Notes for 6.002 Lecture #13, March 20, 2003

Read 13.1-13.3, 13.5

Reprise on Digital Memory (Notes of Lecture #12)

Second Order Responses:

\[ V_{o}(t) = L \frac{di}{dt} + \frac{1}{C} \int i dt \]

The impulse is absorbed by \( L \frac{di}{dt} \) (highest-order term) thus \( i(0^+) = \frac{V_i}{L} \) and energy has been put in the inductor by the impulse.

For \( t > 0 \), after the impulse has passed, the circuit is:

And the D.E. is:

\[ L \frac{di}{dt} + C \int i dt = 0 \]

Equivalently:

\[ LC \frac{di}{dt} + i = 0 \]

This is the D.E. of a harmonic oscillator.

Solutions are of the form:

\[ i(t) = A \sin \omega_0 t + B \cos \omega_0 t \]

where \( \omega_0^2 = \frac{1}{LC} \)

and \( \omega_0 \) is the natural frequency of the oscillator.

The initial condition requires \( i(0^+) = \frac{V_i}{L} \). Thus \( B = \frac{V_i}{L} \) and \( A \) is not determined yet.

\[ t > 0 \]

\[ i(t) = \frac{V_i}{L} \cos \omega_0 t + B \sin \omega_0 t \]

What about the voltage \( U \)?

\[ U(t) = \frac{1}{L C} \sin \omega_0 t - \frac{B}{C} \cos \omega_0 t + C \]

A constant of integration

\[ U(0^+) = 0 \] thus \( B = C = 0 \) and the complete solution is

\[ i(t) = \frac{2}{L} \cos \omega_0 t \]

Orthogonal sine waves

\[ U(t) = 2 \omega_0 \sin \omega_0 t \]
What about stored energies?

\[ E_L = \frac{1}{2} L i^2 = \frac{R^2}{2L} \cos^2 \omega_0 t \]

\[ E_C = \frac{1}{2} C u^2 = \frac{C \omega_0^2 i^2}{2} \sin^2 \omega_0 t \]

or

\[ E_C = \frac{R^2}{2L} \sin^2 \omega_0 t \]

The total stored energy is

\[ E_{total} = \frac{R^2}{2L} \left( \cos^2 \omega_0 t + \sin^2 \omega_0 t \right) \]

\[ E_{total} = \frac{R^2}{2L} \]

With no loss mechanism the circuit oscillates at \( \omega_0 \) indefinitely with total stored energy constant and with the energy flowing back and forth between the inductor and the capacitor.

Introduce a loss mechanism which absorbs a small fraction of the stored energy each cycle; resistance in the inductor winding.

Assume the current and voltage waveforms don't change much over a cycle.

Average loss \( = \int_0^{2\pi} 2Ri^2 \) where \( t \) is the period \( \frac{2\pi}{\omega_0} \)

\[ \text{Avs. Loss} = \frac{2R^2}{2L} \int_0^{2\pi} \cos^2 \omega_0 t \ dt = \frac{R^2 R}{\pi L} \left( \frac{\pi}{2} \right) \]

or

\[ \text{Avs. Loss} = \frac{R^2}{2L} = \frac{R^2}{L} \left( \frac{1}{2L} \right) E_{total} \]

The DE which describes energy decay is:

\[ \frac{dE}{dt} = -\text{Avs. Loss} \]

(logarithms total from \( E \))
OR \( \frac{dE}{dt} + \frac{R}{L} E = 0 \), the solution is \( E(t) = E(0)e^{-\frac{R}{L} t} \).

The current \( i \) and the voltage \( u \) are proportional to \( \sqrt{E} \).

Thus their amplitudes decay with a time constant \( \frac{L}{2R} \).

An exact solution can be found as always by going back to the D.E. with \( R \) present: for \( t \geq 0 \)

\[
L \frac{d^2 i}{dt^2} + Ri + \frac{1}{C} \int idt = 0 \quad \text{KWh}
\]

or

\[
\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = 0
\]

where \( \omega_0^2 = \frac{1}{LC} \), \( \alpha = \frac{R}{2L} \).

Assume \( Ae^{\lambda t} \) as a solution.

The characteristic equation is

\[
\lambda^2 + 2\alpha \lambda + \omega_0^2 = 0
\]

For the oscillatory case: \( \alpha < \omega_0 \) and the roots are:

\[
\lambda_1, \lambda_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}
\]

(Note that in EECs, to avoid confusion with current, \( V / I = j \) not \( i \).)

After a modest amount of algebra and trigonometry and the assumption that \( R \) is small \( \) (more precisely \( \alpha \ll \omega_0 \)):

\[
i(t) = \frac{2}{\omega_0} e^{-\alpha t} \cos \omega_0 t + \frac{2}{\omega_0} e^{-\alpha t} \sin \omega_0 t
\]

which correspond to the drawings above.

This slightly-overshoot oscillator is an important concept.

As a second example consider a parallel RLC circuit excited by a current impulse.
Addendum to (503-039), Notes for Lecture #13, 3/20/03

I) The analysis at the bottom of p1 is incorrect. The expression for \( i(t) \) should read:

\[
\begin{align*}
t > 0: \quad i(t) &= \frac{2}{L} \cos \omega t + A \sin \omega t. \\
\text{The correct expression for } u(t) &\text{ is:} \\
\varepsilon > 0: \quad u(t) &= \frac{2}{\omega_0 L} \sin \omega t - \frac{A}{\omega_0} \cos \omega t + K \quad (K \text{ is const. of integer})
\end{align*}
\]

Because \( u(0^+) = 0 \), \( \frac{di}{dt} (0^+) \) must be zero, as required by the differential equation; at \( t = 0^+ \), \( \int_0^t i(t) \, dt = 0 \) because \( i(t) \) is finite \( \frac{di}{dt} = -\frac{2}{L} \omega_0 \frac{A}{\omega_0} \sin \omega t + \frac{A}{\omega_0} \cos \omega t \) by differentiation.

Thus \( A = 0 \) and \( K = 0 \) to satisfy \( u(t) = 0 \)

My analysis required: \( -\frac{A}{\omega_0 L} + K = 0 \), which is insufficient.
(Courtesy of Prof. Hutchinson)

II) The argument at the end of the hour, not included in the notes, went like this:

Energy loss per period: \( \frac{\lambda^2 R}{L^2} (\pi^2) \) [from \( \int_0^\pi \frac{\lambda^2 R}{L^2} R \, dt \)]

Energy loss per radian = \( \frac{\lambda^2 R}{L^2} (\pi^2) \frac{1}{2\pi} = \frac{\lambda^2}{2L} \left( \frac{R}{L \omega_0} \right) \)

Average stored energy in a period = \( \frac{\lambda^2}{2L} \) [from middle of]

Thus \( \frac{\text{Average stored energy}}{\text{Energy loss per radian}} = \frac{L \omega_0}{R} = Q \), the quality factor

From \( \omega_3 \) middle where \( \frac{R}{L} = \alpha \), \( Q = \frac{\omega_0}{2\alpha} \)

High \( Q \) means longer lasting oscillation.

How great is the decay in \( Q \) regions? i.e. let \( \omega_0 t = Q \left( \frac{2\pi}{Q} \right) \)

\[
\frac{e^{-Q \frac{2\omega_0 \pi}{Q}}}{23} = e^{-\pi t} \approx \frac{1}{23} \text{ or about 4% of initial value}
\]

A rough measure of \( Q \) can be obtained by counting the number of cycles to be down to 4% of the initial amplitude

N.B. This energy-based analysis works only for high \( Q \), e.g., \( Q > 5 \).