as shown below. The simplification is again that the triode region of operation is compressed onto the curve $i_D = Kv_{DS}^2/2$, which becomes a common curve of operation for $v_{GS} - V_T > v_{DS}$.

(A) Determine the range of $v_{IN}$ over which the MOSFET operates in cutoff. Also, determine $v_{OUT}$ for this operating range.

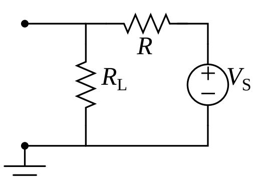
(B) Assuming that the MOSFET operates in its saturation region, determine $v_{OUT}$ as a function of $v_{IN}$. Also, determine the range of $v_{OUT}$ and the range of $v_{IN}$ that correspond to the saturated operation of the MOSFET.

(C) For values of $v_{IN}$ that are above the range found in Part (B), the MOSFET operates in its triode region, which in the model below is compressed onto the curve $i_D = Kv_{DS}^2/2$. Determine $v_{OUT}$ for $v_{IN}$ in this range of operation.




**Answer:** The easiest way to solve this problem is to find the Thevenin equivalent circuit for $V_S$, $R$, and $R_L$, and reuse the results from Problem 4.3. For clarity, these three elements have been re-drawn below.




The Thevenin Voltage of this circuit is formed by the $R_L$, $R$ voltage divider, so $V_{Th} = \frac{R_L}{R+R_L}V_S$. The equivalent resistance is just $R||R_L = \frac{RR_L}{R+R_L}$.

Substituting this equivalent circuit back into the amplifier, we notice that it is the same as the first stage of the amplifier in Problem 4.3. We can simply reuse the answers from it, replacing $R$ with $R||R_L$ and $V_S$ with $\frac{R_L}{R+R_L}V_S$.

(A) In Cutoff for: $v_{\rm IN} < V_{\rm T}$

$v_{\rm OUT}(v_{\rm IN})$ $\quad v_{\rm OUT} = \frac{R_{\rm L}}{R+R_{\rm L}} V_{\rm S}$

(B) In Saturation for: $V_{\rm T} \le v_{\rm IN} \le \frac{-1+\sqrt{1+2\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K\frac{R_{\rm L}}{R+R_{\rm L}}V_{\rm S}}}{\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K} + V_{\rm T}$

$v_{\rm OUT}(v_{\rm IN})$ $\quad v_{\rm OUT} = \frac{R_{\rm L}}{R+R_{\rm L}} V_{\rm S} - \frac{\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K}{2}(v_{\rm IN} - v_{\rm T})^2$

Output Range: $\frac{-1+\sqrt{1+2\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K\frac{R_{\rm L}}{R+R_{\rm L}}V_{\rm S}}}{\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K} \le v_{\rm OUT} \le \frac{R_{\rm L}}{R+R_{\rm L}} V_{\rm S}$

(C) In Triode for: $v_{\rm IN} > \frac{-1+\sqrt{1+2\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K\frac{R_{\rm L}}{R+R_{\rm L}}V_{\rm S}}}{\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K} + V_{\rm T}$

$v_{\rm OUT}(v_{\rm IN})$ $\quad \frac{-1+\sqrt{1+2\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K\frac{R_{\rm L}}{R+R_{\rm L}}V_{\rm S}}}{\frac{R\cdot R_{\rm L}}{R+R_{\rm L}}K}$

**Problem 5.2:** This problem continues to study the two-stage amplifier studied first in Problem 4.3. In this problem, let $v_{\rm IN} = V_{\rm IN} + v_{\rm in}$ and $v_{\rm OUT} = V_{\rm OUT} + v_{\rm out}$, where $V_{\rm IN}$ and $V_{\rm OUT}$ are the large-signal components of $v_{\rm IN}$ and $v_{\rm OUT}$, respectively, and $v_{\rm in}$ and $v_{\rm out}$ are the small-signal components of $v_{\rm IN}$ and $v_{\rm OUT}$, respectively.

(A) Assume that both MOSFETs are biased so that they operate in their saturation regions. Develop a small-signal circuit model for the amplifier that can be used to determine $v_{\rm out}$ as a function of $v_{\rm in}$. In doing so, assume that $V_{\rm IN}$ defines the operating point around which the small-signal model is constructed, and evaluate all small-signal model parameters in terms of $V_{\rm IN}$ as necessary.

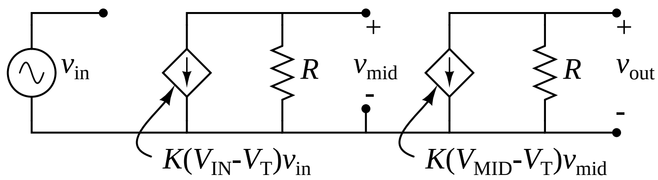
(B) Use the small-signal model to determine $v_{\rm out}$ as a function of $v_{\rm in}$.

(C) Compare the small-signal gain found in Part (B), defined as $v_{\rm out}/v_{\rm in}$, to that found in Part (F) of Problem 4.3. Explain any differences.

(D) Determine the small-signal Thevenin equivalent of the amplifier when it is viewed through its output port.

**Answer:**

(A) The small signal model can be constructed by replacing the MOSFETS with dependent current sources, and removing any DC bias voltages from any voltage sources; the bias voltage sources are incremental short circuits. The resulting circuit is shown below. The first MOSFET's corresponding current source is dependent on $v_{\rm IN}$, so $i_{\rm D} = K(V_{\rm IN} - V_{\rm T})v_{\rm IN}$. The second MOSFET's corresponding current source is dependent on $v_{\rm MID}$, and it's operating point is evaluated at $V_{\rm MID} = V_{\rm S} - \frac{RK}{2}(V_{\rm IN} - V_{\rm T}) - V_{\rm T}$.




(B) We can quickly write that $v_{MID} = -RK(V_{IN} - V_T)v_{IN}$. Defining $V_{MID} = V_S - \frac{RK}{2}(V_{IN} - V_T)^2$, we find that $v_{OUT}(V_{IN}) = R^2K^2(V_{IN} - V_T)(V_{MID} - V_T)v_{IN}$.

(C) The two answers are identical.

(D) We've already found the open circuit voltage at the output port, it's just the product of the gain and $v_{IN}$. Therefore

$$V_{Th} = R^2K^2V_{IN} - V_T)(V_{MID} - V_T)v_{IN}$$

To find $R_{eq}$ we set all the independent sources, including $v_{IN}$, to zero. This means that $v_{mid}$ will be zero, so the current source feeding into the $R$ at the output will be turned off. The only component left at the output port is the resistor R, so

$$R_{eq} = R$$

**Problem 5.3:** Consider again the amplifier described in Exercise 5.1. In this problem, let $v_{IN} = V_{IN} + v_{in}$ and $v_{OUT} = V_{OUT} + v_{out}$, where $V_{IN}$ and $V_{OUT}$ are the large-signal components of $v_{IN}$ and $v_{OUT}$, respectively, and $v_{in}$ and $v_{out}$ are the small-signal components of $v_{IN}$ and $v_{OUT}$, respectively.

(A) Using your result from Exercise 5.1, determine the small signal gain of the amplifier as a function of the input bias voltage $v_{IN}$. That is, determine $v_{out}/v_{in} = dv_{OUT}/dv_{IN}$ evaluated at $V_{IN}$.

(B) Again assume that the MOSFET is biased so that it operates in its saturation region. Develop a small-signal circuit model for the amplifier that can be used to determine $v_{out}$ as a function of $v_{in}$. In doing so, assume that $V_{IN}$ defines the operating point around which the small-signal model is constructed, and evaluate all small-signal model parameters in terms of $V_{IN}$ as necessary.

(C) Use the small-signal model to determine the small-signal gain $v_{out}/v_{in}$. Compare this small-signal gain to that found in Part (A) and explain any differences.

(D) Determine the small-signal Thevenin equivalent of the amplifier.

**Answer:**

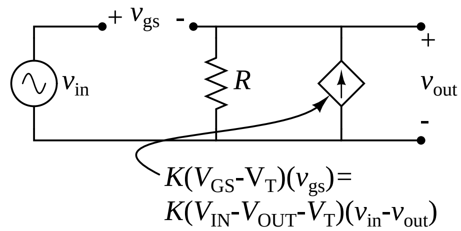
(A) From Exercise 5.1 we know that

$$v_{OUT} = v_{IN} - V_T + \frac{1}{RK} - \sqrt{\frac{1}{R^2K^2} + \frac{2}{RK}(v_{IN} - V_T)}$$

Taking the derivative of this, and evaluating it at $V_{\text{IN}}$ gives

$$\left.\frac{dv_{\text{OUT}}}{dv_{\text{IN}}}\right|_{V_{\text{IN}}} = 1 - \frac{1}{\sqrt{1 + 2RK(V_{\text{IN}} - V_{\text{T}})}}$$

(B) The small signal model is given below. Note that $v_{\text{gs}} = v_{\text{in}} - v_{\text{out}}$ and $V_{\text{GS}} = V_{\text{IN}} - V_{\text{OUT}}$.




The term $K(V_{\text{IN}} - V_{\text{T}} - V_{\text{OUT}})$ for the dependent current source that replaced the MOSFET can be expanded by using the expression for $v_{\text{OUT}}$ above, evaluated at $V_{\text{IN}}$. The value of the current source becomes

$$\frac{1}{R} - \sqrt{\frac{1}{R^2} + \frac{2K}{R}(V_{\text{IN}} - V_{\text{T}})}$$

(C) Using the small-signal model

$$v_{\text{out}} = \left(1 - \sqrt{1 + 2KR(V_{\text{IN}} - V_{\text{T}})}\right)(v_{\text{in}} - v_{\text{out}})$$

Solving for $v_{\text{out}}/v_{\text{in}}$ yields

$$\frac{v_{\text{out}}}{v_{\text{in}}} = 1 - \frac{1}{\sqrt{1 + 2KR(V_{\text{IN}} - V_{\text{T}})}}$$

This answer is identical to the one from Part (A).

(D) We have already found the small-signal open-circuit voltage in Part (C). It is just

$$V_{\text{Th}} = \frac{v_{\text{out}}}{v_{\text{in}}} \cdot v_{\text{in}} = v_{\text{in}}\left(1 - \frac{1}{\sqrt{1 + 2KR(V_{\text{IN}} - V_{\text{T}})}}\right)$$

To find the equivalent resistance, set $v_{\text{in}} = 0$, and apply a test voltage and current, $v_{\text{p}}$ and $i_{\text{p}}$. We find

$$v_{\text{p}} = Ri_{\text{p}} - v_{\text{p}}\left(1 - \frac{1}{\sqrt{1 + 2KR(V_{\text{IN}} - V_{\text{T}})}}\right)$$

Solving this for $v_{\text{p}}/i_{\text{p}} = R_{\text{eq}}$ yields

$$R_{\text{eq}} = \frac{R}{2 - \frac{1}{\sqrt{1+2KR(V_{\text{IN}}-V_{\text{T}})}}}$$