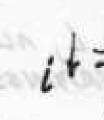
NOTES FOR 6.002 LECTURE # 11, MARCH 13, 2003

READ: 9.5, 10.1–10.3, 10.5 LAB #2 NEXT WEEK!

CIRCUIT ELEMENTS WHICH STORE ENERGY:

CAPACITOR:  Q IS CHARGE ON THE POSITIVE PLATE

$Q = Cv$ C IS THE CAPACITANCE AND IS FOR 6.002 A CONSTANT.

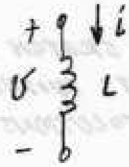
$$i = \frac{dQ}{dt} \quad \text{OR} \quad i = C\frac{dv}{dt}$$

$$v = \frac{1}{C}\int i\,dt$$

DIMENSION: $\frac{\text{AMPERE-SEC}}{\text{VOLT}}$ = FARAD

STORED ENERGY $E = \int (i \cdot v)\,dt = \frac{1}{2}Cv^2$ OR $\frac{1}{2C}Q^2$

CHANGING THE VOLTAGE ON THE CAPACITOR MEANS CHANGING THE STORED ENERGY AND THIS CANNOT HAPPEN INSTANTANEOUSLY UNLESS THE CURRENT IS VERY LARGE.

INDUCTOR:  λ IS THE FLUX LINKAGE IN THE WINDING

$\lambda = Li$ L IS THE INDUCTANCE - A CONSTANT

$$v = \frac{d\lambda}{dt} \quad \text{OR} \quad v = L\frac{di}{dt}$$

DIMENSION: $\frac{\text{VOLT-SEC}}{\text{AMPERE}}$ = HENRY

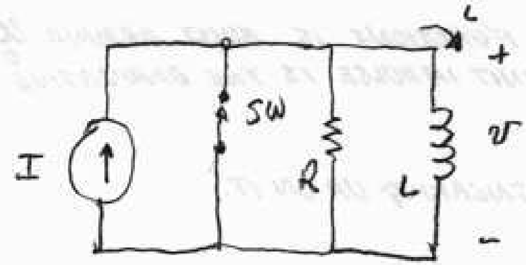
$$i = \frac{1}{L}\int v\,dt$$

STORED ENERGY $E = \int (v \cdot i)\,dt = \frac{1}{2}Li^2$ OR $\frac{1}{2L}\lambda^2$

CHANGING THE CURRENT IN THE INDUCTOR MEANS CHANGING THE STORED ENERGY AND THIS CANNOT HAPPEN INSTANTANEOUSLY UNLESS THE VOLTAGE IS VERY LARGE.

NOTE THE SYMMETRY OF REPRESENTATION

CONSIDER A CIRCUIT WITH AN INDUCTOR:



THE SWITCH IS CLOSED FOR $t < 0$ AND THE CURRENT i IN THE INDUCTOR IS ZERO FOR $t < 0$

WHAT HAPPENS WHEN THE SWITCH OPENS?

THE (KCL) NODE EQUATION FOR v IS: $\frac{1}{L}\int v\,dt + \frac{v}{R} = I \quad t>0$

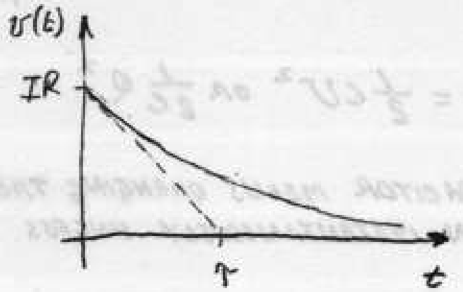
DIFFERENTIATE TO GET THE DIFFERENTIAL EQUATION: $\frac{v}{L} + \frac{1}{R}\frac{dv}{dt} = 0$

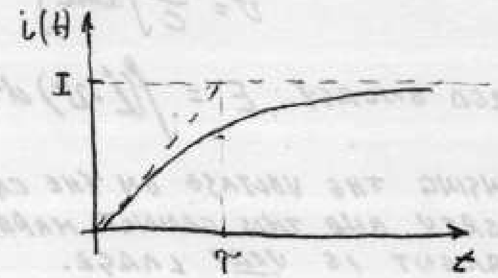
SOLUTION TO THE D.E., WHICH IS HOMOGENEOUS, IS $v(t) = Ae^{-t/\tau} \quad \tau = \frac{L}{R}$

INITIALLY ($t=0+$) THERE IS NO CURRENT IN L, ALL THE CURRENT FLOWS INTO R. THUS $v(0+) = IR$ WHICH DETERMINES A.

$v(t) = IRe^{-t/\tau}$ TO GET $i(t)$: $i = \frac{1}{L}\int_0^t v(t) = \frac{IR}{L}\int_0^t e^{-t/\tau}\,dt$

$i(t) = -Ie^{-t/\tau}\Big|_0^t = I(1-e^{-t/\tau})$

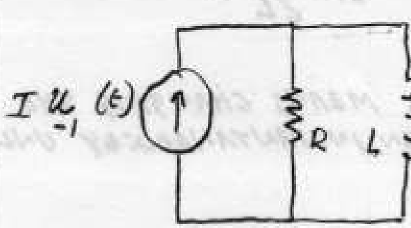




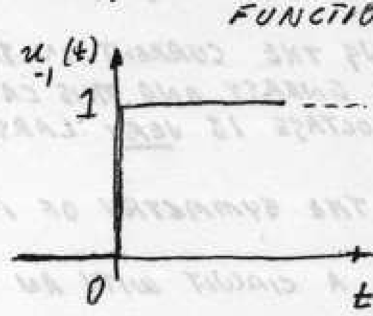
YOU SHOULD ENDEAVOR TO BE ABLE TO SKETCH AND LABEL RESPONSES OF FIRST-ORDER RC OR RL CIRCUITS WITHOUT WORKING THROUGH THE FORMAL SOLUTION. AFTER WHICH THE SOLUTIONS CAN BE WRITTEN FROM THE SKETCHES.

SINGULARITY FUNCTIONS

THE PREVIOUS CIRCUIT COULD HAVE BEEN REPRESENTED AS:



WHERE $u_{-1}(t)$ DENOTES THE UNIT-STEP FUNCTION



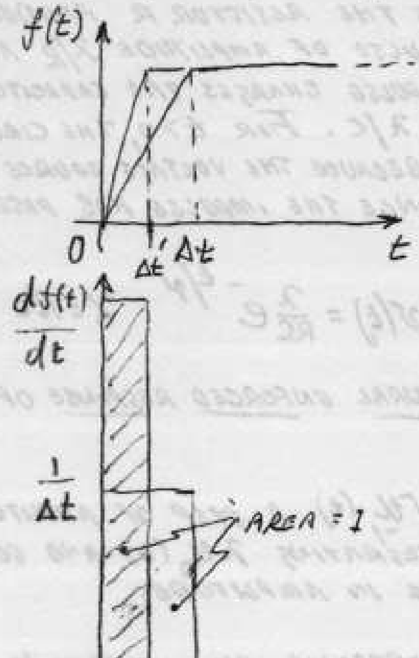
$$u_{-1}(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

INDETERMINATE AT $t=0$

THE FAMILY OF SINGULARITY FUNCTIONS IS BUILT AROUND $u_0(t)$ - THE UNIT IMPULSE. THE UNIT IMPULSE IS THE DERIVATIVE OF THE UNIT STEP.

IT IS BEST UNDERSTOOD BY SNEAKING UP ON IT.

CONSIDER A RAPID CHANGE WHICH IS NOT QUITE A UNIT STEP:



THE DERIVATIVE OF $f(t)$ IS SHOWN BELOW IT

OBVIOUSLY AS Δt IS REDUCED, THE DERIVATIVE GETS THINNER AND TALLER, ILLUSTRATED BY $\Delta t' = \Delta t/2$

IN THE LIMIT, AS $\Delta t \to 0$

$$u_{-1}(t) = \lim_{\Delta t \to 0} f(t) \quad \text{UNIT STEP}$$

AND

$$u_0(t) = \lim_{\Delta t \to 0} \frac{df(t)}{dt} \quad \text{UNIT IMPULSE}$$

AND:
$$u_0(t) = \frac{d}{dt} u_{-1}(t)$$

$$u_{-1}(t) = \int_{t<0}^{t>0} u_0(t)\, dt$$

THE UNIT IMPULSE CAN BE THOUGHT OF AS A TALL RECTANGULAR PULSE OF INFINITESIMAL WIDTH AND INFINITE HEIGHT, AND OF UNIT AREA

AN IMPULSE OF CURRENT $Q u_0(t)$ HAS DIMENSIONS OF AMPERE-SECONDS OR COULOMBS

AN IMPULSE OF VOLTAGE $\lambda u_0(t)$ HAS DIMENSIONS OF VOLT-SECONDS

THE FAMILY OF SINGULARITY FUNCTIONS IS:

$$\cdots u_{-2}(t),\ u_{-1}(t),\ u_0(t),\ u_1(t) \cdots$$

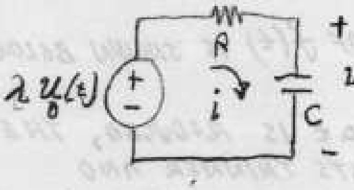
UNIT RAMP	UNIT STEP	UNIT IMPULSE	UNIT DOUBLET

OBTAINED BY INTEGRATION ⟵ ⟶ OBTAINED BY DIFFERENTIATON

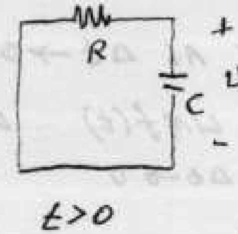
WITHIN THE DOMAIN OF CIRCUITS WHICH SATISFY THE STRICT DEFINITION OF LINEARITY, IF THE RESPONSE TO $f(t)$ IS $g(t)$, THE RESPONSE TO $\frac{df(t)}{dt}$ WILL BE $\frac{dg(t)}{dt}$ AND THE RESPONSE TO $\int f(t)dt$ WILL BE $\int g(t)\, dt$. USEFUL PROPERTIES

A DELAYED (OR ADVANCED) SINGULARITY FUNCTION IS $u_n(t-T)$ [OR $u_n(t+T)$] WHERE T IS THE AMOUNT OF DELAY OR ADVANCEMENT.

CONSIDER A CIRCUIT:



THE VOLTAGE IMPULSE OF AMPLITUDE λ VOLT-SEC APPEARS ACROSS THE RESISTOR R PRODUCING A CURRENT IMPULSE OF AMPLITUDE λ/RC AMPERE-SEC. THE CURRENT IMPULSE CHARGES THE CAPACITOR TO A VOLTAGE OF λ/C. FOR $t>0$, THE CIRCUIT CAN BE SIMPLIFIED BECAUSE THE VOLTAGE SOURCE IS A SHORT CIRCUIT ONCE THE IMPULSE HAS PASSED.



$t>0$

BY INSPECTION: $v(t) = \frac{\lambda}{RC} e^{-t/\tau} \quad \tau = RC \quad (1)$

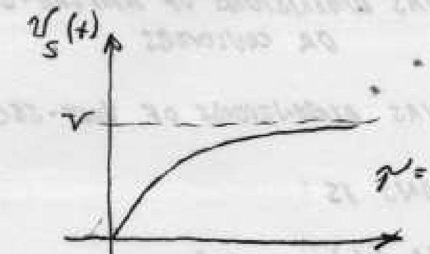
THIS IS THE NATURAL UNFORCED RESPONSE OF THE RC CIRCUIT

SUPPOSE INSTEAD THE SOURCE WAS $V u_{-1}(t)$ A STEP OF AMPLITUDE V. THIS NEW INPUT IS OBTAINED BY INTEGRATING $\lambda u_0(t)$ AND SCALING IT BY V/λ TO REFLECT THE CHANGE IN AMPLITUDE.

THE RESPONSE TO THE NEW INPUT IS OBTAINED BY INTEGRATING AND SCALING (1). E.G.

RESPONSE TO STEP IS $v_s(t) = \frac{V}{\lambda}\int_0^t \frac{\lambda}{RC} e^{-t/\tau} dt = -\frac{V}{RC}\tau e^{-t/\tau}\Big|_0^t$

$v_s(t) = V\left(1 - e^{-t/\tau}\right)$ (WHICH CAN BE CHECKED FROM LECTURE #10)



IF THE ARGUMENT AT THE TOP OF THE PAGE IS NOT PERSUASIVE, DO THIS:

THE D.E. IS $\lambda u_0(t) = iR + \frac{1}{C}\int i\,dt$ INTEGRATE BOTH SIDES FROM 0− TO 0+

$\lambda\int_{0^-}^{0^+} u_0(t) = R\int_{0^-}^{0^+} i\,dt + \frac{1}{C}\underbrace{\int_{0^-}^{0^+}\int i\,dt\,dt}$ $\quad \int_{0^-}^{0^+} u_0\,dt = 1 \quad \int_{0^-}^{0^+} i\,dt = CV_0$ WHERE V_0 IS CAPACITOR VOLTAGE AT 0+

$\lambda = R(CV_0) + 0$ ZERO BECAUSE WHILE THE FIRST INTEGRAL IS FINITE, THE SECOND IS NOT - NOTHING ACCUMULATES IN THE VERY SHORT INTERVAL

THUS $V_0 = \frac{\lambda}{RC}$

AS ABOVE