Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Spring 2002

Homework #6 Solutions

Exercise 6.1: Consider an amplifier with an input-output relation that takes the form $v_{OUT} = V_A(v_{IN}/V_B)^5$, where V_A and V_B are voltage constants. Determine its output bias voltage V_{OUT} and its small-signal gain v_{out}/v_{in} for a given input bias voltage V_{IN}.

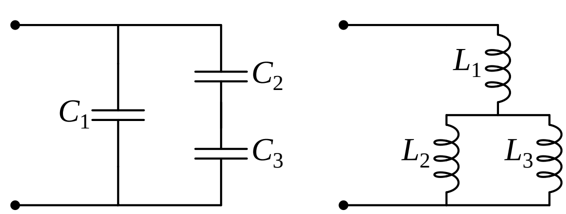
Answer: For a given input bias voltage V_{IN}, we can readily compute the corresponding input bias voltage V_{OUT} as

$$V_{OUT} = V_A \left(\frac{v_{IN}}{V_B}\right)^5 \bigg|_{v_{IN}=V_{IN}} = V_A \left(\frac{V_{IN}}{V_B}\right)^5$$

The small signal gain is given by

$$\frac{v_{out}}{v_{in}} = \frac{dv_{OUT}}{dv_{IN}}\bigg|_{v_{IN}=V_{IN}} = \frac{5V_A v_{IN}^4}{V_B^5}\bigg|_{v_{IN}=V_{IN}} = \frac{5V_A V_{IN}^4}{V_B^5}$$

Exercise 6.2: Find the capacitance of the all-capacitor network, and the inductance of the all-inductor network, shown below.



Answer: Use the facts that series inductances and parallel capacitances add, and series reciprocal capacitances and parallel reciprocal inductances add.

$$\begin{aligned} C_{eq} &= C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} \\ &= C_1 + \frac{C_2 C_3}{C_2 + C_3} \end{aligned}$$

$$\begin{aligned} L_{eq} &= L_1 + \left(\frac{1}{L_2} + \frac{1}{L_3}\right)^{-1} \\ &= L_1 + \frac{L_2 L_3}{L_2 + L_3} \end{aligned}$$

Problem 6.1: This problem studies the propagation delay of digital signals through the inverter shown below. Assume that the MOSFET in the inverter acts as a switch with on-state resistance R_{ON}. The inverter is loaded with a capacitor, having capacitance C_G, that models the combined input capacitance of the logic gates connected to its output. Assume that the inverter obeys the static discipline defined in part by V_{OL} and V_{OH}.

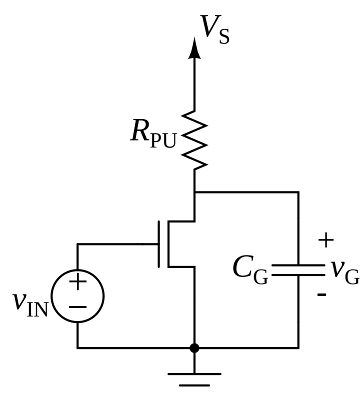
(A) Assume that the MOSFET has been off for a very long time. At $t = 0$, v_{IN} turns the MOSFET on. Determine $v_G(t)$ for $t \geq 0$.

(B) How long does it take $v_G(t)$ to pass by V_{OL}? This delay is the fall time of the inverter.

(C) Assume that the MOSFET has been on for a very long time. At $t = 0$, v_{IN} turns the MOSFET off. Determine $v_G(t)$ for $t \geq 0$.

(D) How long does it take $v_G(t)$ to pass by V_{OH}? This delay is the rise time of the inverter.

(E) How can the fall and rise times be shortened via the design of R_{PU}? What limits the extent to which this design path may be followed?

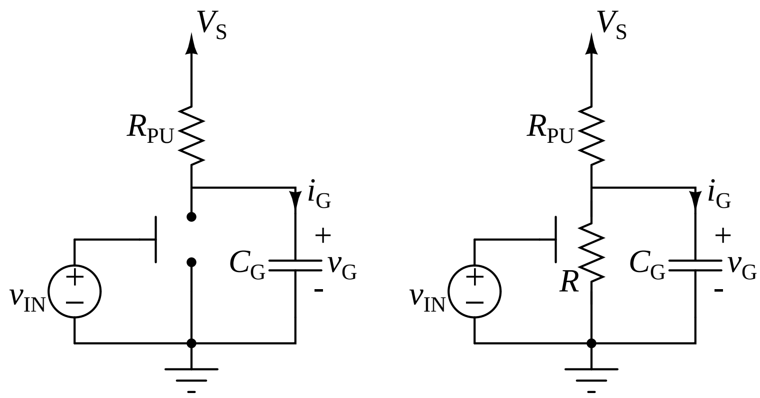


Answer:

Before we begin the problem, let us try to understand what is happening in this inverter circuit. Since we must follow the static discipline, we must have $v_{IN} \leq V_{IL}$ in order for a logical "0" to be represented. Likewise, $v_{IN} \geq V_{IH}$ for a logical "1" to be represented. When an inverter has an input of "1", we get an output of "0", and vice-versa. If the inverter is receiving a "0" at its input that suddenly changes to a "1", then the output changes from a "1" into a "0". This means that v_{IN} traveled from some value below V_{IL} to some value above V_{IH}. Likewise, the output traveled from some value greater than V_{OH} to a value less than V_{OL}. Now, if it takes a non-zero time for the output to change from a "1" into a "0", then there is a delay from the change in the input to a successful change in the output. This will result when the inverter drives a capacitance on the output node, as shown in the above diagram. A similar observation holds for the reverse transistor.

The output of combinational logic circuits are usually always connected to the gates of other logic circuits. Since there is an inherent capacitance associated with the gates of MOSFETs, this accounts for the capacitance shown in the diagram above. As you may recall from your physics classes, for capacitances in parallel, the equivalent capacitance is the sum of all the parallel capacitances. This value, which is connected to the MOSFET output, is modeled by the capacitor, C_G.

The following two figures model the inverter above in its two states. These states correspond to the MOSFET being off and on.



"OFF" State **"ON" State**

(A) Looking at the OFF-State figure above, we see the circuit when the MOSFET is off. We know that a capacitor looks like an open circuit after a very long time if driven by an unvarying, DC signal input. This means that $i_G(t = 0^-) = 0$ since there is no other path for the current to flow to ground. Hence, $v_G(t = 0^-) = V_S$. Since there are no impulses of drive current being introduced within the circuit as it switches on or off, we know that the capacitor voltage will be continuous. Therefore, $v_G(t = 0^+) = v_G(t = 0^-) = V_S$. Now that we have the initial condition for the ON-State behavior, we need to find a differential equation to solve for the time dependence of the capacitor voltage, v_G. The constitutive relation for the capacitor is:

$$C_G \frac{dv_G(t)}{dt} = i_G(t)$$

By examining the "ON" state figure above for $t \geq 0$, we can apply KCL to the MOSFET drain node:

$$\frac{V_S - v_G(t)}{R_{PU}} = \frac{v_G(t)}{R_{ON}} + i_G(t)$$

Substituting the constitutive relation and rearranging terms:

$$\frac{d}{dt}v_G(t) + \frac{1}{C_G}\frac{R_{ON} + R_{PU}}{R_{ON}R_{PU}}v_G(t) = \frac{V_S}{R_{PU}C_G}$$

Assuming the form:

$$v_G(t) = Ae^{-\frac{t}{\tau}} + K$$

and inserting our guess into our differential equation, we get:

$$-\frac{A}{\tau}e^{-\frac{t}{\tau}} + \frac{1}{C_G}\frac{R_{ON} + R_{PU}}{R_{ON}R_{PU}}Ae^{-\frac{t}{\tau}} + \frac{1}{C_G}\frac{R_{ON} + R_{PU}}{R_{ON}R_{PU}}K = \frac{V_S}{R_{PU}C_G}$$

In order for this equation to have a non-trivial solution, we must have:

$$\frac{1}{\tau} = \frac{1}{C_G}\frac{R_{ON} + R_{PU}}{R_{ON}R_{PU}}$$

solving for τ:

$$\tau = C_G\frac{R_{ON}R_{PU}}{R_{ON} + R_{PU}} = C_G(R_{ON} \parallel R_{PU})$$

The constant portions must also be equivalent:

$$\frac{1}{C_{\mathrm{G}}}\frac{R_{\mathrm{ON}}+R_{\mathrm{PU}}}{R_{\mathrm{ON}}R_{\mathrm{PU}}}K=\frac{V_{\mathrm{S}}}{R_{\mathrm{PU}}C_{\mathrm{G}}}$$

Solving for K:

$$K=\frac{V_{\mathrm{S}}}{R_{\mathrm{PU}}}\left(\frac{R_{\mathrm{ON}}R_{\mathrm{PU}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}\right)=V_{\mathrm{S}}\frac{R_{\mathrm{ON}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}$$

Using the initial condition:

$$v_{\mathrm{G}}(t=0)=V_{\mathrm{S}}=Ae^{-\frac{0}{\tau}}+K$$

Solving for A:

$$A=V_{\mathrm{S}}-K=V_{\mathrm{S}}-V_{\mathrm{S}}\frac{R_{\mathrm{ON}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}$$

Finally, our complete solution for $t>0$ is:

$$v_{\mathrm{G}}(t>0)=V_{\mathrm{S}}e^{-\frac{t}{\tau}}+V_{\mathrm{S}}\frac{R_{\mathrm{ON}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}(1-e^{-\frac{t}{\tau}})$$

Where $\tau=C_{\mathrm{G}}(R_{\mathrm{ON}}\parallel R_{\mathrm{PU}})$. This makes intuitive sense since $v_{\mathrm{G}}(t=0)=V_{\mathrm{S}}$. The second term is zero and only the first term contributes to v_{G}. Also, $v_{\mathrm{G}}(t=\infty)=V_{\mathrm{S}}\frac{R_{\mathrm{ON}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}$. The first term vanishes and only the second term contributes. At long times, the capacitor acts like an open circuit, so the inverter v_{G} can be solved using a voltage divider formula.

(B) At $t=0$, $v_{\mathrm{G}}(t=0)=V_{\mathrm{S}}$, which corresponds to a logical "1" as long as $V_{\mathrm{OH}}<V_{\mathrm{S}}$. When the MOSFET turns on, v_{G} begins to drop and head towards V_{OL}. The time it takes for v_{G} to reach V_{OL} is the fall time, t_{f}.

$$v_{\mathrm{G}}(t=t_{\mathrm{f}})=Ae^{-\frac{t_{\mathrm{f}}}{\tau}}+K=V_{\mathrm{OL}}$$

solving for t_{f}:

$$t_{\mathrm{f}}=\tau\ln\left(\frac{V_{\mathrm{S}}-K}{V_{\mathrm{OL}}-K}\right)=C_{\mathrm{G}}\frac{R_{\mathrm{ON}}R_{\mathrm{PU}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}\ln\left(\frac{V_{\mathrm{S}}-V_{\mathrm{S}}\frac{R_{\mathrm{ON}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}}{V_{\mathrm{OL}}-V_{\mathrm{S}}\frac{R_{\mathrm{ON}}}{R_{\mathrm{ON}}+R_{\mathrm{PU}}}}\right)$$

We can graphically show the significance of t_{f}. The left graph below represents v_{IN} changing from low to high. The graph on the right shows v_{OUT} changing from high to low. The time it takes for v_{OUT} to become a valid "0", referenced to the time v_{IN} changed, is defined as t_{f}.

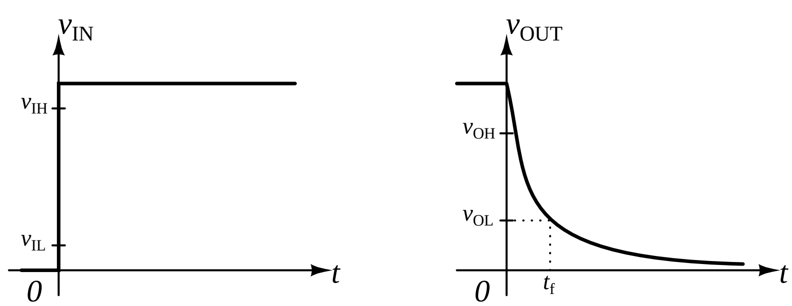


Figure for Problem 6.1 Part (B)

(C) For this part, we assume the inverter has been on for a long period of time. The voltage across the capacitor, v_G, can be found by evaluating the solutions for v_G from Part (A) for $t = \infty$:

$$v_G(t=\infty) = V_S e^{-\frac{\infty}{\tau}} + V_S \frac{R_{ON}}{R_{ON}+R_{PU}}(1-e^{-\frac{\infty}{\tau}}) = V_S \frac{R_{ON}}{R_{ON}+R_{PU}}$$

since this describes v_G at the output of the inverter in the "ON" state after very long times. This then becomes the initial condition for the rising transient of v_G at $t = 0$:

$$v_G(t=0) = V_S \frac{R_{ON}}{R_{ON}+R_{PU}}$$

For $t > 0$, the MOSFET turns off and the circuit looks like the "OFF" state figure above. Not that this figure can be obtained from the On-State figure with $R_{ON} \to \infty$. Therefore, the governing differential equation can too. It is

$$\frac{dv_G}{dt} + \frac{1}{C_G R_{PU}} v_G(t) = \frac{V_S}{R_{PU} C_G}$$

Again, assuming a solution of the form

$$v_G(t) = Ae^{-t/\tau} + K$$

and substituting this solution into the differential equation yields

$$-\frac{A}{\tau}e^{-t/\tau} + \frac{A}{C_G R_{PU}}e^{-t/\tau} + \frac{K}{C_G R_{PU}} = \frac{V_S}{C_G R_{PU}}$$

Since this equation must hold for all time, we can match constant and exponential terms separately. This yields

$$\begin{aligned} \tau &= R_{PU} C_G \\ K &= V_S \end{aligned}$$

Next, substitution of the initial condition yields

$$A = V_S \frac{R_{ON}}{R_{ON}+R_{PU}} - V_S = -\frac{R_{PU}}{R_{ON}+R_{PU}}$$

Finally, all terms may be combined to yield

$$v_G(t) = V_S - \frac{R_{PU}}{R_{ON}+R_{PU}} V_S \left(e^{-\frac{t}{R_{PU}C_G}}\right)$$

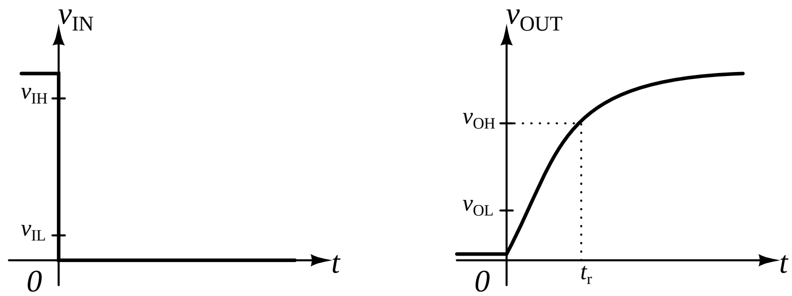
(D) The rise time, t_r is the time it takes for the output voltage to rise from its initial "low" value to the minimum value of a valid "high". This minimum valid "high" value is called V_{OH}. Using the general time-dependent solution for $v_G(t)$ found it part (C):

$$v_G(t=t_r) = V_S - \frac{R_{PU}}{R_{ON}+R_{PU}} V_S \left(e^{-\frac{t_r}{R_{PU}C_G}}\right) = V_{OH}$$

Solving for t_r, we get:

$$t_r = R_{PU} C_G \ln \left[\frac{V_S \frac{R_{PU}}{R_{ON}+R_{PU}}}{V_S - V_{OH}}\right]$$

The meaning of t_r can be seen in the same fashion as in Part (C). The figure below shows a time-domain representation of v_{IN} and v_{OUT}:



(E) If you lower $R_{\rm PU}$ then the time constant, τ shortens and the time it takes to rise and fall drops. However, the value of $R_{\rm PU}$ is limited by the fact that we want $v_{\rm G}$ to register as a valid "1" or "0". $R_{\rm PU}$ has no effect on $v_{\rm G}$ as a valid "1" since if the MOSFET turns off, $v_{\rm G}$ will equal $V_{\rm S}$, which is greater than the minimum value for a valid "1": $V_{\rm OH}$. However, the steady-state value for $v_{\rm G}$ being used as a logical "0" is determined by the equation:

$$v_{\rm G} = V_{\rm S}\frac{R_{\rm ON}}{R_{\rm ON} + R_{\rm PU}}$$

If $R_{\rm PU}$ is made too small, then $v_{\rm G}$ won't ever fall less than $V_{\rm OL}$ and we will then have violated the static discipline. Thus, as a general constraint to remember for a logical "0" output:

$$v_{\rm G} = V_{\rm S}\frac{R_{\rm ON}}{R_{\rm ON} + R_{\rm PU}} < V_{\rm OL}$$

Therefore,

$$R_{\rm PU} > R_{\rm ON}\frac{V_{\rm S} - V_{\rm OL}}{V_{\rm OL}}$$

Also, as $R_{\rm PU}$ is made smaller the inverter dissipates more and more power when the MOSFET is in its "ON" state. Temperature rise might therefore pose a significant limit to the reduction of $R_{\rm PU}$.

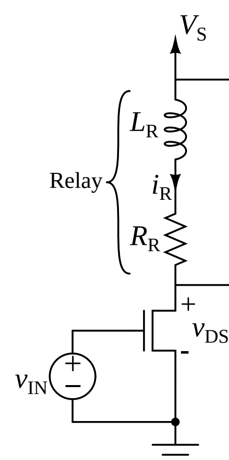
Problem 6.2: In the circuit shown below, a MOSFET and an external resistor having resistance R_X are used to control the current i_R in the winding of a relay. Here, the relay is modeled as a series inductor and resistor having inductance L_R and resistance R_R, respectively. The MOSFET may be modeled as an ideal switch.

(A) At $t = 0$, v_{IN} turns the MOSFET on so that $v_{DS} = 0$. Determine $i_R(t)$ for $t \geq 0$ given that $i_R(t = 0) = 0$.

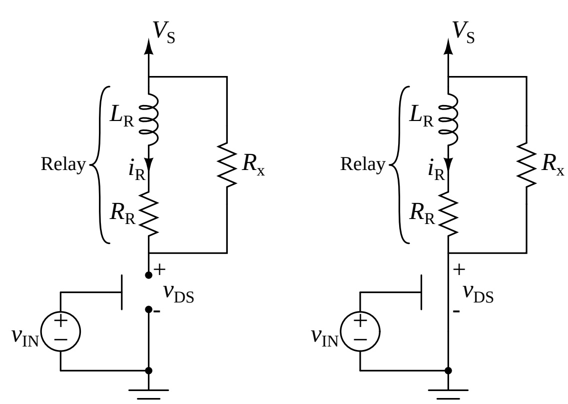
(B) Next, at $t = T$, v_{IN} turns the MOSFET off. Determine both $i_R(t)$ and $v_{DS}(t)$ for $t \geq T$. Hint: $i_R(t)$ is continuous at $t = T$.

(C) Sketch and clearly label graphs of both $i_R(t)$ and $v_{DS}(t)$ for $t \geq 0$ assuming that $T \approx 5L_R/R_R$ and $R_X = R_R$.

(D) The relay control circuit would be less expensive without the external resistor, which may be "removed" from the circuit by considering the limit $R_X \to \infty$. Why might such a cost reduction be unwise?



Answer:



"OFF" State **"ON" State**

(A) When the MOSFET is "ON", the circuit can be redrawn as in the right-hand figure above.

Our initial condition is given:

$$i_{\mathrm{R}}(t=0) = 0$$

Our constitutive equation for the inductor's voltage and current is:

$$v_{\mathrm{L}}(t) = L_{\mathrm{R}} \frac{di_{\mathrm{R}}(t)}{dt}$$

where $v_{\mathrm{L}}(t)$ is the voltage difference from the top of the inductor to the bottom of the inductor. Performing KVL along the left-hand branch of the relay:

$$V_{\mathrm{S}} - v_{\mathrm{L}}(t) - i_{\mathrm{R}}(t) R_{\mathrm{R}} = 0$$

Substituting the constitutive relation and putting the differential equation into standard form:

$$\frac{d}{dt} i_{\mathrm{R}}(t) + \frac{R_{\mathrm{R}}}{L_{\mathrm{R}}} i_{\mathrm{R}}(t) = \frac{V_{\mathrm{S}}}{L_{\mathrm{R}}}$$

The homogeneous solution solves the differential equation when the right-hand side is 0:

$$\frac{d}{dt} i_{\mathrm{R_H}}(t) + \frac{R_{\mathrm{R}}}{L_{\mathrm{R}}} i_{\mathrm{R_H}}(t) = 0$$

We assume the form:

$$i_{\mathrm{R_H}}(t) = Ae^{-\frac{t}{\tau}}$$

Substituting into the differential equation, we get:

$$-\frac{1}{\tau} Ae^{-\frac{t}{\tau}} + \frac{R_{\mathrm{R}}}{L_{\mathrm{R}}} Ae^{-\frac{t}{\tau}} = 0$$

We can divide by $Ae^{-\frac{t}{\tau}}$ since we know this will never be zero for finite time. Then, solving for τ:

$$\tau = \frac{L_{\mathrm{R}}}{R_{\mathrm{R}}}$$

The particular solution gives us a "particular" solution for the general differential equation:

$$\frac{d}{dt} i_{\mathrm{R_P}}(t) + \frac{R_{\mathrm{R}}}{L_{\mathrm{R}}} i_{\mathrm{R_P}}(t) = \frac{V_{\mathrm{S}}}{L_{\mathrm{R}}}$$

It should take the form:

$$i_{\mathrm{R_H}}(t) = K$$

Substituting this into the differential equation, we get:

$$\frac{R_{\mathrm{R}}}{L_{\mathrm{R}}} K = \frac{V_{\mathrm{S}}}{L_{\mathrm{R}}}$$

Solving for K, we get:

$$K = \frac{V_{\mathrm{S}}}{R_{\mathrm{R}}}$$

Combining the homogeneous and particular solution, out general solution to the differential equation is:

$$i_{\mathrm{R}}(t) = i_{\mathrm{R_H}}(t) + i_{\mathrm{R_P}}(t) = Ae^{-\frac{t}{\tau}} + \frac{V_{\mathrm{S}}}{R_{\mathrm{R}}}$$

where $\tau = \frac{L_R}{R_R}$. We can now solve for A with the initial condition by evaluating our function of $i_R(t)$ at the initial time

$$i_R(t=0) = Ae^{-\frac{0}{\tau}} + \frac{V_S}{R_R} = A + \frac{V_S}{R_R} = 0$$

So,

$$A = -\frac{V_S}{R_R}$$

Finally, our answer is:

$$i_R(t) = \frac{V_S}{R_R}(1 - e^{-\frac{t}{\tau}})$$

where $\tau = \frac{L_R}{R_R}$ as stated above.

(B) We know that i_R must be continuous since there are no impulse inputs driven into the circuit to cause discontinuities. Therefore, since i_R must be continuous at $t = T$, we can use the general result from Part (A) to find $i_R(t = T)$ for the "ON" MOSFET. We evaluate the above function of $i_R(t)$ at $t = T$ to obtain our initial condition:

$$i_R(t=T) = \frac{V_S}{R_R}(1 - e^{-\frac{T}{\tau_1}})$$

where we have relabeled the τ of Part (A) as τ_1. For $t > T$, the MOSFET drain-to-source connection opens and v_{DS} is no longer zero. Using KVL around the top loop:

$$v_L(t) + i_R(t)R_R + i_R(t)R_X = 0$$

Substituting our constitutive relation and putting the differential equation into standard form:

$$\frac{d}{dt}i_R(t) + \frac{R_R + R_X}{L_R}i_R(t) = 0$$

Since the right-hand side is zero, this differential equation only has a non-trivial homogeneous solution, since the particular solution is 0. Therefore, the total solution is the homogeneous solution. We assume a solution of the form:

$$i_R(t) = Ae^{-\frac{t}{\tau_2}}$$

Substituting our solution into the differential equation and solving for τ as in Part (A), we get:

$$\tau_2 = \frac{L_R}{R_R + R_X}$$

Using our initial condition, we can solve for A:

$$i_R(t=T) = Ae^{-\frac{T}{\tau_2}} = \frac{V_S}{R_R}(1 - e^{-\frac{T}{\tau_1}})$$

So,

$$A = \frac{V_S}{R_R}(1 - e^{-\frac{T}{\tau_1}})e^{\frac{T}{\tau_2}}$$

Our final answer is:

$$i_R(t > T) = \frac{V_S}{R_R}(1 - e^{-\frac{T}{\tau_1}})e^{-\frac{(t-T)}{\tau_2}}$$

where τ_1 and τ_2 are as above.

(C) Plugging in values, $T = \frac{5L_\mathrm{R}}{R_\mathrm{R}}$ and $R_\mathrm{X} = R_\mathrm{R}$:

$$i_\mathrm{R}(t > T) = \frac{V_\mathrm{S}}{R_\mathrm{R}}(1 - e^{-\frac{5L_\mathrm{R}}{R_\mathrm{R}}\frac{R_\mathrm{R}}{L_\mathrm{R}}})e^{\frac{5L_\mathrm{R}}{R_\mathrm{R}}\frac{R_\mathrm{R}+R_\mathrm{R}}{L_\mathrm{R}}}e^{-t\frac{R_\mathrm{R}+R_\mathrm{R}}{L_\mathrm{R}}}$$

Simplifying, we get:

$$i_\mathrm{R}(t > T) = \frac{V_\mathrm{S}}{R_\mathrm{R}}(1 - e^{-5})e^{10}e^{-2t\frac{R_\mathrm{R}}{L_\mathrm{R}}}$$

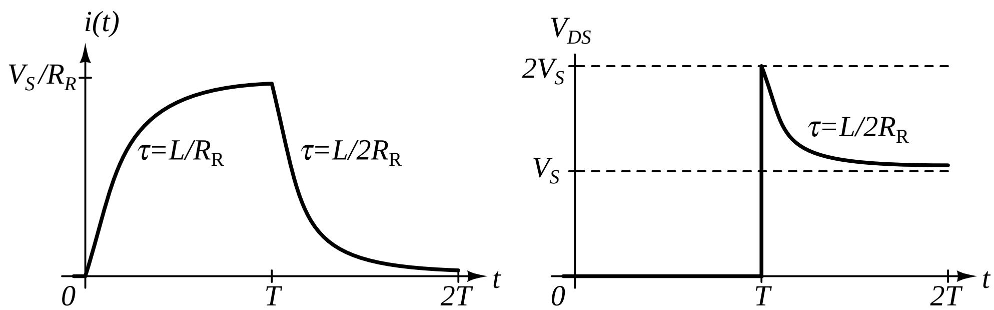
We can find $v_\mathrm{DS}(t > T)$ using our results for $i_\mathrm{R}(t)$ by applying $V = IR$ to the R_X resistor:

$$v_\mathrm{DS}(t) = V_\mathrm{S} + R_\mathrm{X} i_\mathrm{R}(t) = V_\mathrm{S} + R_\mathrm{X}\frac{V_\mathrm{S}}{R_\mathrm{R}}(1 - e^{-\frac{T}{\tau_1}})e^{-\frac{(t-T)}{\tau_2}}$$

Again, plugging in the given values, we obtain:

$$v_\mathrm{DS}(t > T) = V_\mathrm{S}[1 + (1 - e^{-5})e^{10}e^{-2t\frac{R_\mathrm{R}}{L_\mathrm{R}}}]$$

The figures below show our results on a time-domain graph.



(D) If $R_\mathrm{X} \to \infty$, the circuit will fail to function. When turned on, the relay performs as in Part (A), since the value of R_X does not show itself in the differential equation for (A). But, when the MOSFET turns off, there is no longer a path to ground provided by the MOSFET through its Drain to Source. As shown in Part (B), the continuous current at $t = T$ will begin to flow through the resistor R_X. If $R_\mathrm{X} \to \infty$, there will be no way for the current to sustain its continuity and the current drops to zero. This can also be seen by approximating the time constant in Part (B) as $\tau_2 = \frac{L_\mathrm{R}}{R_\mathrm{R}+R_\mathrm{X}} = 0$ which means the time delay for the current to reach its steady-state value is 0 and the graph takes on a step-like characteristic.

Physically, what would happen is that when the MOSFET opens and the inductor tries to keep the current continuous, there would be a build-up of potential at the point of least resistance to ground. Most likely, this would happen at the MOSFET drain, with the current flowing through a parasitic capacitance to ground. This would cause an immense electric field between the MOSFET's drain and source, which would in turn cause an electric breakdown that could destroy the MOSFET.

Problem 6.3: At $t = 0^-$, the networks shown below have zero initial state. That is, the capacitor voltage $v(t)$ and the inductor current $i(t)$ are both zero at $t = 0^-$. At $t = 0$, the voltage source produces an impulse of area Λ, and the current source produces an impulse of area Q.

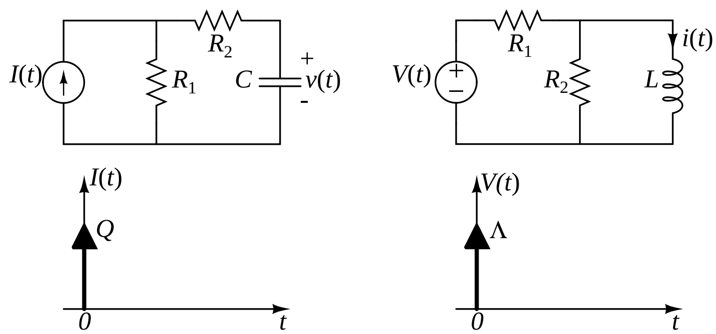
(A) Derive the differential equation that relates $v(t)$ to $I(t)$ and $i(t)$ to $V(t)$. Hint: consider using Thevenin or Norton equivalent networks to simplify the work.

(B) Find the capacitor voltage $v(t)$ and the inductor current $i(t)$ at both $t = 0^+$ and $t = \infty$. One way to find the states at $t = 0^+$ is to integrate the corresponding differential equations from $t = 0^-$ to $t = 0^+$ under the assumption that each state remains finite during that time; you should justify this assumption. Then, substitute the initial conditions at $t = 0^-$ into the results to determine the states at $t = 0^+$. Try to determine the states at $t = \infty$ through physical, rather than mathematical, reasoning.

(C) Next, find the time constant by which each state goes from its initial value at $t = 0^+$ to its final value at $t = \infty$.

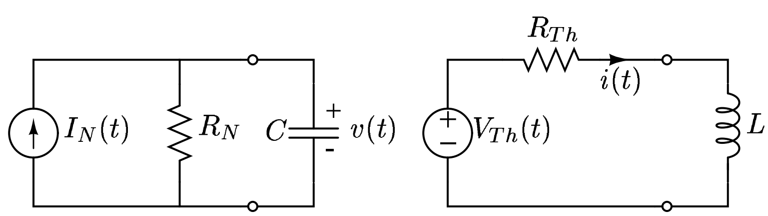
(D) Using the previous results, and without necessarily solving the differential equations directly, construct $v(t)$ and $i(t)$ for $t \geq 0$.

(E) Verify that the solutions to Part (D) are correct by substituting them into the differential equation found in Part (A).



Answer:

(A) First we reduced the circuit to their Norton and Thevenin equivalents respectively as shown below where



$$\begin{aligned} R_{\text{N}} &= R_1 + R_2 \\ I_{\text{N}}(t) &= \tfrac{R_1}{R_1+R_2} I(t) \end{aligned}$$

for the capacitor circuit and

$$R_{\text{Th}} = R_1 \parallel R_2$$

$$V_{\mathrm{Th}}(t) = \tfrac{R_2}{R_1+R_2}V(t)$$

for the inductor circuit.

By applying KCL to the capacitor circuit we obtain the following differential equation:

$$\begin{aligned} \frac{1}{R_{\mathrm{N}}}v(t) + C\frac{\mathrm{d}v(t)}{\mathrm{d}t} - I_{\mathrm{N}}(t) &= 0 \quad \text{or} \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{R_{\mathrm{N}}C}v(t) &= \tfrac{1}{C}I_{\mathrm{N}}(t) \end{aligned}$$

Similarly by applying KVL to the inductor circuit we obtain the following differential equation:

$$\begin{aligned} R_{\mathrm{Th}}i(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} - V_{\mathrm{Th}}(t) &= 0 \quad \text{or} \\ \frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R_{\mathrm{Th}}}{L}i(t) &= \tfrac{1}{L}V_{\mathrm{Th}}(t) \end{aligned}$$

(B) Integrating both sides of the differential equation from Part (A) for the capacitor circuit from $t = 0^-$ to $t = 0^+$ under the assumption that $v(t)$ is finite during that time we obtain

$$\begin{aligned} \int_{v(0^-)}^{v(0^+)} \mathrm{d}v + \frac{1}{R_{\mathrm{N}}C}\int_{t=0^-}^{t=0^+} v(t)\mathrm{d}t &= \tfrac{1}{C}\textstyle\int_{t=0^-}^{t=0^+} I_{\mathrm{N}}(t)\mathrm{d}t \\ v(0^+) - v(0^-)\nearrow^0 &= \tfrac{1}{C}\left(\tfrac{R_1}{R_1+R_2}\right)Q \\ v(0^+) &= \left(\tfrac{R_1}{R_1+R_2}\right)\tfrac{Q}{C} \quad \text{[Volts]} \end{aligned}$$

Note that Q has the unit of Coulombs.

As $t \to \infty$ the capacitor looks like open circuit. Also, $I_{\mathrm{N}} = 0$ at $t = \infty$. Thus, the voltage across R_{N} becomes zero or

$$v(\infty) = 0 \tag{1}$$

For the inductor, integrate both sides of the differential equation again from $t = 0^-$ to $t = 0^+$ under the assumption that $i(t)$ is finite during that time to obtain

$$\begin{aligned} \int_{t=0^-}^{t=0^+} \mathrm{d}i + \frac{R_{\mathrm{Th}}}{L}\int_{t=0^-}^{t=0^+} i(t)\mathrm{d}t &= \tfrac{1}{L}\textstyle\int_{t=0^-}^{t=0^+} V_{\mathrm{Th}}(t)\mathrm{d}t \\ i(0^+) - i(0^-)\nearrow^0 &= \tfrac{1}{L}\left(\tfrac{R_2}{R_1+R_2}\right)\Lambda \\ i(0^+) &= \left(\tfrac{R_2}{R_1+R_2}\right)\tfrac{\Lambda}{L} \quad \text{[Amps]} \end{aligned}$$

Note that Λ has the unit of Webers, or volt-seconds, in this case.

As $t \to \infty$ the inductor looks like short circuit. Also $V_{\mathrm{Th}} = 0$ at $t = 0$. Thus, the current through R_{Th} becomes zero or

$$i(\infty) = 0$$

(C) The time constants can be readily deduced as

$$\tau_{cap} = R_{\mathrm{N}}C \tag{2}$$

and

$$\tau_{ind} = \frac{L}{R_{\mathrm{Th}}} \tag{3}$$

from the Norton and Thevenin equivalent circuits respectively with the source set to zero.

(D) Putting the answers obtained in parts (**a**) and (**b**) together we construct $v(t)$ and $i(t)$ for $t \geq 0$ as

$$\begin{aligned} v(t) &= \left(\tfrac{R_1}{R_1+R_2}\right) \tfrac{Q}{C}\, e^{-\frac{t}{\tau_{cap}}} \quad \text{[Volts]} \\ i(t) &= \left(\tfrac{R_2}{R_1+R_2}\right) \tfrac{\Lambda}{L}\, e^{-\frac{t}{\tau_{ind}}} \quad \text{[Amps]} \end{aligned}$$

To find an expression for all time for the capacitor voltage and inductor current, both expressions can be multiplied by the unit step function, which is 0 for $t < 0$, and unity for $t \geq 0$. This gives

$$\begin{aligned} v(t) &= \left(\tfrac{R_1}{R_1+R_2}\right) \tfrac{Q}{C}\, e^{-\frac{t}{\tau_{cap}}} u_{-1}(t) \quad \text{[Volts]} \\ i(t) &= \left(\tfrac{R_2}{R_1+R_2}\right) \tfrac{\Lambda}{L}\, e^{-\frac{t}{\tau_{ind}}} u_{-1}(t) \quad \text{[Amps]} \end{aligned}$$

(E) For the capacitor circuit, substituting the answer from Part (D) into the differential equation from Part (A) and differentiating using the chain rule gives

$$\begin{aligned} \left(\frac{1}{C}\right) I_{\mathrm{N}}(t) = \quad & -\left(\tfrac{R_1}{R_1+R_2}\right)\left(\tfrac{1}{t_{cap}}\right) \tfrac{Q}{C} e^{-\frac{t}{\tau_{cap}}} u_{-1}(t) \\ & + \left(\tfrac{R_1}{R_1+R_2}\right) \tfrac{Q}{C}\, e^{-\frac{t}{\tau_{cap}}} u_0(t) \\ & + \left(\tfrac{1}{R_{\mathrm{N}}C}\right)\left(\tfrac{R_1}{R_1+R_2}\right) \tfrac{Q}{C}\, e^{-\frac{t}{\tau_{cap}}} u_{-1}(t) \end{aligned}$$

Noting that $\tau_{cap} = R_{\mathrm{N}}C$ the first and last terms on the right hand side cancel. Also, $u_0(t)$ only has a value at $t = 0$, so the exponential in the second term can be evaluated at $t = 0$ to be one. Substituting in $I_{\mathrm{N}}(t) = \frac{R_1}{R_1+R_2} Q u_0(t)$ gives

$$\left(\frac{Q}{C}\right)\left(\frac{R_1}{R_1+R_2}\right) u_0(t) = \left(\frac{Q}{C}\right)\left(\frac{R_1}{R_1+R_2}\right) u_0(t)$$

Similarly, for the inductor circuit, substituting the answer from Part (D) into the differential equation from Part (A) and differentiating using the chain rule gives

$$\begin{aligned} \left(\frac{1}{L}\right) V_{\mathrm{Th}}(t) = \quad & -\left(\tfrac{R_2}{R_1+R_2}\right)\left(\tfrac{1}{t_{ind}}\right) \tfrac{\Lambda}{L} e^{-\frac{t}{\tau_{ind}}} u_{-1}(t) \\ & + \left(\tfrac{R_2}{R_1+R_2}\right) \tfrac{\Lambda}{L}\, e^{-\frac{t}{\tau_{ind}}} u_0(t) \\ & + \left(\tfrac{R_{\mathrm{Th}}}{L}\right)\left(\tfrac{R_2}{R_1+R_2}\right) \tfrac{\Lambda}{L}\, e^{-\frac{t}{\tau_{ind}}} u_{-1}(t) \end{aligned}$$

Noting that $\tau_{ind} = \frac{L}{R_{\text{Th}}}$ the first and last terms on the right hand side cancel. Also, $u_0(t)$ only has a value at $t = 0$, so the exponential in the second term can be evaluated at $t = 0$ to be one. Substituting in $V_{\text{Th}}(t) = \frac{R_2}{R_1+R_2}\Lambda u_0(t)$ gives

$$\left(\frac{R_2}{R_1 + R_2}\right)\frac{\Lambda}{L}u_0(t) = \left(\frac{R_2}{R_1 + R_2}\right)\frac{\Lambda}{L}u_0(t)$$