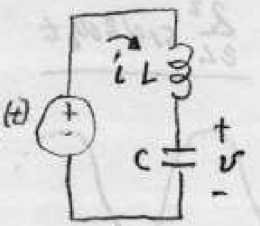
NOTES FOR 6.002 LECTURE #13, MARCH 20, 2003

READ 13.1 - 13.3, 13.5

REPRISE ON DIGITAL MEMORY (NOTES OF LECTURE #12)

SECOND ORDER RESPONSES:



FOR $t < 0$

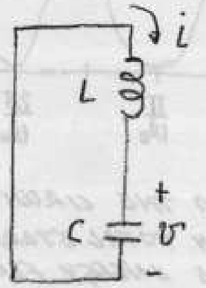
$i = v = 0$

(NO STORED ENERGY)

KVL AROUND LOOP: $\lambda u_0(t) = L\frac{di}{dt} + \frac{1}{C}\int i\,dt$

THE IMPULSE IS ABSORBED BY $L\frac{di}{dt}$ (HIGHEST-ORDER TERM) THUS $i(0+) = \lambda/L$ AND ENERGY HAS BEEN PUT IN THE INDUCTOR BY THE IMPULSE.

FOR $t > 0$, AFTER THE IMPULSE HAS PASSED, THE CIRCUIT IS:



AND THE D.E. IS: $L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$

EQUIVALENTLY: $LC\frac{d^2i}{dt^2} + i = 0$

THIS IS THE D.E. OF A HARMONIC OSCILLATOR

SOLUTIONS ARE OF THE FORM:

$i(t) = A\sin\omega_0 t + B\cos\omega_0 t$

WHERE $\omega_0^2 = 1/LC$

AND ω_0 IS THE NATURAL FREQUENCY OF THE OSCILLATOR,

THE INITIAL CONDITION REQUIRES $i(0+) = \lambda/L$. THUS $B = \lambda/L$ AND A IS NOT DETERMINED YET:

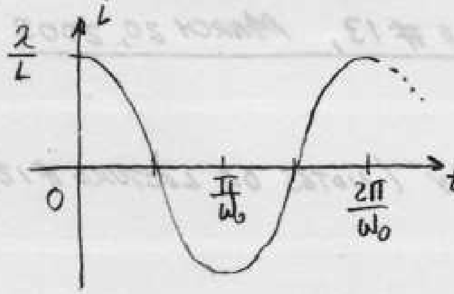
$t > 0 \quad i(t) = \lambda/L \cos\omega_0 t + B\sin\omega_0 t$

WHAT ABOUT THE VOLTAGE v? $v = \frac{1}{C}\int i\,dt$

$v(t) = \frac{\lambda}{\omega_0 LC}\sin\omega_0 t - \frac{B}{\omega_0 C}\cos\omega_0 t + C$ ← CONSTANT OF INTEGRATION

$v(0+) = 0$ THUS $B = C = 0$ AND THE COMPLETE SOLUTION IS

$$\left.\begin{aligned} i(t) &= \frac{\lambda}{L}\cos\omega_0 t \\ v(t) &= \lambda\omega_0\sin\omega_0 t \end{aligned}\right\} \text{ ORTHOGONAL SINE WAVES}$$



WHAT ABOUT STORED ENERGIES?

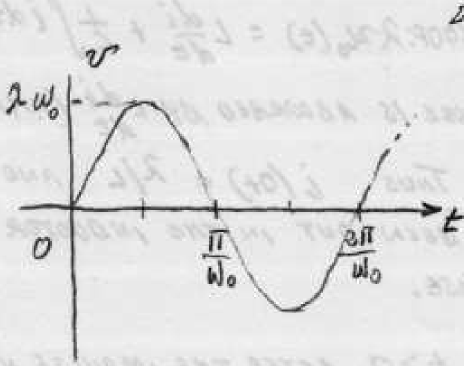
$$E_L = \frac{1}{2} L i^2 = \frac{\lambda^2}{2L}\cos^2\omega_0 t$$

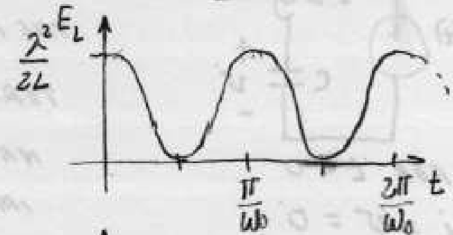
$$E_C = \frac{1}{2} C v^2 = \frac{C\lambda^2\omega_0^2}{2}\sin^2\omega t$$

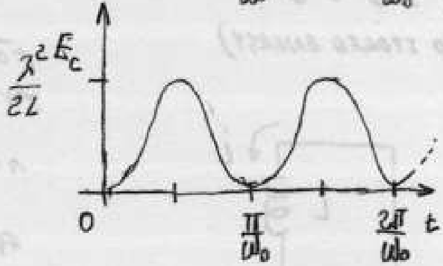
OR

$$E_C = \frac{\lambda^2}{2L}\sin^2\omega_0 t$$

ETC.







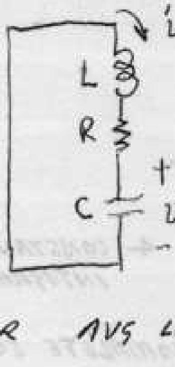
THE TOTAL STORED ENERGY IS

$$E_{TOTAL} = \frac{\lambda^2}{2L}\left(\cos^2\omega t + \sin^2\omega t\right)$$

$$E_{TOTAL} = \frac{\lambda^2}{2L}$$

WITH NO LOSS MECHANISM THE CIRCUITS OSCILATES AT ω_0 INDEFINITELY WITH TOTAL STORED ENERGY CONSTANT AND WITH THE ENERGY FLOPPING BACK AND FORTH BETWEEN THE INDUCTOR AND THE CAPACITOR.

INTRODUCE A LOSS MECHANISM WHICH ABSORBS A SMALL FRACTION OF THE STORED ENERGY EACH CYCLE: RESISTANCE IN THE INDUCTOR WINDING.



ASSUME THE CURRENT AND VOLTAGE WAVEFORMS DONT CHANGE MUCH OVER A CYCLE

AVERAGE LOSS $= \frac{1}{T}\int_0^T i^2 R\,dt$ WHERE T IS THE PERIOD $\frac{2\pi}{\omega_0}$

$$\text{AVG. LOSS} = \frac{\lambda^2 R}{TL^2}\int_0^{2\pi/\omega_0}\cos^2\omega_0 t\,dt = \frac{\lambda^2 R}{TL^2}\left(\frac{\pi}{\omega_0}\right)$$

OR $\text{AVG LOSS} = \frac{R\lambda^2}{2L^2} = \frac{R}{L}\left(\frac{\lambda^2}{2L}\right) \leftarrow E_{TOTAL}$

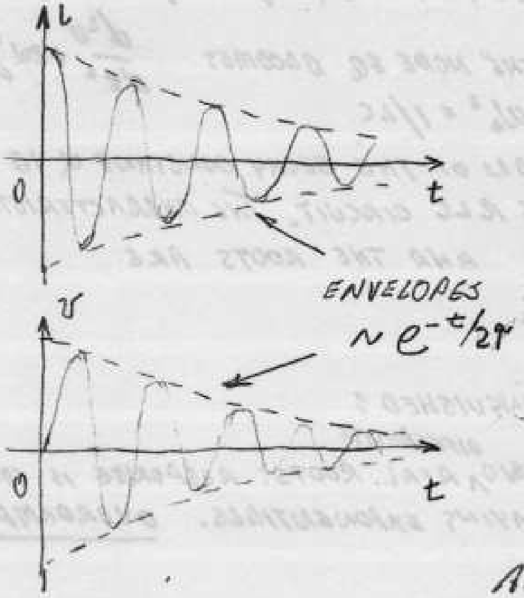
THE D.E. WHICH DESCRIBES ENERGY DECAY IS: $\frac{dE}{dt} = -$ AVG LOSS

(DROPPING TOTAL FROM E)

OR $\frac{dE}{dt} + \frac{R}{L}E = 0$ THE SOLUTION IS $E(t) = E(0)e^{-t/\tau}$ $\tau = \frac{L}{R}$

THE CURRENT i AND THE VOLTAGE v ARE PROPORTIONAL TO $\sqrt{E}$

THUS THEIR AMPLITUDES DECAY WITH A TIME CONSTANT $\frac{L}{2R}$



AN EXACT SOLUTION CAN BE FOUND AS ALWAYS BY GOING BACK TO THE D.E. WITH R PRESENT: FOR $t>0$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0 \quad \text{KVL}$$

OR

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \omega_0^2 i = 0$$

WHERE $\omega_0^2 = \frac{1}{LC}$, $\alpha = \frac{R}{2L}$

ASSUME Ae^{st} AS A SOLUTION.

THE CHARACTERISTIC EQUATION IS $s^2 + 2\alpha s + \omega_0^2 = 0$

FOR THE OSCILLATORY CASE $\alpha < \omega_0$ AND THE ROOTS ARE:

$$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} \quad \text{COMPLEX ROOTS}$$

(NOTE THAT IN EECS, TO AVOID CONFUSION WITH CURRENT, $\sqrt{-1} = j$ NOT i)

AFTER A MODEST AMOUNT OF ALGEBRA AND TRIGONOMETRY AND THE ASSUMPTION THAT R IS SMALL (MORE PRECISELY $\alpha \ll \omega_0$):

$$\left.\begin{aligned} i(t) &\cong \frac{\lambda}{L} e^{-\alpha t}\cos\omega_0 t \\ v(t) &\cong \lambda\omega_0 e^{-\alpha t}\sin\omega t \end{aligned}\right\}$$ WHICH CORRESPOND TO THE DRAWINGS ABOVE.

THIS <u>LIGHTLY-DAMPED</u> OSCILLATOR IS AN IMPORTANT CONCEPT

AS A SECOND EXAMPLE CONSIDER A PARALLEL RLC CIRCUIT EXCITED BY A CURRENT IMPULSE

ADDENDUM TO (503-039), NOTES FOR LECTURE #13, 3/20/03

I) THE ANALYSIS AT THE BOTTOM OF p1 IS INCORRECT. THE EXPRESSION FOR $i(t)$ SHOULD READ:

$t>0: i(t) = \frac{\lambda}{L}\cos\omega_0 t + A\sin\omega_0 t$. THE CORRECT EXPRESSION FOR $v(t)$ IS:

$t>0: v(t) = \frac{\lambda}{\omega_0 LC}\sin\omega_0 t - \frac{A}{\omega_0 C}\cos\omega_0 t + K$ (K IS CONST. OF INTEG.)

BECAUSE $v(0+) = 0$, $\frac{di}{dt}(0+)$ MUST BE ZERO, AS REQUIRED BY THE DIFFERENTIAL EQUATION; AT $t=0+$, $\int i(t)dt = 0$ BECAUSE $i(t)$ IS FINITE

$$\frac{di}{dt} = -\frac{\lambda\omega_0}{L}\sin\omega_0 t + A\omega_0\cos\omega_0 t \quad \text{BY DIFFERENTIATION.}$$

THUS $A=0$ AND $K=0$ TO SATISFY $v(0+)=0$

MY ORIGINAL ALGEBRA REQUIRED: $-\frac{A}{\omega_0 C} + K = 0$, WHICH IS INSUFFICIENT.

(COURTESY OF PROF. HUTCHINSON)

II) THE ARGUMENT AT THE END OF THE HOUR, NOT INCLUDED IN THE NOTES, WENT LIKE THIS:

ENERGY LOSS PER PERIOD := $\frac{\lambda^2 R}{L^2}\left(\frac{\pi}{\omega_0}\right)$ $\left[\text{FROM } \int_0^{T} i(t)^2 R\,dt\right]$

ENERGY LOSS PER RADIAN $= \frac{\lambda^2 R}{L^2}\left(\frac{\pi}{\omega_0}\right)\frac{1}{2\pi} = \frac{\lambda^2}{2L}\left(\frac{R}{L\omega_0}\right)$

AVERAGE STORED ENERGY IN A PERIOD $= \frac{\lambda^2}{2L}$ [FROM MIDDLE OF p2]

THUS $\frac{\text{AVG. STORED ENERGY}}{\text{ENERGY LOSS/RADIAN}} = \frac{L\omega_0}{R} = Q$, THE QUALITY FACTOR

FROM p3 MIDDLE WHERE $\frac{R}{2L} = \alpha$, $Q = \frac{\omega_0}{2\alpha}$

HIGH Q MEANS LONGER LASTING OSCILLATION.

HOW GREAT IS THE DECAY IN Q PERIODS? I.E. LET $\omega_0 t = Q(2\pi)$

$$e^{-\alpha t} = e^{-\left[\alpha\frac{Q(2\pi)}{\omega_0}\right]t} = e^{-\pi} \cong \frac{1}{23} \text{ OR ABOUT 4\% OF INITIAL VALUE}$$

A ROUGH MEASURE OF Q CAN BE OBTAINED BY COUNTING THE NUMBER OF CYCLES TO BE DOWN TO 4% OF THE INITIAL AMPLITUDE

N.B. THIS ENERGY-BASED ANALYSIS WORKS ONLY FOR HIGH Q. E.G., $Q>5$.