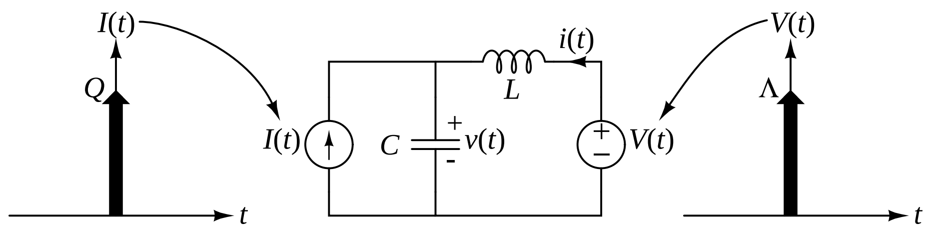
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Spring 2002

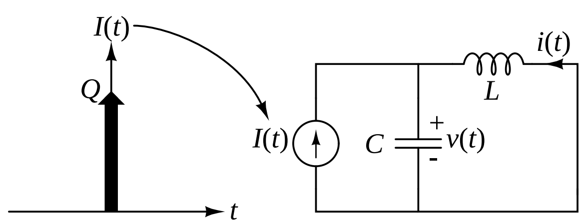
Homework #9 Solutions

Exercise 9.1: Determine $i(t)$ and $v(t)$ for $t \geq 0^+$ in the network shown below. Note that the inductor and capacitor both have nonzero states at $t = 0$. Hint: use superposition to establish initial conditions at $t = 0^+$.



Answer: Because the circuit is made of linear elements, we can find the system response to each input separately, and combine them using superposition to find the total response of the circuit to both inputs.

First, consider the circuit with only the current source, as shown in the figure below.



For short periods of time, namely the duration of the impulse, the capacitor behaves like a short circuit. Consequently, all of the current impulse goes through it. Therefore, Q Coulombs is injected into the capacitor, stepping its voltage at $t = 0^+$ to

$$v(0^+) = \frac{Q}{C} \text{ Volts}$$

We know that the circuit will ring with a natural frequency $\omega = \sqrt{LC}$. Because there are no inputs into the system besides the impulse, and there are no elements that dissipate power, the resulting response will be a sinusoid with amplitude $\frac{Q}{C}$. The capacitor voltage starts at it's maximum, so we find

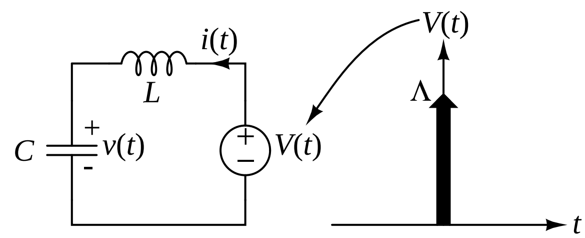
$$v_{\mathrm{I}}(t) = \frac{Q}{C} \cos\left(\frac{t}{\sqrt{LC}}\right)$$

We will define v_{I} to be the portion of the total response contributed by the current source.

Using the capacitor's v-i relationship, $i_{\mathrm{C}} = C\frac{d}{dt}v_{\mathrm{C}}$, we can find the current component for the circuit driven only by the current source:

$$\begin{aligned} i_{\mathrm{I}}(t) &= C\frac{d}{dt}v_{\mathrm{C}} \\ &= C\frac{Q}{C}\frac{d}{dt}\cos\left(\frac{t}{\sqrt{LC}}\right) \\ &= -\frac{Q}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \end{aligned}$$

Next, consider the circuit with only the voltage source as shown in the figure below.



We can follow a similar analysis to find the $v_{\mathrm{V}}(t)$ and $i_{\mathrm{V}}(t)$ components from the voltage impulse input.

For short periods of time, namely the duration of the impulse, the inductor bahaves like an open circuit, and so the voltage impulse falls across it. Therefore, Λ Volt·seconds is injected into the inductor, stepping its current at $t = 0^+$ to $i(0^+) = \frac{\Lambda}{L}$.

After the impulse occurs the circuit is reduced to a single inductor and capacitor. The inductor current starts at it's maximum and begins to fall, so we expect it to be a cosine function, just like the capacitor's voltage with the current source input.

$$i_{\mathrm{V}}(t) = \frac{\Lambda}{L}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

Using the inductor's v-i relationship, $v_{\mathrm{L}} = L\frac{d}{dt}i_{\mathrm{L}}$, we can find the voltage component for the circuit driven only by the voltage source. Note that because of the way v and i are defined in the circuit, $v = -v_{\mathrm{L}}$.

$$\begin{aligned} v_{\mathrm{V}}(t) &= -L\frac{d}{dt}i_{\mathrm{L}} \\ &= -L\frac{\Lambda}{L}\frac{d}{dt}\cos\left(\frac{t}{\sqrt{LC}}\right) \\ &= \frac{\Lambda}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \end{aligned}$$

The total response to both inputs is just the sum of the individual responses. This yields

$$
\begin{aligned}
v(t) &= \frac{Q}{C}\cos\left(\frac{t}{\sqrt{LC}}\right) + \frac{\Lambda}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \\
i(t) &= -\frac{Q}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{\Lambda}{L}\cos\left(\frac{t}{\sqrt{LC}}\right)
\end{aligned}
$$

Problem 9.1: The network shown below is driven in steady state by the sinusoidal input voltage $v_{\mathrm{I}}(t) = V_{\mathrm{I}}\cos(\omega t)$. The output of the network is the voltage $v_{\mathrm{O}}(t)$, which takes the form $v_{\mathrm{O}}(t) = V_{\mathrm{O}}\cos(\omega t + \phi)$. Find V_{O} and ϕ as functions of ω as follows.

(A) Using the Taylor Series expansions for e^x, $\cos(x)$ and $\sin(x)$, show that $e^{jx} = \cos(x) + j\sin(x)$. Following this, recognize that $\cos(x) = \Re\{e^{jx}\}$.

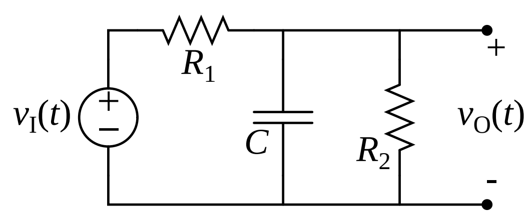
(B) Show that $A + Bj = \sqrt{A^2 + B^2}e^{j\arctan(B/A)}$. Thus, the magnitude and phase of $A + Bj$ are $\sqrt{A^2 + B^2}$ and $\arctan(B/A)$, respectively.

(C) Find a differential equation that can be solved for $v_{\mathrm{O}}(t)$ given $v_{\mathrm{I}}(t)$.

(D) Following Part A, let $v_{\mathrm{I}}(t) = V_{\mathrm{I}}e^{j\omega t}$. Also, let $v_{\mathrm{O}}(t) = \hat{V}_{\mathrm{O}}e^{j\omega t}$ where $\hat{V}_{\mathrm{O}}$ is a complex function of the circuit parameters, ω and V_{I}. With these substitutions, use the differential equation to find $\hat{V}_{\mathrm{O}}$.

(E) Following Parts A and B, first express v_{O} from Part (D) in the form $v_{\mathrm{O}}(t) = |\hat{V}_{\mathrm{O}}|e^{j(\omega t + \angle\hat{V}_{\mathrm{O}})}$, and determine $|\hat{V}_{\mathrm{O}}|$ and $\angle\hat{V}_{\mathrm{O}}$ as functions of the circuit parameters, ω and V_{I}. Then, find V_{O} and ϕ for the original cosine input, again both as functions of the circuit parameters, ω and V_{I}.

(F) Sketch and clearly label $V_{\mathrm{O}}/V_{\mathrm{I}}$ and ϕ as functions of ω. Identify the low-frequency and high-frequency asymptotes on the sketch.



Answer:

(A) Recall that the Taylor series for an exponential is

$$
e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots
$$

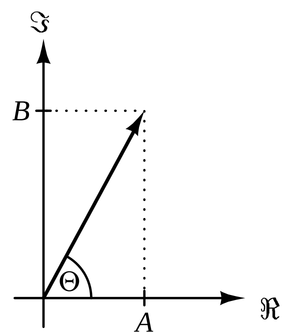
If we substitute $t = jx$ into this series, we get

$$
e^{jx} = \sum_{n=0}^{\infty}\frac{(jx)^n}{n!}
$$

$$= 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \frac{jx^5}{5!} - \cdots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

The two series in the last line are the Taylor series polynomials for $\cos(x)$ and $\sin(x)$, respectively. Therefore, $e^{jx} = \cos(x) + j\sin(x)$, and $\cos(x) = \Re\{e^{jx}\}$.

(B) Consider the imaginary plane, with an arbitrary complex number $A + Bj$ plotted as a vector from the origin, as shown below.



The length of the above vector is $\sqrt{A^2 + B^2}$ by the Pythagorean theorem. It points in the direction of $e^{j\theta}$, where $\tan(\theta) = \frac{B}{A}$. Therefore, we can write that $A + Bj = \sqrt{A^2 + B^2}e^{j\arctan(B/A)}$.

(C) Using the node method we can write

$$\frac{v_I - v_O}{R_1} = C\frac{dv_O}{dt} + \frac{v_O}{R_2}$$

Rearranging terms yields the first order differential equation relating v_O to v_I below.

$$\frac{dv_O}{dt} + \frac{v_O}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{v_I}{R_1 C}$$

(D) Substituting in $v_I = V_I e^{j\omega t}$ and $v_O = \hat{V_O} e^{j\omega t}$ into the differential equation above, we find that all of the $e^{j\omega t}$ terms cancel, and we are left with

$$j\omega\hat{V_O} + \frac{\hat{V_O}}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_I}{R_1 C}$$

Renaming $\frac{R_1 R_2}{R_1+R_2} = R_P$ and rearranging some terms yields

$$\hat{V_O} = \frac{V_I \frac{R_P}{R_1}}{j\omega R_P C + 1}$$

(E) The magnitude of $\hat{V_O}$ is the magnitude of the numerator divided by the magnitude of the denominator.

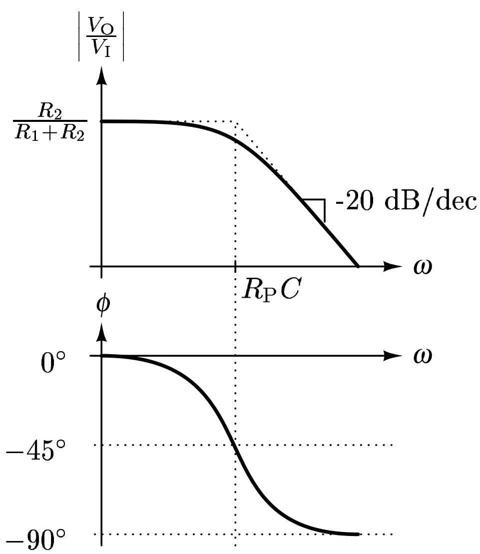
$$|\hat{V_O}| = \frac{V_I \frac{R_C}{R_1}}{\sqrt{1 + (\omega R_P C)^2}}$$

The phase of $\hat{V_O}$ is just the phase of the numerator minus the phase of the denominator.

$$\angle \hat{V_O} = 0 - \arctan(\omega R_P C) = -\arctan(\omega R_P C)$$

The output constants V_O and ϕ are just the magnitude and the phase of the output, which we found above. That is to say $V_O = |\hat{V_O}|$ and $\phi = \angle \hat{V_O}$.

(F) For $\omega \ll R_P C$, $\frac{V_O}{V_I} \approx \frac{R_2}{R_1+R_2}$. As ω increases far past $R_P C$, $\frac{V_O}{V_I} \rightarrow \frac{V_I}{\omega R_1 C}$. For $\omega \ll R_P C$, $\phi \approx 0$. As ω reaches $R_P C$, $\phi \rightarrow -45°$. When $\omega \gg R_P C$, $\phi \approx -90°$. These are graphed below. Note that $\left|\frac{\hat{V_O}}{V_I}\right|$ is graphed on log-log paper, and that $\angle \frac{\hat{V_O}}{V_I}$ is graphed on semi-logx paper.



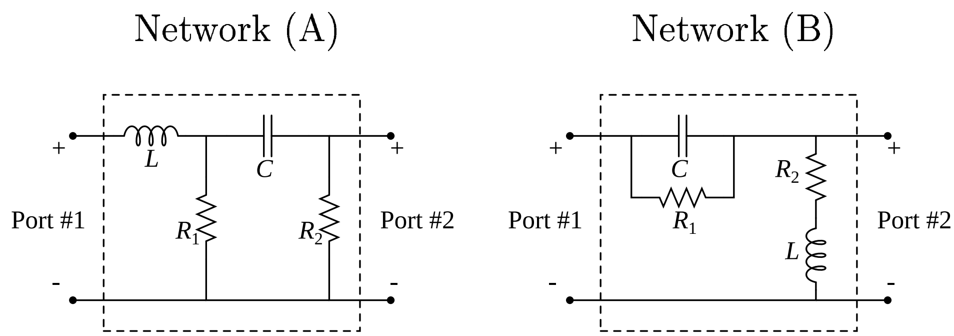
Problem 9.2: This problem concerns the sinusoidal-steady-state behavior of the networks shown below, both of which have two ports.

(A) Determine the impedance of each network as viewed into Port #1 under the assumption that Port #2 is open.

(B) Assume that Port #1 of each network is driven in sinusoidal steady state by the voltage $V_1 \cos(\omega t)$, and that Port #2 is open. Determine the current into the positive terminal of each network at Port #1. Express the current in the form $I_1 \cos(\omega t + \phi_1)$ where I_1 is an amplitude and ϕ_1 is a phase angle.

(C) Assume that Port #1 of each network is again driven in sinusoidal steady state by the voltage $V_1 \cos(\omega t)$, and that Port #2 is again open. Determine the voltage which appears at Port #2. Express the voltage in the form $V_2 \cos(\omega t + \phi_2)$ where V_2 is an amplitude and ϕ_2 is a phase angle.

Note that the results of this problem are useful when completing the pre-lab exercises to Lab #3.



Answer: Since we are dealing with circuits in sinusoidal steady state (SSS), we know that we are only looking for a particular solution to our differential equation system. In other words, all transients have died away and therefore we no longer concern ourselves with initial conditions. Our input will take the form $e^{j\omega t}$, where ω is the characteristic frequency of our complex input. For real inputs, one would merely take the cosine or sine term of the complex exponent, using Euler's law. With out SSS input assumption, we are allowed to solve our system using the impedance technique. Knowing the impedance transformations for linear resistors, capacitors, and inductors, we can redraw the two networks in impedances which can be easily reduced using out resistor laws. For a resistor of value R, the impedance is:

$$Z = R$$

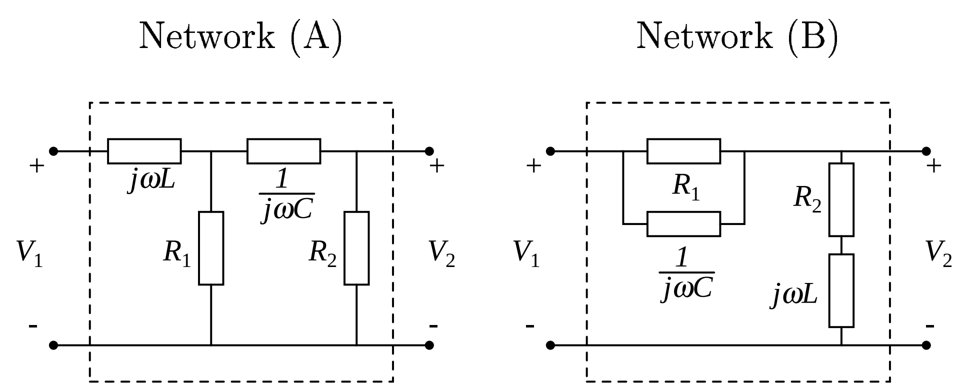
For a capacitor of value C, the impedance is:

$$Z = \frac{1}{j\omega C}$$

For an inductor of value L the impedance is:

$$Z = j\omega L$$

Our redraw circuits are shown in the figure below.



(A) Given Port #2 is open, i.e. nothing is connecting the two nodes together except the circuit elements in the network. We can determine a total impedance with our parallel and series resistor laws. In Network (A), the equivalent impedance is:

$$\tilde{Z} = j\omega L + R_1 || \left(\frac{1}{j\omega C} + R_2 \right) = \frac{R_1 - \omega^2 (R_1 + R_2) LC + j\omega (L + R_1 R_2 C)}{1 + j\omega (R_1 + R_2) C}$$

For Network (B) we have:

$$\tilde{Z} = R_1 || \left(\frac{1}{j\omega C}\right) + R_2 + j\omega L = \frac{R_1 + R_2 - \omega^2 RLC + j\omega(R_1R_2C + L)}{1 + j\omega R_1 C}$$

(B) The current going into the positive terminal of Port #1 is easy to determine since the impedance defines the ratio of the complex port voltage over the complex current. We assume the complex voltage across Port #1 is $V_1(t) = V_1 e^{j\omega t}$ where V_1 is a real value. Therefore, we know that the current will have the form $I_1(t) = \tilde{I}_1 e^{j\omega t}$ where $\tilde{I}_1$ is assumed to be complex. For Network (A) we have:

$$\tilde{Z} = \frac{V_1(t)}{I_1(t)} = \frac{V_1 e^{j\omega t}}{\tilde{I}_1 e^{j\omega t}}$$

or

$$\tilde{I}_1 = \frac{V_1}{\tilde{Z}} = V_1 \frac{1 + j\omega(R_1 + R_2)C}{R_1 - \omega^2(R_1 + R_2)LC + j\omega(L + R_1R_2C)}$$

We can represent $\tilde{I}_1$ in polar form:

$$\tilde{I}_1 = |\tilde{I}_1| e^{j\angle\tilde{I}_1}$$

where the magnitude is equal to the magnitude of the numerator divided by the magnitude of the denominator:

$$|\tilde{I}_1| = V_1 \sqrt{\frac{1 + [\omega(R_1 + R_2)C]^2}{[R_1 - \omega^2(R_1 + R_2)LC]^2 + [\omega(L + R_1R_2C)]^2}}$$

and the phase is equal to the phase of the numerator minus the phase of the denominator:

$$\angle\tilde{I}_1 = \tan^{-1}(\omega(R_1 + R_2)C) - \tan^{-1}\left(\frac{\omega(L + R_1R_2C)}{R_1 - \omega^2(R_1 + R_2)LC}\right)$$

Our total complex answer turns out to be:

$$i(t) = \tilde{I}_1 e^{j\omega t} = |\tilde{I}_1| e^{j\angle\tilde{I}_1} e^{j\omega t} = |\tilde{I}_1| e^{j(\omega t + \angle\tilde{I}_1)}$$

The real input, $V_1(t) = V_1 \cos(\omega t)$, is found by taking the real part of the complex input $V_1(t) V_1 e^{j\omega t}$. Therefore, the real output, $\tilde{I}(t) = |\tilde{I}_1| \cos(\omega t + \angle\tilde{I}_1)$, can be found by taking the real part of the complex input response:

$$i(t) = I_1 \cos(\omega t + \phi_1)$$

where $I_1 \equiv |\tilde{I}_1|$ and $\phi_1 \equiv \angle\tilde{I}_1$ as derived above.

For Network (B), we use the same process to obtain a similar solution:

$$\tilde{I}_1 = \frac{V_1}{\tilde{Z}} = V_1 \frac{1 + j\omega R_1 C}{R_1 + R_2 - \omega^2 R_1 LC + j\omega(R_1R_2C + L)}$$

so

$$i(t) = I_1 \cos(\omega t + \phi_1)$$

where

$$I_1 \equiv |\tilde{I}_1| = V_1 \sqrt{\frac{1 + (\omega R_1 C)^2}{(R_1 + R_2 - \omega^2 R_1 LC)^2 + [\omega(R_1R_2C + L)]^2}}$$

and

$$\phi_1 \equiv \angle\tilde{I}_1 = tan^{-1}(\omega R_1 C) - tan^{-1}\left(\frac{\omega(R_1R_2C + L)}{R_1 + R_2 - \omega^2 R_1 LC}\right)$$

(C) We now consider a voltage source connecting the Port #1 terminals. The source is driven at $V(t) = V_1 \cos(\omega t)$ which can be transformed into the complex input $V(t) = V_1 e^{j\omega t}$. For Network (A), we can combine the capacitor and the two resistors together as

$$\tilde{Z}_1 = R_1 || \left(\frac{1}{j\omega C} + R_2 \right) = \frac{R_1 \left(\frac{1}{j\omega C} + R_2 \right)}{\frac{1}{j\omega C} + R_1 + R_2} = \frac{R_1(1 + j\omega R_2 C)}{1 + j\omega(R_1 + R_2)C}$$

So, the node voltage located between the inductor and capacitor with respect to the negative input terminal can be found using a voltage divider relation:

$$\tilde{V}_{\text{node}} = V_1 \frac{\tilde{Z}_1}{\tilde{Z}_1 + j\omega L} = V_1 \frac{\frac{R_1(1+j\omega R_2 C)}{1+j\omega(R_1+R_2)C}}{\frac{R_1(1+j\omega R_2 C)}{1+j\omega(R_1+R_2)C} + j\omega L} = V_1 \frac{R_1(1 + j\omega R_2 C)}{R_1(1 + j\omega R_2 C) + j\omega L[1 + j\omega(R_1 + R_2)C]}$$

Returning to the full impedance circuit model, we can now use another voltage divider relation to find the Port #2 voltage:

$$\tilde{V}_2 = \tilde{V}_{\text{node}} \frac{R_2}{\frac{1}{j\omega C} + R_2} = \tilde{V}_{\text{node}} \frac{j\omega R_2 C}{1 + j\omega R_2 C} = V_1 \frac{j\omega R_1 R_2 C}{R_1 - \omega^2(R_1 + R_2)LC + j\omega(L + R_1 R_2 C)}$$

with the complex output response being

$$V_2(t) = \tilde{V}_2 e^{j\omega t} = |\tilde{V}_2| e^{j\angle\tilde{V}_2} e^{j\omega t} = |\tilde{V}_2| e^{j(\omega t + \angle\tilde{V}_2)}$$

where

$$|\tilde{V}_2| = V_1 \frac{\omega R_1 R_2 C}{\sqrt{[R_1 - \omega^2(R_1 + R_2)LC]^2 + [\omega(L + R_1 R_2 C)]^2}}$$

and

$$\angle\tilde{V}_2 = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_1 - \omega^2(R_1 + R_2)LC} \right)$$

So the output response of Port #2 with respect to the real cosine input stated above is

$$V_2(t) = V_2 \cos(\omega t + \phi_2)$$

where $V_2 \equiv |\tilde{V}_2|$ and $\phi_2 \equiv \angle\tilde{V}_2$.

For Network (B), we apply the same principles to solve the voltage divider:

$$\tilde{V}_2 = V_1 \frac{R_2 + j\omega L}{R_2 + j\omega L + R_1 || \left(\frac{1}{j\omega C} \right)} = \frac{(R_2 + j\omega L)(1 + j\omega R_1 C)}{R_1 + R_2 - \omega^2 R_1 LC + j\omega(L + R_1 R_2 C)}$$

So the response to the real cosine input is

$$V_2(t) = V_2 \cos(\omega t + \phi_2)$$

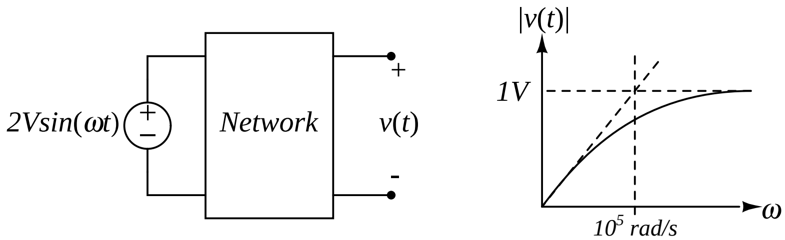
where

$$V_2 \equiv \tilde{V}_2 = \sqrt{\frac{[R_2^2 + (\omega L)^2][1 + (\omega R_1 C)^2]}{(R_1 + R_2 - \omega^2 R_1 LC)^2 + [\omega(L + R_1 R_2 C)]^2}}$$

and

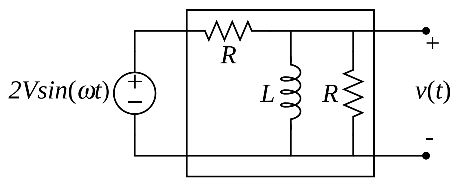
$$\phi_2 \equiv \angle\tilde{V}_2 = \tan^{-1}\left(\frac{\omega L}{R_2} \right) + \tan^{-1}(\omega R_1 C) - \tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_1 + R_2 - \omega^2 R_1 LC} \right)$$

Problem 9.3: Using a 1-mH inductor and two resistors, design a two-port network that has the following input-output relation in the sinusoidal steady state. Note that the relation is defined for the case of an unloaded, or open-circuited, output port.



Answer: At high frequencies, the inductor will function as an open circuit. We know that the amplitude of $v(t)$ for large ω is half of the input amplitude. This means that at high frequencies, the network should look like a resistive divider with two equal resistors.

At low frequencies, the inductor functions as a short, and $v(t)$ is small. This means the the inductor must be connected across the output terminals, as shown in the figure below.



Using impedances to find $v(t)$ yields

$$|v(t)| = \frac{\omega L(2 \text{ Volts})}{\sqrt{R^2 + 4L^2\omega^2}}$$

The breakpoint of this function occurs when $R^2 = 4\omega^2 L^2$. Consequently we find

$$\omega = \frac{R}{2L} = 10^5 \text{ rad/s}$$

Substituting 1 mH in for L yields

$$R = 200\Omega$$

The resulting circuit is