NOTES FOR 6.002 LECTURE #15, APRIL 3, 2003

READ 14.1 – 14.1

SINUSOIDAL STEADY STATE ANALYSIS OF LINEAR CIRCUITS

ACTUAL INPUT: $v(t) = V\cos(\omega t + \phi)$

REPLACE WITH $\bar{v}(t) = \bar{V}e^{j\omega t}$ WHERE $\bar{V} = Ve^{j\phi}$

USING THE ALGEBRA OF COMPLEX NUMBERS FIND AN OUTPUT OF THE FORM

$\bar{v}_{OUT}(t) = \bar{V}_{OUT}e^{j\omega t}$ THE REAL PART OF WHICH IS THE ACTUAL

OUTPUT $v_{OUT} = V_{OUT}\cos(\omega t + \phi + \theta)$ WHERE $\bar{V}_{OUT} = V_{OUT}e^{j(\phi+\theta)}$

CONSIDER THE BASIC PASSIVE ELEMENTS WITH EXPONENTIAL EXCITATION

LET $\bar{v}(t) = \bar{V}e^{st}$, $\bar{i}(t) = \bar{I}e^{st}$ WHERE $s = j\omega$ FOR CONVENIENCE

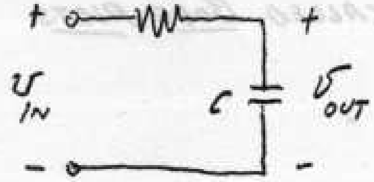
$\bar{V}$ AND $\bar{I}$ ARE COMPLEX AMPLITUDES

RESISTANCE	CAPACITANCE	INDUCTANCE
$\bar{v}(t) = R\bar{i}(t)$	$\bar{I}e^{st} = C\frac{d}{dt}\bar{V}e^{st}$	$\bar{V}e^{st} = L\frac{d}{dt}\bar{I}e^{st}$
$\bar{V} = R\bar{I}$	$= Cs\bar{V}e^{st}$	$\bar{V} = Ls\bar{I}$
$\bar{Z}_R = \frac{\bar{V}}{\bar{I}} = R$	$\bar{Y}_C = \frac{\bar{I}}{\bar{V}} = Cs$	$\bar{Z}_L = \frac{\bar{V}}{\bar{I}} = Ls$
$\bar{Y}_R = 1/R = G$	$\bar{Z}_C = \frac{1}{Cs}$	$\bar{Y}_L = 1/Ls$

$\bar{Z}$ IS THE IMPEDANCE OF THE ELEMENT; $\bar{Y}$ IS THE ADMITTANCE.

ONCE THE ELEMENTS ARE REPRESENTED BY THEIR IMPEDANCES OR ADMITTANCE THE CIRCUIT CAN BE ANALYZED JUST LIKE A PURELY RESISTIVE NETWORK! THE ALGEBRA OF COMPLEX NUMBERS IS EMPLOYED.

EXAMPLE I



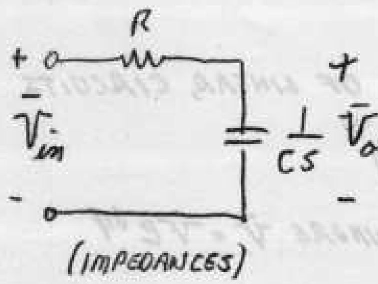
$v_{IN} = V_{in}\cos\omega t$

FIND v_{OUT}

EXPLORE FREQUENCY DEPENDENCE OF v_{OUT}

FIRST CONSIDER THE ASYMPTOTES

ASSUME AN EXCITATION OF $\bar{V}_{in} e^{st}$ WHERE $\bar{V}_{in} = V e^{j0} = V$



JUST LIKE A RESISTIVE VOLTAGE DIVIDER:

$$\bar{V}_{out} = \bar{V}_{in} \frac{\frac{1}{CS}}{R + \frac{1}{CS}} = \bar{V}_{in} \frac{1}{1 + RCS}$$

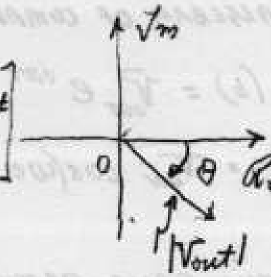
LET $S = j\omega$ AND

PUT $\bar{V}_{out}$ IN POLAR FORM: $\bar{V}_{out} = V_{in} \frac{1}{\sqrt{1+(RC\omega)^2}} e^{-j\theta}$

$\theta = \tan^{-1}(RC\omega)$

THEN

$$v_{OUT} = \mathrm{Re}\left[V e^{j0} \frac{1}{\sqrt{}} e^{-j\theta} e^{j\omega t} \right]$$



$$v_{OUT} = \frac{V}{\sqrt{}} \cos(\omega t - \theta)$$

THE OUTPUT IS, OF COURSE, A COSINE OF FREQUENCY ω, DELAYED WITH RESPECT TO THE INPUT BY θ AND ATTENUATED BY $1/\sqrt{}$. BOTH θ AND THE AMPLITUDE $V/\sqrt{}$ ARE FUNCTIONS OF ω.

THE DEPENDENCE OF THE ATTENUATION AND PHASE LAG ON ω IS BEST EXPLORED BY FOCUSING ON THE VOLTAGE TRANSFER RATIO $\bar{A}$

$$\bar{A}(\omega) = \frac{\bar{V}_{out}}{\bar{V}_{in}} = \frac{1}{1 + jRC\omega} = \frac{1}{\sqrt{1+(RC\omega)^2}} e^{-j\theta} \qquad \theta = \tan^{-1}(RC\omega)$$

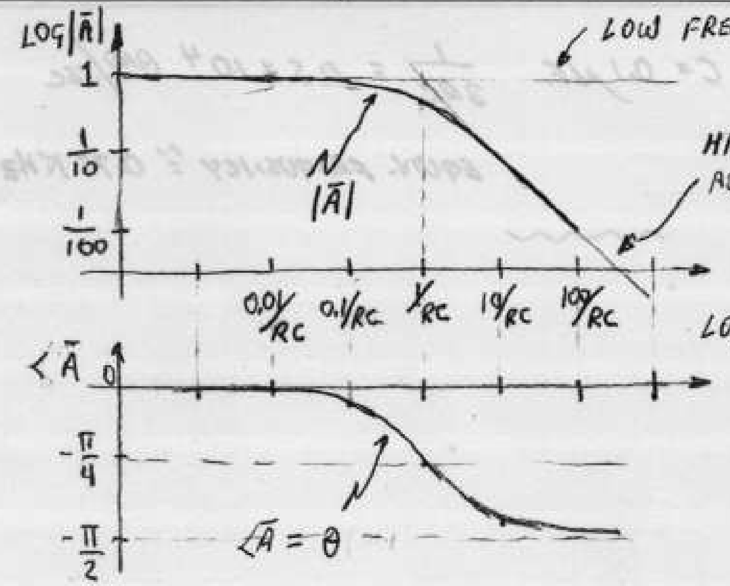
FOR LOW FREQUENCIES $\left(\omega \ll \frac{1}{RC}\right)$ $\bar{A}(\omega) \cong 1$ WHY?

FOR HIGH FREQUENCIES $\left(\omega \gg \frac{1}{RC}\right)$ $\bar{A}(\omega) \cong \frac{1}{RC\omega} e^{-j\pi/2}$ WHY?

BECAUSE ω IN MANY CASES MAY VARY OVER MANY DECADES IT IS CONVENIENT TO PLOT $|\bar{A}(\omega)|$ ON A LOGARITHMIC SCALE FOR ω.

FOR REASONS THAT WILL SOON BE CLEAR, $|\bar{A}(\omega)|$ IS ALSO PLOTTED ON A LOGARITHMIC SCALE WHILE $\theta = \angle \bar{A}(\omega)$ IS PLOTTED ON A LINEAR SCALE.

THE RESULTING DIAGRAMS ARE CALLED <u>BODE PLOTS</u>



FOR HIGH FREQUENCIES

$|\bar{A}| \cong \frac{1}{RC} \times \frac{1}{\omega}$

$\log|\bar{A}| = \log\frac{1}{RC} - \log\omega$

A STRAIGHT LINE WITH A SLOPE OF −1 ON $\log|\bar{A}|$ VS $\log\omega$ COORDINATES

WHEN $\omega = \frac{1}{RC}$ $|\bar{A}| = 1$

WHICH LOCATES THE ASYMPTOTE

DEMO $R = 1K\Omega$ $C = 0.1\mu F$

$1/RC = 10^4$ RAD/SEC WHERE THE ASYMPTOTES INTERSECT

THE EQUIVALENT FREQUENCY IS $\frac{10^4}{2\pi} \cong 1.5\ KHz$ A LOW-PASS FILTER

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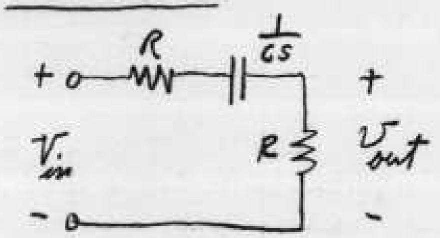
EXAMPLE II



(IMPEDANCES)

AGAIN A VOLTAGE DIVIDER:

$$\bar{A}(\omega) = \frac{R}{R + R + \frac{1}{CS}} = \frac{RCS}{1 + 2RCS}$$

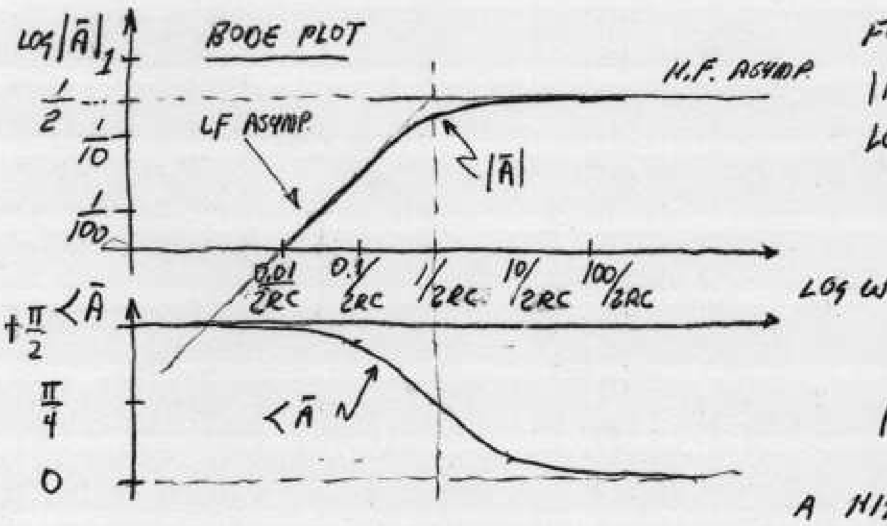
$$|\bar{A}(\omega)| = \frac{RC\omega}{\sqrt{1 + (2RC\omega)^2}} \qquad \angle\bar{A}(\omega) = \frac{\pi}{2} - \theta$$

$$\theta = \tan^{-1}(2RC\omega)$$

FOR LOW FREQUENCIES $|\bar{A}| \to 0$ WHY?

FOR HIGH FREQUENCIES $|\bar{A}| \cong \frac{1}{2}$ WHY?

BODE PLOT



FOR LOW FREQUENCIES

$|\bar{A}| = RC \times \omega$

$\log|\bar{A}| = \log RC + \log\omega$

STRAIGHT LINE OF SLOPE +1

FOR HIGH FREQUENCIES

$|\bar{A}| \cong \frac{1}{2}$

A HIGH-PASS FILTER
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DEMO $R = 1K$, $C = 0.1\,\mu F$. $\frac{1}{2RC} = 0.5 \times 10^4$ RAD/SEC

EQUIV. FREQUENCY $\cong 0.75$ KHz