NOTES FOR 6.002 LECTURE #4 FEBRUARY 13, 2003

P.E. GRAY'S OFFICE HOURS: EVERY WEDNESDAY 3-5 PM IN E8-344
OR BY ARRANGEMENT (3-4665)

READINGS THIS WEEK: SECTIONS 3.1-3.3, 3.5

NOW, FOLLOWING TOOLS FOR ANALYSIS OF LUMPED ELECTRIC CIRCUITS, TAKE UP TWO THEOREMS APPLICABLE TO NETWORKS COMPRISED OF LINEAR ELEMENTS.

**STRICT DEFINITION OF LINEARITY:**

\[
\begin{align*}
\text{INPUT} & \quad \text{ELEMENT, PASSIVE CIRCUIT, SYSTEM} & \quad \text{OUTPUT} \\
V & \quad \rightarrow & \quad V_1 & \quad \rightarrow & \quad V_1 \\
V_2 & \quad \rightarrow & \quad V_2 & \quad \rightarrow & \quad V_2
\end{align*}
\]

THE BOX IS LINEAR --- IF: \( aV_1 + bV_2 \rightarrow aV_1 + bV_2 \)

WHERE \( a, b \) ARE CONSTANTS

\[
\begin{align*}
\text{\( V = IR \) OBVIOUSLY LINEAR}
\end{align*}
\]

\[
\begin{align*}
\text{\( U = IR + V \) NOT LINEAR (INNARDS NOT}
\end{align*}
\]

PASSIVE --- AN INDEPENDENT SOURCE IS PRESENT

WE WILL SOON INTRODUCE OTHER LINEAR ELEMENTS: INDUCTORS, CAPACITORS, DEPENDENT SOURCES.

**DEVELOP THE SUPERPOSITION PRINCIPLE, FORMALLY A THEOREM**

WHAT FOLLOWS IS A PLACABILITY ARGUMENT

**CONSIDER CIRCUIT OF LECTURE #3:**

\[
\begin{align*}
\text{\( V_1, V_2, V_3 \)} & \quad \text{\( G_1, G_2, G_3 \)} & \quad \text{\( G_4, G_5, G_6 \)} & \quad \text{\( V_B \)}
\end{align*}
\]

\[
\begin{align*}
\text{\( I \)} & \quad \text{\( G_1, G_2, G_3 \)} & \quad \text{\( G_4, G_5, G_6 \)} & \quad \text{\( V_B \)}
\end{align*}
\]
The 3 KCL equations are:

\[(\mathbf{V}_2 - \mathbf{V}_3) \mathbf{g}_1 + \mathbf{V}_3 \mathbf{g}_2 + (\mathbf{V}_2 - \mathbf{V}_3) \mathbf{g}_3 = 0\]

\[(\mathbf{V}_2 - \mathbf{V}_3) \mathbf{g}_3 + (\mathbf{V}_2 - \mathbf{V}_3) \mathbf{g}_5 + \left[\mathbf{V}_2 - \left(\mathbf{V}_2 - \mathbf{V}_3\right)\right] \mathbf{g}_4 = 0\]

\[(\mathbf{V}_3 - \mathbf{V}_2) \mathbf{g}_5 + (\mathbf{V}_3 - \mathbf{V}_2) \mathbf{g}_6 - \mathbf{I} + \left[\mathbf{V}_3 - \mathbf{V}_2\right] \mathbf{g}_4 = 0\]

Re-written, collecting terms:

\[
\mathbf{V}_1 \left(\mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3\right) - \mathbf{V}_2 \left(\mathbf{g}_3\right) = \mathbf{V}_A \mathbf{g}_1
\]

\[- \mathbf{V}_1 \left(\mathbf{g}_3\right) + \mathbf{V}_2 \left(\mathbf{g}_3 + \mathbf{g}_5 + \mathbf{g}_6\right) - \mathbf{V}_3 \left(\mathbf{g}_4 + \mathbf{g}_5\right) = \mathbf{V}_B \mathbf{g}_4
\]

\[- \mathbf{V}_2 \left(\mathbf{g}_4 + \mathbf{g}_5\right) + \mathbf{V}_3 \left(\mathbf{g}_4 + \mathbf{g}_5 + \mathbf{g}_6\right) = \mathbf{I} + \mathbf{V}_A \left(\mathbf{g}_4 + \mathbf{g}_5\right)
\]

Note symmetry across the diagonal and nature of coefficients.

Note also that the network responses \((\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)\) are linearly related to the three source values. If all source values are multiplied by \(k\), all responses are \(k\) times as big.

Because the sources appear only in linear combinations (at most) the responses for each source acting alone are readily determined. Three single-source equations are:

1. Responses with \(\mathbf{V}_A\) acting, \(\mathbf{V}_D, \mathbf{I} = 0\)
2. Responses with \(\mathbf{V}_D\) acting, \(\mathbf{V}_A, \mathbf{I} = 0\)
3. Responses with \(\mathbf{I}\) acting, \(\mathbf{V}_A, \mathbf{V}_D = 0\)
Clearly the complete responses are: 
\[ V = V_m \left| \frac{1}{V_A} + \frac{1}{V_B} \right| + V_m \left| I \right| \]

I.e. the sum of the component responses

**Superposition Principle:** The response of a network comprised of linear elements which contains \( n \) independent sources is equal to the response of \( n \) different networks each of which has only one source acting, with the other sources shut off.

Consider an algebraically simpler example:

![Circuit Diagram]

\[ KCL \text{ at } V: \quad (V + V_{in}) g_1 + V g_2 + (V - V_{in}) g_3 = I = 0 \]

\[ (g_1 + g_2 + g_3) V = -V g_1 + V g_1 + I \]

\[ V = \frac{-V g_1 + V g_3 + I}{g_1 + g_2 + g_3} \]

The network response

Now consider the network under three conditions, each with one source acting, the other two turned off.

What does it mean to turn a source off?

**Voltage Source**

\[ V = 0 \]

A short circuit

**Current Source**

\[ I = 0 \]

An open circuit
CASE A: \( V_A \) ACTIVE, \( V_B, I \) OFF

\( (R_0 = 1/\sigma) \)

\[
R_2 \equiv \frac{R_2 R_3}{R_2 + R_3}
\]

\[
V = -\frac{V_A}{R_2 + R_3 + R_1}
\]

BY INSPECTION:

\[
V = \frac{R_2 R_3}{R_2 + R_3 + R_1} \quad \text{MULTIPLY NURGATOR AND DENOMINATOR BY} \quad R_2 R_3
\]

\[
V = -\frac{V_A \sigma_1}{\sigma_1 + \sigma_3 + \sigma_2}
\]

CASE B: \( V_B \) ACTIVE, \( V_A, I \) OFF

SAME TOPOLOGY AS CASE A WITH R \(_4\) INTERCHANGED

\[
V = \frac{V_B \sigma_3}{\sigma_1 + \sigma_3 + \sigma_2}
\]

\( V_A \) REPLACED BY A SHORT

I REMOVED

CASE C: I ACTIVE, \( V_A, V_B \) OFF

THREE CONDUCTANCES IN PARALLEL

\[
V = \frac{I}{\sigma_1 + \sigma_2 + \sigma_3}
\]

\( V_A, V_B \) REPLACED BY SHORTS

THE THREE COMPONENTS ADDED TOGETHER GIVE THE COMPLETE RESPONSE