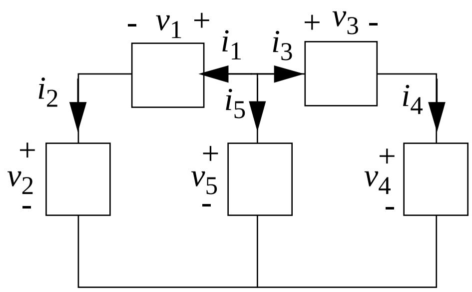
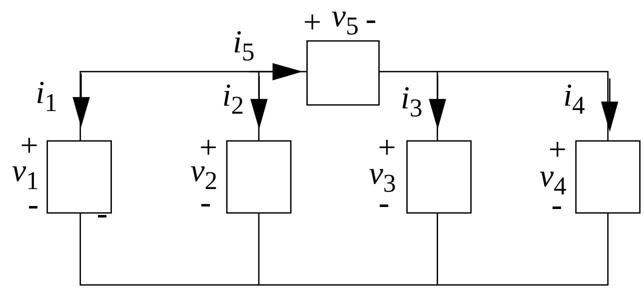
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Spring 2002

Homework #1 Solutions

Exercise 1.1: Both networks shown below have five branches, each with defined voltages and currents. Several of the branch voltages and currents are specified. Using KVL and KCL, find the unknown branch voltages and currents.





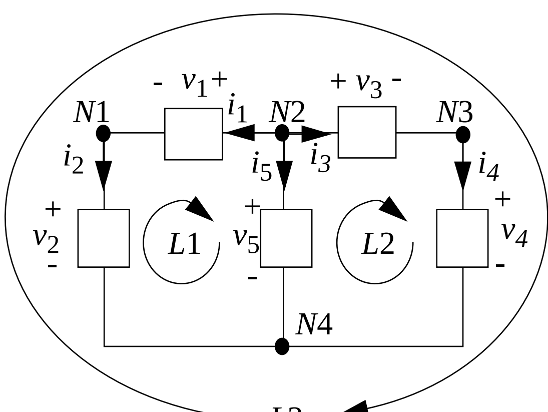
v_1=1V v_5=5V v_4=2V

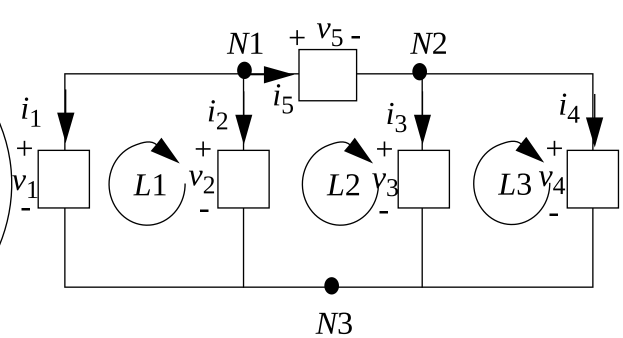
i_2=2A i_3=1A

i_1=1A i_3=3A i_5=5A

v_2=2V v_3=4V

Answer: We wish to find all the missing values in each circuit. The circuit on the left has 3 loops and 4 nodes. We are missing 5 values: v_2, v_3, i_1, i_4, and i_5. We need at least 5 independent equations containing these 5 unknowns. To solve for these values, it is usually helpful to look for equations containing as few unknowns as possible (well, except for those containing 0 unknowns).





Let us try a KVL equation. Looking at the leftmost loop L1 (containing elements 1, 2, and 5), and starting at the bottom node N4, we trace a path to add up the voltages:

$$-v_2 - v_1 + v_5 = 0$$

Note that in tracing the path of the loop, we take the polarity associated with the first sign of the voltage defined for each element. Substituting the known values, we get $-v_2 - 1V + 5V = 0$, or $v_2 = 4V$.

Now let us try the KCL equation for the upper left node N1 (the junction between elements 1 and 2). We shall sum the currents going into the node:

$$i_1 - i_2 = 0$$

Substituting the known values, we get $i_1 - 2A = 0$, or $i_1 = 2A$. This is rather obvious, because elements 1 and 2 are in series, so they have the same current.

We can repeat this operation for the node N3: $i_3 - i_4 = 0$, so substituting the known i_3, we get $i_4 = 1A$.

To find i_5, we can choose either node N2 (connecting elements 1, 3, and 5) or node N4 (connecting elements 2, 4, and 5). Let us work with the lower node. Do not be confused by the shape of the node; the bottom line of this circuit is all one node. Finding the equation for this node, we have $i_2 + i_5 + i_4 = 0$. Substituting the known currents, we get $2A + i_5 + 1A = 0$, or $i_5 = -3A$. We could check our results using the redundant equation associated with node N2.

We must still find v_3. Choosing the outside loop L3 (containing elements 1, 2, 3, and 4), we get the equation $-v_2 - v_1 + v_3 + v_4 = 0$. Substituting the knowns, we have $-4V - 1V + v_3 + 2V = 0$, or $v_3 = 3V$. This circuit is now "solved".

The circuit on the right is the *dual* topology to that on the left. This means that the KVL equations for one circuit match the KCL equations for the other, and vice versa. Instead of 4 nodes and 3 loops, we now have 4 loops and 3 nodes.

We again have 5 unknowns to solve for. However, we can observe some very simple equations; namely the far left loop L1 and far right loop L3. These both correspond to elements in parallel. Elements in parallel (across the same two nodes) must have the same voltage, so we have $v_1 = v_2 = 2V$ (KVL, left loop) and $v_4 = v_3 = 4V$ (KVL, right loop).

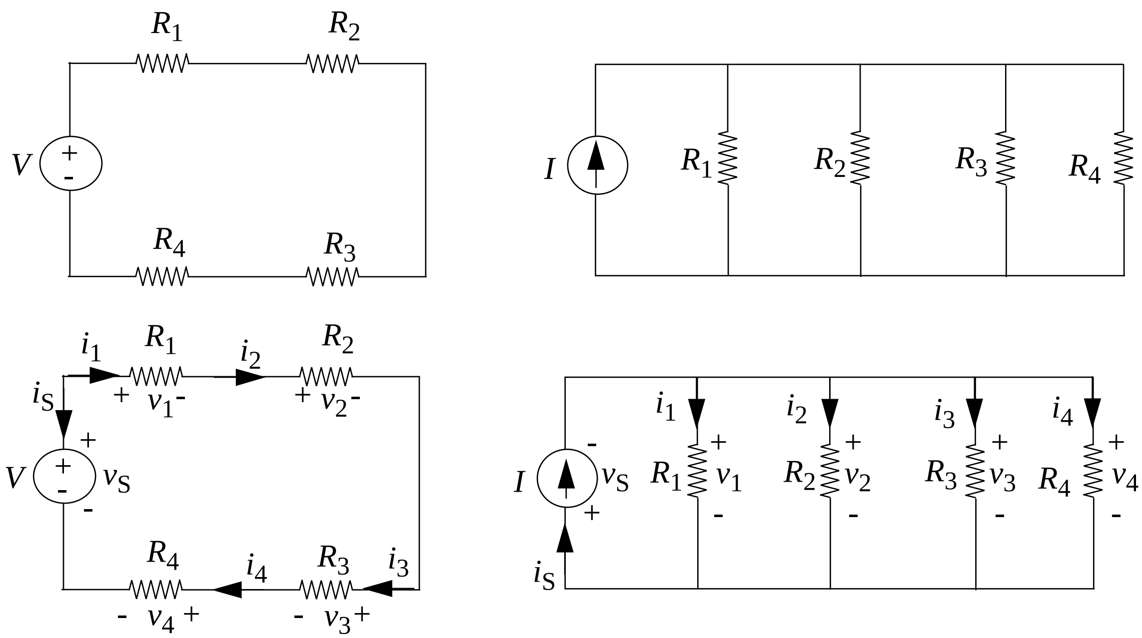
To find v_5, we perform KVL with either the middle or outside loops (they both contain element 5). Choosing the middle loop, we have $-v_2 + v_5 + v_3 = 0$. Substituting knowns and collecting, we have $v_5 = v_2 - v_3 = 2V - 4V = -2V$.

We now need to find i_2 and i_4. We will solve for i_2 using KCL at the upper-left node N1: $-i_1 - i_2 - i_5 = 0$. So, $i_2 = -i_1 - i_5 = -1A - 5A = -6A$. We can solve for i_4 using either of the other two nodes. Using KCL at the upper right node, we get $i_5 - i_3 - i_4 = 0$, resulting in $i_4 = i_5 - i_3 = 5A - 3A = 2A$.

Exercise 1.2: For both networks shown below, find the voltage across, and the current through each element in the network. Be sure to make the polarity of the voltages and currents clear.

Answer: Here are some labeled circuits:

Let us first look at the left circuit, which consists of 1 big loop. Identifying this loop is what makes the problem fall apart quickly. This is a simple series circuit, so the current through each



element must be the same. Let us define this current to be i. Using our voltages and currents labeled in the figure, we have $-i_S = i_1 = i_2 = i_3 = i_4 = i$. KVL around our single loop gives us $v_S = v_1 + v_2 + v_3 + v_4$. Substituting from our constituent relations, we get $V = i_1R_1 + i_2R_2 + i_3R_3 + i_4R_4$. But the current is the same for each component, so we have $V = i(R_1 + R_2 + R_3 + R_4)$. This should be clear because we know resistors in series add. So,

$$i = \frac{V}{R_1 + R_2 + R_3 + R_4}$$

$$v_1 = iR_1 = \frac{R_1}{R_1 + R_2 + R_3 + R_4}V$$

and similarly, $v_2 = iR_2$, $v_3 = iR_3$, and $v_4 = iR_4$.

It is worth noting that the resistor voltage values correspond to a *voltage divider.*

The right circuit is the dual of the left, and has only two nodes. This is a simple parallel combination, so each element has the same voltage across it. Let us define this voltage to be v. From our shown voltages and currents, we have $-v_S = v_1 = v_2 = v_3 = v_4 = v$. KCL at one of the nodes gives us $i_S = i_1 + i_2 + i_3 + i_4$. Substituting from our constituent relations we get $I = \frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3} + \frac{v_4}{R_4} = v(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4})$. We solve to get

$$v = \frac{R_1R_2R_3R_4}{R_2R_3R_4 + R_1R_3R_4 + R_1R_2R_4 + R_1R_2R_3}I$$

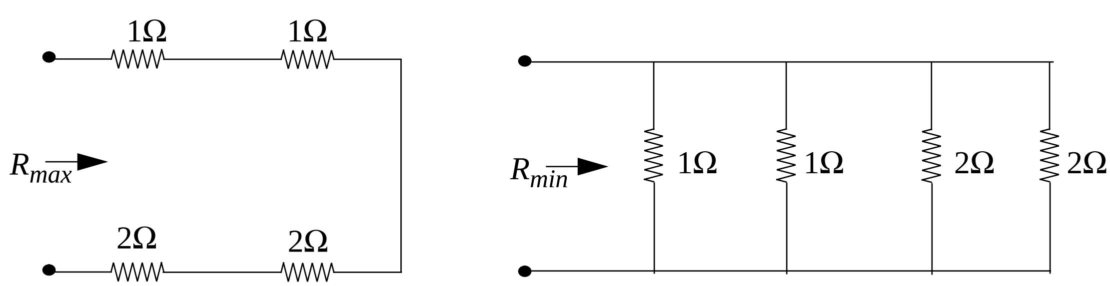
$$i_1 = \frac{v}{R_1} = \frac{R_2R_3R_4}{R_2R_3R_4 + R_1R_3R_4 + R_1R_2R_4 + R_1R_2R_3}I$$

and similarly, $i_2 = \frac{v}{R_2}$, $i_3 = \frac{v}{R_3}$, and $i_4 = \frac{v}{R_4}$.

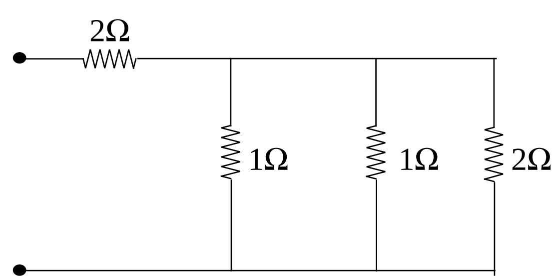
Note that the resistor currents correspond to a *current divider.* Also, it is worth trying this problem with conductances rather than resistances.

Exercise 1.3: A collection of four resistors includes two 1-Ω resistors and two 2-Ω resistors. What is the largest-valued resistor that can be synthesized using one or more resistors from the collection? What is the smallest-valued resistor that can be synthesized using one or more resistors from the collection? How can a 2.4-Ω resistor be synthesized using one or more resistors from the collection?

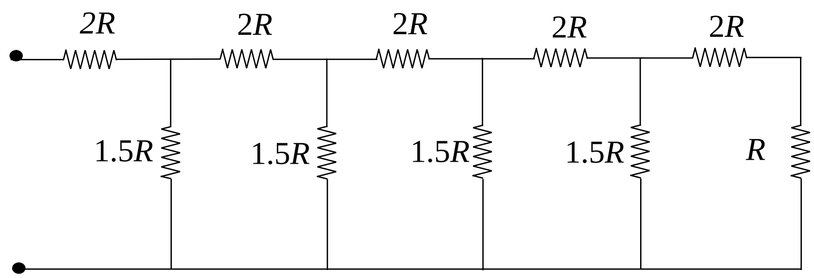
Answer: Resistors in series combine to form a resistor larger than (in fact, equal to the sum of) each of the component resistances. So, the largest (finite) resistance we can create with these four resistors is their series combination, or $2\Omega + 2\Omega + 1\Omega + 1\Omega = 6\Omega$. Conversely (by duality, via conductances), resistors in parallel combine to form a resistor smaller than each of the component resistances. So, the smallest (nonzero) resistance we can create from these components is $2\Omega \parallel 2\Omega \parallel 1\Omega \parallel 1\Omega = \frac{2\cdot2\cdot1\cdot1}{2\cdot1\cdot1+2\cdot1\cdot1+2\cdot2\cdot1+2\cdot2\cdot1}\Omega = \frac{1}{3}\Omega$. The combinations are shown below.

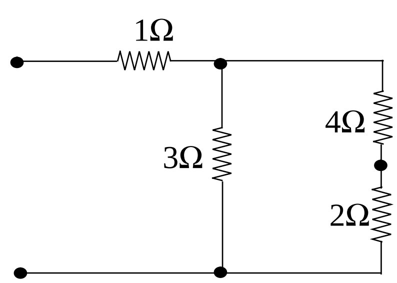


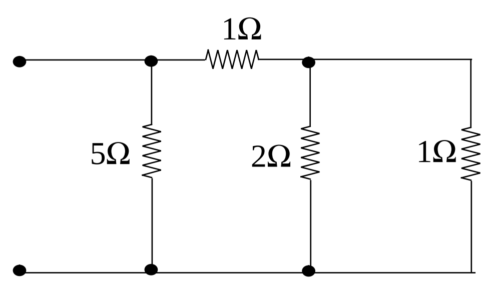
To form a 2.4Ω resistor from our components, it is perhaps easiest to use trial and error. If we take both 1Ω resistors and one 2Ω resistor, and put the three in parallel, we get $2\Omega \parallel 1\Omega \parallel 1\Omega = \frac{2\cdot1\cdot1}{2\cdot1+2\cdot1+1\cdot1}\Omega = 0.4\Omega$. We have a 2Ω resistor remaining. So, by putting the 2Ω resistor in series with the 0.4Ω one we created, we arrive at 2.4Ω. This is shown below.



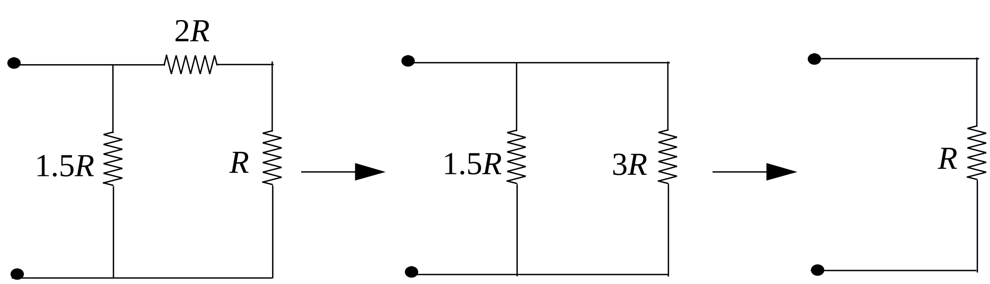
Problem 1.1: Find the equivalent resistance of the following networks as viewed from their ports.





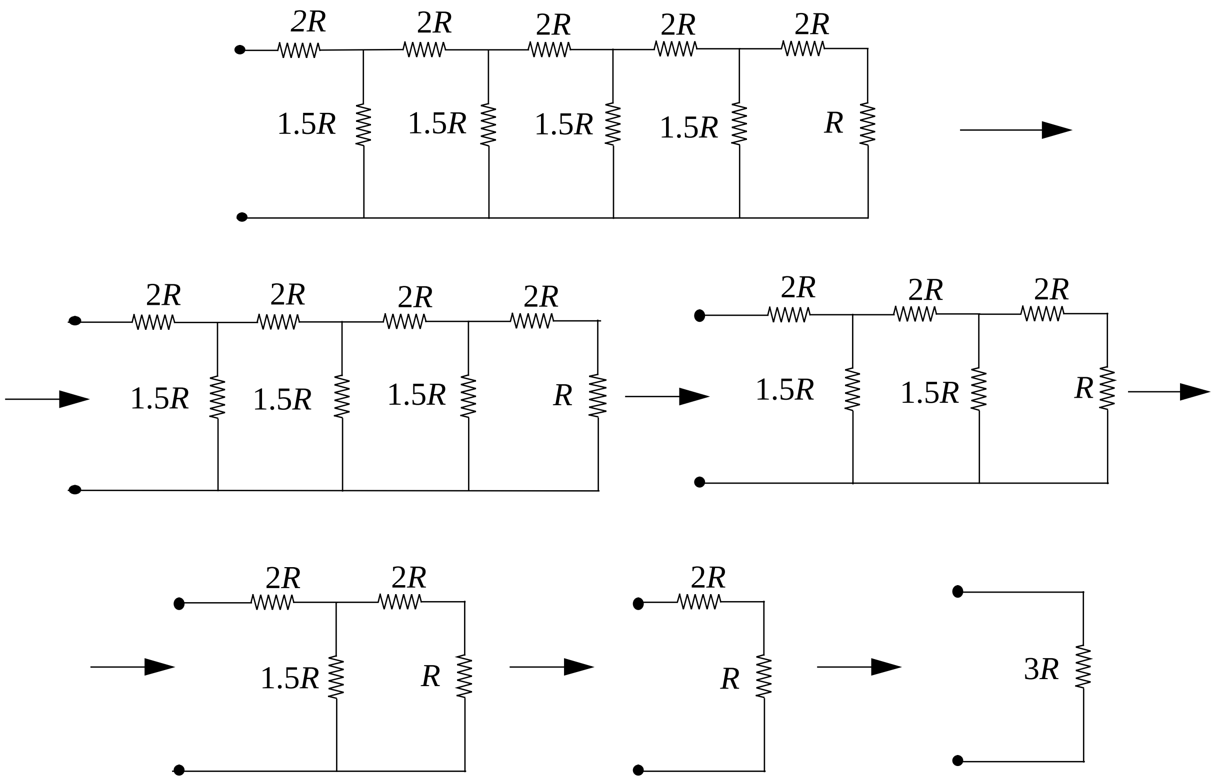


Answer: For the first circuit, it is easier to look first at the combination of the rightmost 3 resistors:



Here we have a series combination of the $2R$ and R resistors, forming a $3R$ resistor. Then we have a parallel combination $1.5R \parallel 3R = \frac{(1.5R)(3R)}{1.5R+3R} = R$.

Next, we notice that our overall network collapses as we reduce these little blocks on the right, as shown in the figure below:



So, we see the network has an equivalent resistance of $R_{eq1} = 3R$ as seen from the terminals.

Looking now at the second circuit, we solve this by viewing series (+) and parallel (||) combining as operators (like multiplication and addition, but missing some properties). We have a 4Ω and 2Ω series combination. There is a 3Ω in parallel with that, and then we have a 1Ω in series with the whole mess. So,

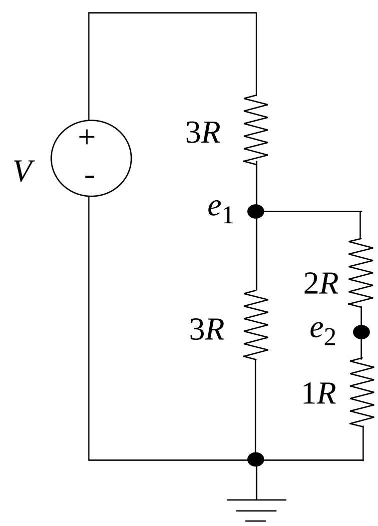
$$R_{eq2} = 1\Omega + [3\Omega \parallel (4\Omega + 2\Omega)] = 1\Omega + (3\Omega \parallel 6\Omega) = 1\Omega + 2\Omega = 3\Omega.$$

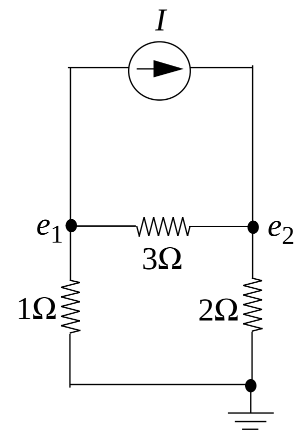
We solve the last circuit in a similar manner:

$$R_{eq3} = 5\Omega \parallel [1\Omega + (2\Omega \parallel 1\Omega)] = 5\Omega \parallel (1\Omega + \frac{2}{3}\Omega) = 5\Omega \parallel \frac{5}{3}\Omega = \frac{5}{4}\Omega.$$

Problem 1.2: Using node analysis, find the unknown node voltages in both networks shown below. Note the definition of the ground node in both networks.

Answer: Nodal analysis is a highly structured method for solving circuits, where all nodes are labeled with variables representing voltages relative to ground, and KCL equations are written using only these variables and the constituent relations of the components between the variables. This strict procedure makes nodal analysis well suited to software implementation.





The circuits have already been labeled with node voltage variables. However, looking at the first circuit, we notice that only two of four nodes have been labeled! As humans, we like to avoid extra work. So we eliminate the node we chose as ground, and avoid writing the redundant equation $e_3 = 0$. Now, the ideal voltage source would result in the trivial equation $e_4 = V$. We will simply use V in our node equations.

Writing the node equations based on the sum of currents exiting each node equalling zero, we get

$$\frac{(e_1 - V)}{3R} + \frac{(e_1 - 0)}{3R} + \frac{(e_1 - e_2)}{2R} = 0$$
$$\frac{(e_2 - e_1)}{2R} + \frac{(e_2 - 0)}{R} = 0$$

and collecting terms, we put the equations in the form $\overline{G}\overline{e} = \overline{I}$, where $\overline{G}$ is the *conductance matrix*, $\overline{e}$ is the node voltage vector, and $\overline{I}$ is a source current vector.

$$\begin{pmatrix} \frac{1}{3R} + \frac{1}{3R} + \frac{1}{2R} & -\frac{1}{2R} \\ -\frac{1}{2R} & \frac{1}{2R} + \frac{1}{R} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{V}{3R} \\ 0 \end{pmatrix}$$

$$\frac{1}{R} \begin{pmatrix} \frac{7}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{R} \begin{pmatrix} \frac{V}{3} \\ 0 \end{pmatrix}$$

We solve this equation using a method like Cramer's rule, Gaussian elimination, or matrix inversion to arrive at

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{54} \begin{pmatrix} 9 & 3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 2V \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{V}{3} \\ \frac{V}{9} \end{pmatrix}$$

Similarly, for the second circuit, we write the node equations:

$$\frac{(e_1 - e_2)}{3\Omega} + \frac{(e_1 - 0)}{1\Omega} + I = 0$$

$$\frac{(e_2 - e_1)}{3\Omega} + \frac{(e_2 - 0)}{2\Omega} - I = 0$$

Collecting terms and standardizing the form;

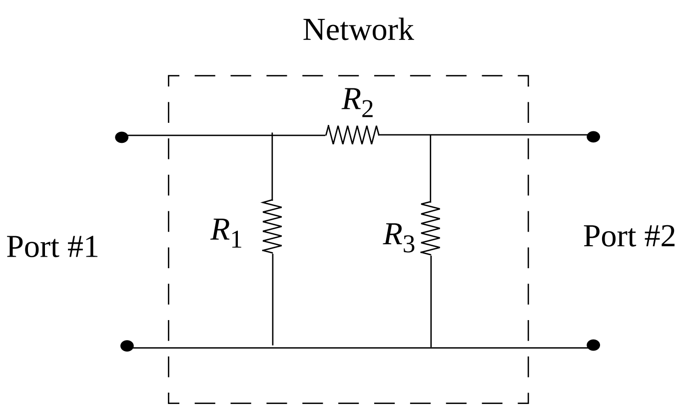
$$\begin{pmatrix} \frac{1}{3\Omega} + \frac{1}{1\Omega} & -\frac{1}{3\Omega} \\ -\frac{1}{3\Omega} & \frac{1}{3\Omega} + \frac{1}{2\Omega} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -I \\ I \end{pmatrix}$$

and solve to find

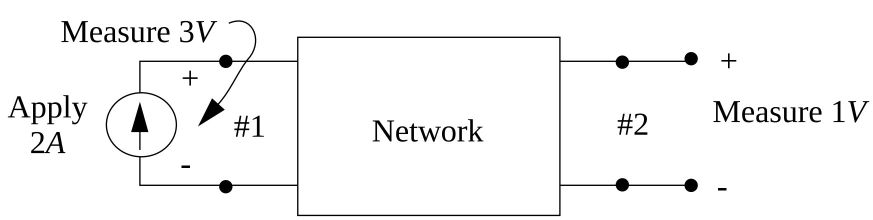
$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix} \Omega \cdot \begin{pmatrix} -I \\ I \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\Omega \\ 1\Omega \end{pmatrix} I$$

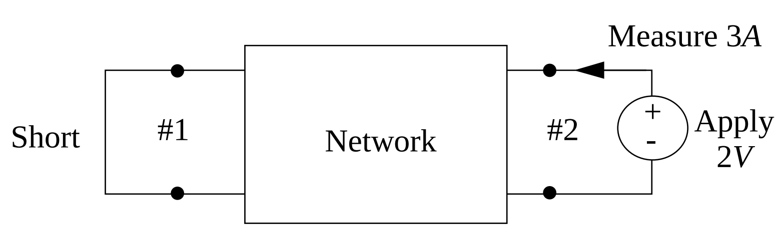
Note that with both circuits, the solved variables e_1 and e_2 have the correct units of voltage.

Problem 1.3: The following network has two ports and three resistors. The resistor values R_1, R_2 and R_3 are unknown.

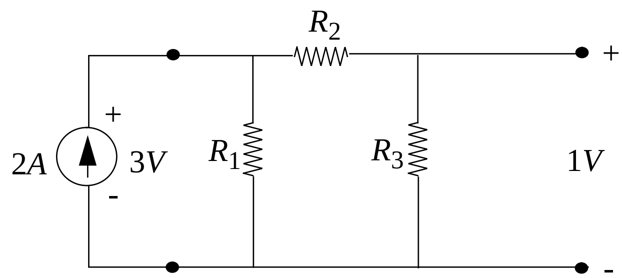


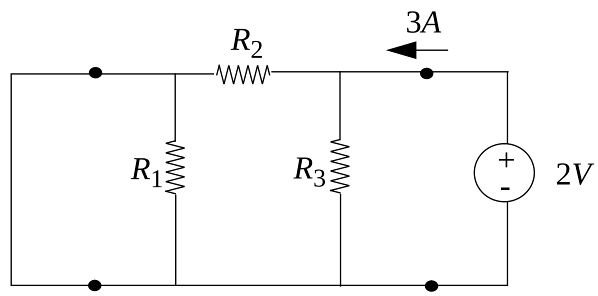
Using the results of the following two experiments performed on the network, find the unknown values of the three resistors.





Answer: This three-resistor circuit, the "pi-equivalent circuit", is an important model, with a similar interpretation as with the Thevenin equivalent circuit. We insert our three-resistor circuit into each experiment, resulting in the figure below.



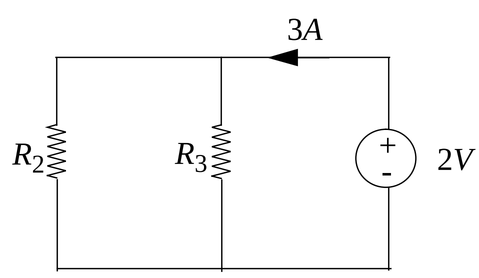


We treat the applied source values and the measured values as knowns, and try to solve for R_1, R_2, and R_3.

Looking at the first experiment, we see we have 3 Volts across the port, and 2 Amperes going in. The equivalent resistance as seen from Port #1 with Port #2 an open circuit is $R_{eq1} = R_1 \parallel (R_2 + R_3) = \frac{3V}{2A} = 1.5\Omega$.

Also looking at the first experiment, we see the voltage at Port #2 is $1V$, while the voltage at Port #1 is $3V$. Noting that R_2 and R_3 are in series when viewed from Port #1, we use the voltage divider to say $\frac{R_3}{R_2+R_3}(3V) = 1V$, or $R_2 = 2R_3$.

Turning to the second experiment, we see that with a short circuit at Port #1, applying $2V$ at Port #2 results in $3A$ drawn by the 3-resistor network. The short circuit reduces the network to the one shown below:



because the parallel combination of R_1 and a short is simply a short. We see that the equivalent resistance as seen from Port #2 with Port #1 shorted is $R_{eq2} = R_2 \parallel R_3 = \frac{2V}{3A} = \frac{2}{3}\Omega$.

We now have the following 3 equations:

$$R_1 \parallel (R_2 + R_3) = 1.5\Omega$$

$$R_2 = 2R_3$$

$$R_2 \parallel R_3 = \frac{2}{3}\Omega$$

Substituting equation 2 in equation 3, we get

$$2R_3 \parallel R_3 = \frac{2}{3}R_3 = \frac{2}{3}\Omega$$

or $R_3 = 1\Omega$.

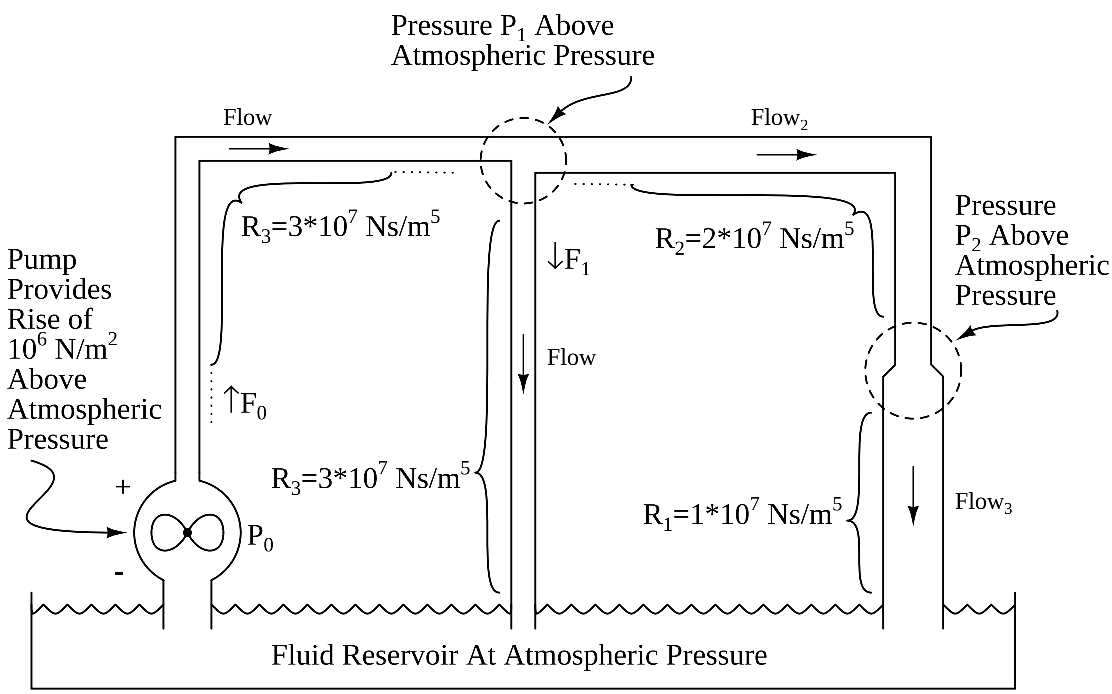
Plugging this back into equation 2, we get $R_2 = 2R_3 = 2\Omega$.

Finally, we plug these values into equation 1:

$$R_1 \parallel (2\Omega + 1\Omega) = R_1 \parallel 3\Omega = 1.5\Omega$$

which we solve to see that $R_1 = 3\Omega$.

Problem 1.4: Electronic circuit theory can be adapted to solve problems in other fields of engineering, as illustrated by this problem which studies the incompressible flow of a fluid through pipes. Assume that the volumetric flow rate F of an incompressible fluid through a pipe, measured in m^3/s, is proportional to the pressure drop P across the pipe from one end of the pipe to the other, measured in $\mathrm{N/m}^2$. Note that positive flow is defined in the direction of decreasing pressure along the pipe. The flow resistance R for the pipe, measured in $\mathrm{Ns/m}^5$, is then defined by $R = P/F$. This behavior can be modeled by a resistor if pressure is modeled by voltage, flow rate is modeled by current, and flow resistance is modeled by electrical resistance. Similarly, a pump that acts as a pressure source can be modeled electrically as a voltage source. Finally, KCL adapted for incompressible fluid flow states that the sum of the flows into a joint connecting two or more pipes is zero. Similarly, KVL adapted for incompressible fluid flow states that the sum of the pressure drops around a loop of pipes and pumps is zero. Given this analogy, find the pressures P_1 and P_2 above atmospheric pressure in the flow circuit shown below. you may find it convenient to think of atmospheric pressure as the equivalent of electrical ground. (Hint: see Problem 1.2).



Answer: To begin solving this as an electrical problem, the flow circuit must be turned into an electrical circuit. The flow circuit moves volumes of water around a network of pipes, just as an electrical circuit moves charge around a network of components and wires. Water is measured in m^3, and charge is measured in C (Coulombs). Be replacing m^3 with C, we should be able to find an electrical analog for fluid flow.

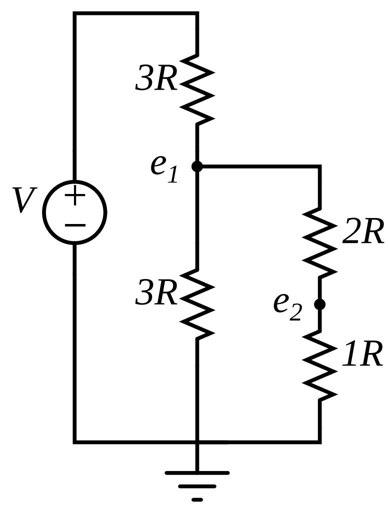
A pump operates as a pressure source, providing the energy to move water through the circuit, just as a voltage source provides the energy to move charge around an electrical network. In creating our electrical network, then, the units of voltage and pressure should be equivalent. That is, $\mathrm{N/m}^2 = \mathrm{Nm/m}^3 = \mathrm{J/m}^3 \Leftrightarrow \mathrm{V} = \mathrm{J/C}$. The substitution of C for m^3 above provides the analogy as for flow.

The volumetric flow rate, in units of m^3/s, measures the volume of water that flows past a particular point in the network over one second. Change m^3 to C again, and we find that flow is equivalent to current in an electrical network: $m^3/s \Leftrightarrow C/s = A$. (Remember from 8.02, current is a measure of the amount of charge that flows past a particular point every second).

The flow resistance of a pipe is defined as $R = P/F$. If we replace pressure with voltage, and flow with current, we find $R = V/I$, which is just Ohm's law, $V = IR$! Each pipe in our flow circuit can therefore be modeled as an electrical resistor. To check, do the units match? Ohms are $V/A = (J/C)/(C/s) = Js/C^2 = Nms/C^2$. Convert to flow units by replacing C with m^3, and we find that $V/A \Leftrightarrow Nms/m^6 = Ns/m^5$!

The fluid reservoir acts as a single node at a constant pressure in our flow circuit. The pressures P_1 and P_2 are measured relative to the pressure at the fluid resevoir. Changing pressure to voltage, we can say that the fluid reservoir acts as the ground node. (Remember, the ground in a circuit is just a node which acts as a reference voltage for all the other voltages in the circuit.)

With these equivalents drawn between flow networks and electrical networks, we can construct the electrical network shown below to model the flow problem.



This circuit is the same one found in problem 1.2, with $R = 10^7$ and $V = 10^6$. Using the result:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{V}{3} \\ \frac{V}{9} \end{pmatrix}$$

we can find the solution to the flow network by susbstituting in the value of V, and we find

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \frac{10^6}{3} \\ \frac{10^6}{9} \end{pmatrix} \frac{N}{m^2}$$