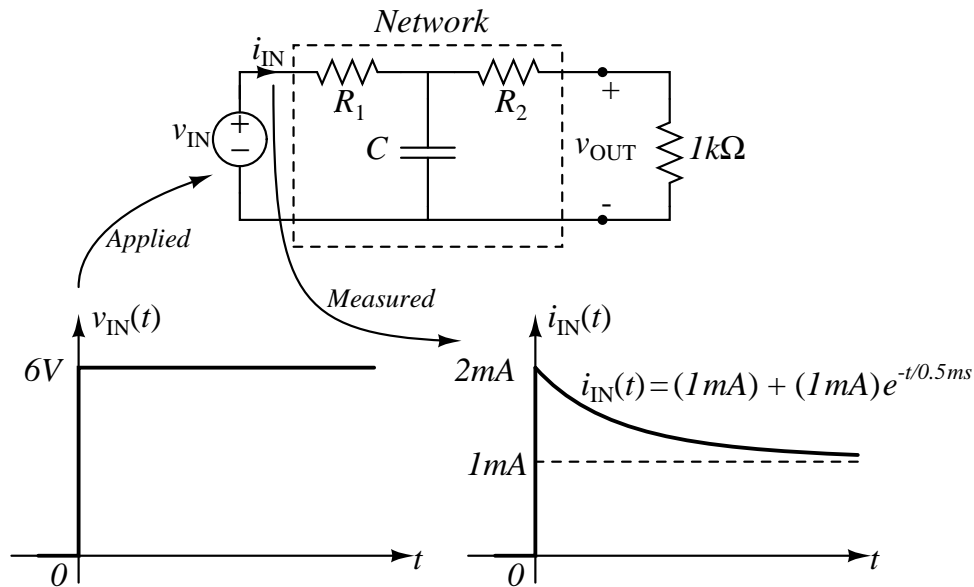


Massachusetts Institute of Technology  
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits  
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Homework #7 Solutions

**Exercise 7.1:** The network shown below has two ports, two resistors and one capacitor. The resistor values  $R_1$  and  $R_2$ , and the capacitor value  $C$ , are all unknown. Also shown below is the result of an experiment performed on the network in which one port is driven with the voltage step  $v_{IN}$  at  $t = 0$ , and the other port is loaded with a  $1\text{ k}\Omega$  resistor. Using the experimental result, which consists of the measured current  $i_{IN}$ , find the values  $R_1$ ,  $R_2$  and  $C$ . Also, find the voltage  $v_{OUT}$  across the load resistor for  $t \geq 0$ . In doing so, assume that the network capacitor is uncharged prior to  $t = 0$ .



**Answer:** Since the network capacitor is uncharged prior to  $t = 0$ , and no impulses are applied to the circuit, the capacitor voltage must be continuous at  $t = 0$ . This means that  $v_C(0^-) = v_C(0^+) = 0$ .

At  $t = 0$  the capacitor looks like a short circuit. Thus, the circuit reduces to  $v_{in}(t)$  in series with  $R_1$ , where  $v_{in} = 6[\text{V}]$  and  $i_{in} = 2[\text{mA}]$ . Therefore,  $R_1 = 3[\text{k}\Omega]$ .

As  $t \rightarrow \infty$  the capacitor looks like an open circuit. Thus, the circuit reduces to  $v_{in}(t)$  in series with  $R_1$ ,  $R_2$ , and the  $1\text{K}\Omega$  resistor, where  $v_{in} = 6[\text{V}]$  and  $i_{in} = 1[\text{mA}]$ . Therefore,  $R_2 = 2[\text{k}\Omega]$

Looking in from the capacitor port the Thevenin resistance is given by

$$R_{\text{eq}} = R_1 \parallel (R_2 + R_3), \quad \text{where } R_3 = 1\text{k}\Omega$$

Therefore,

$$R_{\text{eq}} = \frac{3}{2} [\text{k}\Omega]$$

We can relate the time constant to the product of capacitance and Thevenin resistance of the circuit, i.e.

$$\begin{aligned} \tau &= 0.5 \text{ [ms]} \\ &= R_{\text{eq}} C \\ \Rightarrow C &= \frac{1}{3} \mu\text{F} \end{aligned}$$

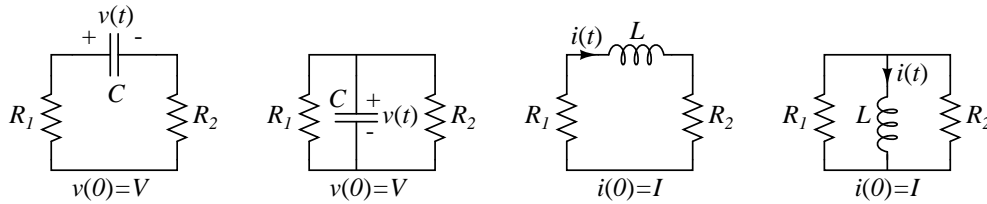
Finally,  $v_{\text{OUT}}$  is given as

$$v_{\text{OUT}} = \frac{1}{3} v_C$$

where

$$\begin{aligned} v_c(t) &= V_{in}(t) - R_1 i_{in}(t) \\ &= 3 \left( 1 - e^{-\frac{t}{0.5\text{ms}}} \right) [\text{V}] \end{aligned}$$

**Exercise 7.2:** Each network shown below has a non-zero initial state at  $t = 0$ , as indicated. Find the network states for  $t \geq 0$ . Hint: what equivalent resistance is in parallel with each capacitor or inductor, and what decay time results from this combination?



**Answer:** Each circuit above is a simple first-order system with no forcing term (or input). The expression for the state variable (either  $v(t)$  or  $i(t)$ ) is then of the form

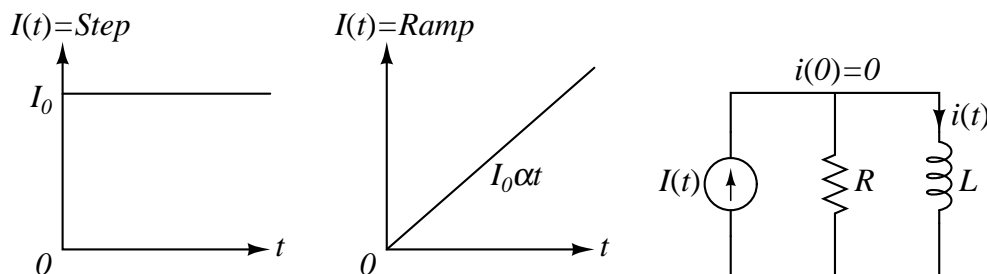
$$Ae^{-t/\tau}$$

Because, for each of the 4 circuits given, the capacitor voltage or inductor current at  $t = \infty$  is 0. We also know that the decay time,  $\tau$ , is  $CR_{\text{eq}}$  for the capacitor circuits, and  $L/R_{\text{eq}}$  for inductor circuits. Using these facts, it is easy to find the network states for  $t \geq 0$ .

$$\begin{aligned} v(t) &= V e^{-\frac{t}{C(R_1+R_2)}} \\ v(t) &= V e^{-\frac{t}{C \frac{R_1 R_2}{R_1+R_2}}} \\ i(t) &= I e^{-\frac{t(R_1+R_2)}{L}} \\ i(t) &= I e^{-\frac{t \frac{R_1 R_2}{R_1+R_2}}{L}} \end{aligned}$$

**Problem 7.1:** This problem examines the relation between transient responses of linear systems. The network shown below is first driven by a current step at  $t = 0$ , then driven by a current ramp at  $t = 0$ , and finally driven by the current step plus the current ramp at  $t = 0$ . In the first two cases, the inductor has zero initial current, as indicated.

- (A) Find the inductor current  $i(t)$  in response to the current step shown below. Assume that  $i(0) = 0$ .
- (B) Find the inductor current  $i(t)$  in response to the current ramp shown below. Again assume that  $i(0) = 0$ .
- (C) The step input can be constructed from the ramp input according to  $I_{\text{Step}}(t) = \frac{1}{\alpha} \frac{d}{dt} I_{\text{Ramp}}(t)$ . Show that their respective responses are related in a similar manner.
- (D) Would the result from Part C hold if  $i(0) \neq 0$ ? Why or why not?
- (E) Finally, find the inductor current  $i(t)$  in response to the current step plus the current ramp, that is to  $I(t) = I_o(1 + \alpha t)$  for  $t \geq 0$ . This time assume that  $i(0) = i_o$ . Hint: think superposition.



**Answer:** The differential equation for the circuit can be written as

$$\frac{L}{R} \frac{di}{dt} + i(t) = I(t)$$

- (A) Solving the differential equation above for a particular solution and the homogeneous one given the step input we obtain

$$\begin{aligned} i_{\text{P}}(t) &= I_o \\ i_{\text{H}}(t) &= -I_o e^{-\frac{t}{(L/R)}} \end{aligned}$$

Therefore, the current response to the step is given by

$$i(t) = I_o \left( 1 - e^{-\frac{t}{(L/R)}} \right) u_{-1}(t)$$

- (B) Guess the particular solution to the differential equation, given the ramp input, to be

$$i_{\text{P}}(t) = At + B$$

Substitution of this solution into the differential equation then yields

$$\frac{L}{R} A + At + B = \alpha I_o t$$

Since this equation must hold for all time,

$$\begin{aligned} A &= \alpha I_o \\ B &= -\frac{L}{R}\alpha I_o \end{aligned}$$

Solving for the homogeneous solution to match the initial condition leads to

$$i_H(t) = \frac{L}{R}\alpha I_o e^{-\frac{t}{(L/R)}}$$

Therefore, the current response to the ramp is given by

$$i(t) = \left[ \alpha I_o t - \frac{L}{R}\alpha I_o \left( 1 - e^{-\frac{t}{(L/R)}} \right) \right] u_{-1}(t)$$

(C) From the solution to Part (B)

$$\begin{aligned} \frac{1}{\alpha} \frac{d}{dt} I_{\text{Ramp}}(t) &= \frac{1}{\alpha} \frac{d}{dt} \left[ \alpha I_o t - \frac{L}{R}\alpha I_o \left( 1 - e^{-\frac{t}{(L/R)}} \right) \right] u_{-1}(t) \\ &= \left[ I_o - I_o e^{-\frac{t}{(L/R)}} \right] u_{-1}(t) + \left[ I_o t - \frac{L}{R} I_o \left( 1 - e^{-\frac{t}{(L/R)}} \right) \right] u_0(t) \\ &= I_o \left( 1 - e^{-\frac{t}{(L/R)}} \right) u_{-1}(t) = I_{\text{Step}}(t) \end{aligned}$$

(D) The result does not hold if  $i(0) \neq 0$ . When we differentiate the input, we only differentiate the external source, since the initial condition is an internal source. However, when we differentiate the output, we differentiate both the homogeneous and the particular solutions. Mathematically, the term corresponding to initial condition of  $I_{\text{Ramp}}(t)$  gets multiplied by the time constant when differentiated to yield  $I_{\text{Step}}(t)$ .

(E) Guess the particular solution to the differential equation, given the ramp plus the step input, to be

$$i_P(t) = At + B$$

Then the differential equation reduces to

$$\frac{L}{R}A + At + B = I_o + \alpha I_o t$$

Thus,

$$\begin{aligned} A &= \alpha I_o \\ B &= I_o \left( 1 - \frac{L}{R}\alpha \right) \\ i_P(t) &= I_o \left( 1 + \alpha t - \frac{L}{R}\alpha \right) \end{aligned}$$

Note that this particular solution is just the sum of the two particular solutions to each separate input. The same answer could have been found using *superposition* of the answers from Parts (A) and (B), since the input is the superposition of the corresponding step and ramp inputs.

The homogeneous solution does not change for a change in input, so it is still of the form

$$i_H(t) = Ae^{-\frac{t}{(L/R)}}$$

where A is a scaling factor used to satisfy the initial condition.

So, for  $t \geq 0$ , we have

$$i(t) = I_o \left( 1 + \alpha t - \frac{L}{R} \alpha \right) + Ae^{-\frac{t}{(L/R)}}$$

Given that  $i(0) = i_o$  we can write

$$i_o = I_o \left( 1 - \frac{L}{R} \alpha \right) + A$$

Which we can solve for A to find

$$A = i_o - I_o \left( 1 - \frac{L}{R} \alpha \right)$$

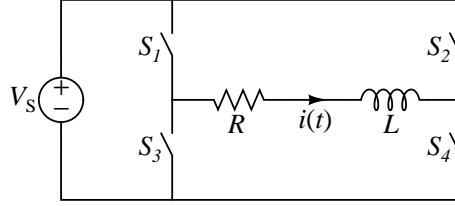
The final expression for  $i(t)$  for  $t \geq 0$  is then

$$i(t) = I_o \left( 1 + \alpha t - \frac{L}{R} \alpha \right) + \left( i_o - I_o \left( 1 - \frac{L}{R} \alpha \right) \right) e^{-\frac{t}{(L/R)}}$$

Note that the resulting homogeneous solution, like the particular solution, is just the sum of the two homogeneous solutions for the two separate inputs, plus the decaying initial condition. We could have used superposition and just summed the two answers from Parts (A) and (B), and added in the decaying  $i_o$  term.

**Problem 7.2:** The circuit shown below can be used to regulate the current through an inductor. Typical applications include the regulation of currents in motors, solenoids and loud speakers, all of which have inductive windings. We will analyze the circuit assuming that it operates in a cyclic manner with switching period  $T$ . During the first part of each period, which lasts for a duration  $DT$ , switches S1 and S4 are on while switches S2 and S3 are off. During the second part of each switching period, which lasts for a duration  $(1 - D)T$ , switches S1 and S4 are off while switches S2 and S3 are on. Note that  $0 \leq D \leq 1$ .

- (A) Assume that  $D$  is constant and that the circuit has been operating long enough to reach a cyclic steady state by  $t = 0$ , at which point a new switching period begins. In terms of the unknown  $i(0)$ , determine  $i(t)$  for  $0 \leq t \leq T$ .
- (B) Use your result from Part (A), and the fact that the circuit operates in a cyclic steady state to determine  $i(0)$ . Note that with this result, and that from Part (A),  $i(t)$  is completely determined.
- (C) Find the average value of  $i(t)$  over the period  $0 \leq t \leq T$ . Hint: is it necessary to average the result from Part A, or is there a faster method to find the average?
- (D) Suppose that the circuit has been operating with  $D \equiv D_1$  for a time long enough to reach a cyclic steady state by  $t = 0$ . Suppose that  $D$  switches to  $D = D_2$  at  $t = 0$ , just as a new switching period begins. In this case, determine  $i(t)$  for  $t \geq 0$ . Hint: can you use your result from Parts (A) and (B) as a particular solution over the interval  $0 \leq t$ ?



**Answer:**

- (A) Over the period  $0 \leq t \leq DT$ ,  $V_S$  is applied to the resistor and inductor. Over the period  $DT \leq t \leq (1-D)T$ ,  $-V_S$  is applied to the resistor and inductor.

We know that starting at  $t = 0$ , the current through the inductor will decay exponentially from its initial value  $i(0)$  towards  $\frac{V_S}{R}$  with a time constant  $\tau = \frac{L}{R}$ . This gives

$$i(t, 0 \leq t \leq DT) = \frac{V_S}{R} + \left( i(0) - \frac{V_S}{R} \right) e^{-t \frac{R}{L}}$$

Notice that when  $t = 0$ , this expression evaluates to  $i(0)$ , and as  $t \rightarrow \infty$ ,  $i(t) \rightarrow \frac{V_S}{R}$ .

For the second part of the switching cycle, when  $DT \leq t \leq T$ , the current decays exponentially from its value  $i(DT)$  towards  $-\frac{V_S}{R}$ . Substituting in for  $i(DT)$  from the equation above gives

$$\begin{aligned} i(t, DT \leq t \leq T) &= -\frac{V_S}{R} + \left( \frac{V_S}{R} + i(DT) \right) e^{-(t-DT) \frac{R}{L}} \\ &= -\frac{V_S}{R} + \left( \frac{V_S}{R} + \left[ \frac{V_S}{R} + \left( i(0) - \frac{V_S}{R} \right) e^{-DT \frac{R}{L}} \right] \right) e^{-(t-DT) \frac{R}{L}} \\ &= -\frac{V_S}{R} + 2 \frac{V_S}{R} e^{-(t-DT) \frac{R}{L}} + \left( i(0) - \frac{V_S}{R} \right) e^{-t \frac{R}{L}} \end{aligned}$$

For the purposes of this solution, let us define  $i_L(t; D)$  to be

$$i_L(t; D) = \begin{cases} \frac{V_S}{R} + \left( i(0) - \frac{V_S}{R} \right) e^{-t \frac{R}{L}} & \text{when } nT \leq t \leq (D+n)T \\ -\frac{V_S}{R} + 2 \frac{V_S}{R} e^{-(t-DT) \frac{R}{L}} + \left( i(0) - \frac{V_S}{R} \right) e^{-t \frac{R}{L}} & \text{when } (D+n)T \leq t \leq (n+1)T \end{cases}$$

Where  $n$  is some positive integer that indexes each switching cycle. The  $n$  is needed because the piece-wise solution above must apply to all switching cycles for  $t \geq 0$ , not just the first one.

- (B) If the circuit is operating in a cyclic steady state, then  $i(T)$  must be equal to  $i(0)$ . Setting the expression found in Part (A) equal to  $i(0)$  and solving for yields

$$\begin{aligned} i(0) &= i(T) \\ i(0) &= -\frac{V_S}{R} + 2 \frac{V_S}{R} e^{-(T-DT) \frac{R}{L}} + \left( i(0) - \frac{V_S}{R} \right) e^{-T \frac{R}{L}} \\ (1 - e^{-T \frac{R}{L}}) i(0) &= \frac{V_S}{R} \left( 2e^{(D-1)T \frac{R}{L}} - (1 + e^{-T \frac{R}{L}}) \right) \\ i(0) &= \frac{V_S}{R} \left( \frac{2e^{(D-1)T \frac{R}{L}} - (1 + e^{-T \frac{R}{L}})}{1 - e^{-T \frac{R}{L}}} \right) \end{aligned}$$

We can check to see if this answer makes sense by evaluating it at  $D = 1$ ,  $D = 0$ , and  $D = .5$ , assuming  $T \ll \frac{L}{R}$ . When  $D = 1$ , we expect  $i(t)$ , and hence  $i(0)$ , to be  $\frac{V_S}{R}$ . Likewise, when  $D = 0$ , we expect  $i(0) = -\frac{V_S}{R}$ . At  $D = .5$  with a fast switching cycle (the period,  $T$ , is much shorter than the time constant  $\frac{L}{R}$ ), we should expect  $i(0) = 0$ . All of these predictions are satisfied by the equation above.

For the purposes of this solution, let us define  $i_o(D)$  to be

$$i_o(D) = \frac{V_S}{R} \left( \frac{2e^{(D-1)T\frac{R}{L}} - (1 + e^{-T\frac{R}{L}})}{1 - e^{-T\frac{R}{L}}} \right)$$

- (C) To find the average value of  $i(t)$  over the period  $0 \leq t \leq T$  is not necessary to go through the trouble of averaging the result from Part (A). Consider, for a moment, the average voltage applied across the inductor. We know that  $v_L(t) = L\frac{di}{dt}$ . Averaging this over the period  $T$  requires taking the integral of both sides with respect to  $t$ , and dividing the result by  $T$ . However, the integral on the right evaluates to  $L(i(T) - i(0))$ , which we know to be 0, because  $i(T) = i(0)$  by definition. The average voltage across an inductor whose terminal current is in cyclic steady state is 0. This means that the average voltage applied to the resistor/inductor pair must appear entirely across the resistor. The average voltage applied is  $\frac{V_S DT + (-V_S)(1-D)T}{T} = V_S(2D - 1)$ . The average value of  $i(t)$  is then

$$\overline{i(t)} = \frac{V_S(2D - 1)}{R}$$

We expect this to be 0 for  $D = .5$ ,  $\frac{V_S}{R}$  for  $D = 1$ , and  $-\frac{V_S}{R}$  for  $D = 0$ , which are all true.

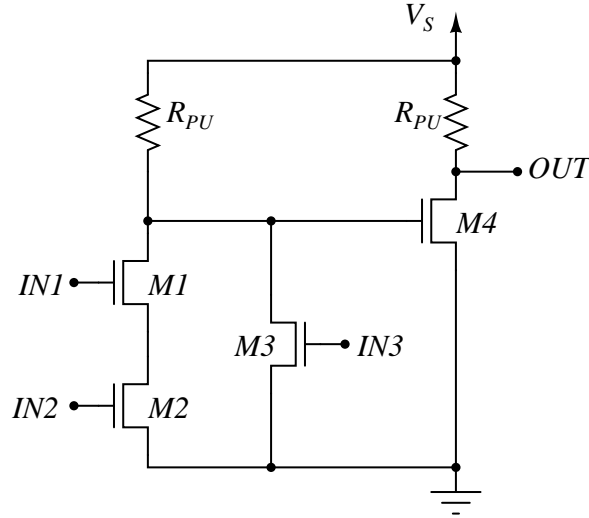
- (D) The particular solution for this input,  $D_2$ , is just the answer from Part (A) evaluated at  $D = D_2$ , which is  $i_L(t; D_2)$ . The homogeneous solution is just an exponential decay times some constant, used to satisfy the initial conditions. In this case, the initial condition is  $i(0) = i_o(D_1)$ . Because  $i_L(0; D_2) = i_o(D_2)$ , the homogeneous solution is just

$$i_H(t) = (i_o(D_1) - i_o(D_2)) e^{-tR/L}$$

The full solution is the sum of the particular and homogeneous solutions:

$$i(t) = i_L(t; D_2) + (i_o(D_1) - i_o(D_2)) e^{-tR/L}$$

**Problem 7.3:** Consider the digital logic circuit from Problem 3.1. Model each MOSFET with the switch-resistor model, and let the on-state resistance  $R_{ON}$  satisfy  $R_{ON} \ll R_{PU}$ . Further assume that MOSFET M4 has a gate-to-source capacitance  $C_{GS}$ . Given that the inputs IN1, IN2 and IN3 cycle through the combinations 000, 001, 010, 011, 100, 101, 110, 111, determine the average power dissipated by the logic circuit. Assume that each input combination is held for the period  $T$  with  $T \gg R_{PU}C_{GS}$ . Make appropriate simplifications based on the inequalities for  $R_{ON}$  and  $T$ .



**Answer:** Because  $R_{ON} \ll R_{PU}$ , we can approximate  $R_{PU} + nR_{ON} \approx R_{PU}$ , where  $n$  is small ( $\leq 2$  for this problem). Also, we can say that the capacitor  $C_{GS}$  will fully charge or discharge in each switching period  $T$ , because  $T \gg R_{PU}C_{GS}$ .

First, let's consider the static power dissipated by the pullup resistors  $R_{PU}$ . When the gate of M4 is pulled high, the output is low, and only the  $R_{PU}$  on the right is dissipating power. When the gate of M4 is low, only the  $R_{PU}$  on the left is dissipating power. The circuit must be in one of these two states all the time, so it is constantly dissipating  $\frac{V_S^2}{R_{PU}}$  Watts.

Now, let us consider the dynamic power dissipated. Dynamic power comes from the fact that the Gate-Source capacitance,  $C_{GS}$  of M4 must be charged and discharged every time the output changes state. If we look at the sequence of inputs in the truth table below, we see that the output changes state 6 times, which means the capacitor charges and discharges 3 times.

IN1	IN2	IN3	OUT
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Every time M4's  $C_{GS}$  is charged,  $\frac{1}{2}C_{GS}V_S^2$  Joules flow into the capacitor through  $R_{PU}$  on the left.



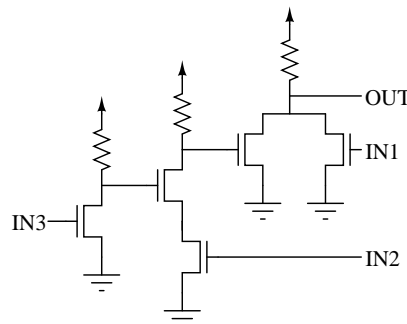
At the same time, an equal amount of energy is lost in  $R_{PU}$ . Every time it is discharged, the energy previously stored in  $C_{GS}$  is dissipated in the  $R_{ONS}$  of M1, M2, and/or M3. The total energy dissipated, then, is  $\frac{1}{2}C_{GS}V_S^2 * [\text{number of changes}]$  Joules. There are six logic transitions in our case. The power is just the energy dissipated per unit time, which is, for this circuit and combination of inputs  $\frac{3}{8T}C_{GS}V_S^2$  Joules.

The total power dissipation is the sum of the dynamic and static dissipations, which is

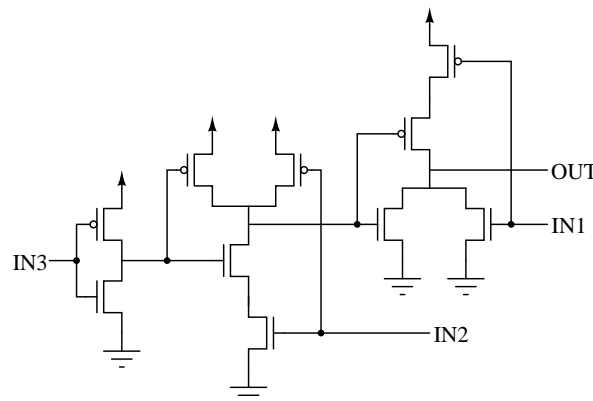
$$P_D = V_S^2 \left( \frac{1}{R_{PU}} + \frac{3C_{GS}}{8T} \right)$$

**Problem 7.4:** Develop a CMOS logic circuit that implements  $OUT = \overline{IN1 + \overline{IN2} \cdot \overline{IN3}}$ . Hint, review your answer to Exercise 4.1 first.

**Answer:** The CMOS logic circuit can be constructed from the answer to Exercise 4.1. Replace each pullup resistor with complementary logic. An inverter's pullup resistor, then, is replaced with a PFET. A NOR gate's pullup resistor, then, is replaced with with two PFETs in series (because the NFETs are in parallel). A NAND gate's pullup resistor is replaced with two PFETs in parallel, because it's NFETs are in series. This leads to the circuit:



NMOS Circuit from Exercise 4.1



CMOS Equivalent Circuit