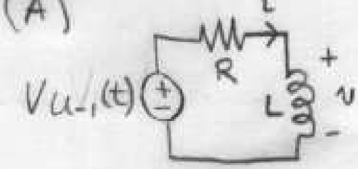
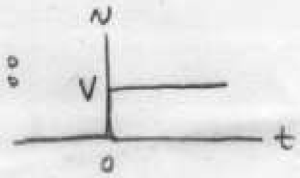
# 6.002 HOMEWORK #7
## SOLUTIONS

## PROBLEM 7.1

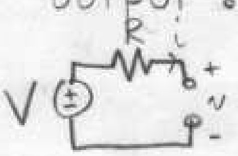
(A)



input:

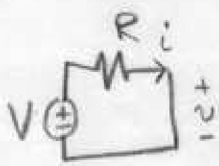


output: at $t=0^+$ the inductor looks like an open circuit, so



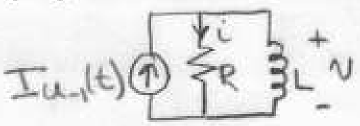
$$\boxed{t=0^+ \quad i=0 \quad v=V}$$

at $t\to\infty$ the inductor looks like a short circuit, so

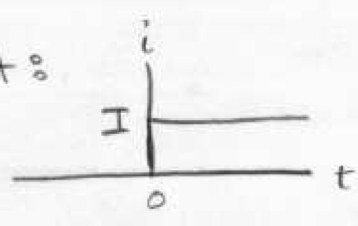


$$\boxed{t\to\infty \quad i=\frac{V}{R} \quad v=0}$$

(B)

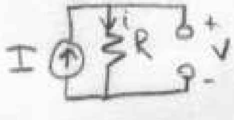


input:



output:

at $t=0^+$, the indutor is open, all current goes through R.



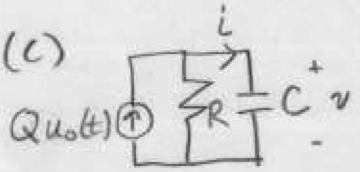
$$\boxed{t=0^+ \quad i=I \quad v=I/R}$$

at $t\to\infty$, the inductor is a short, all current goes through the short.

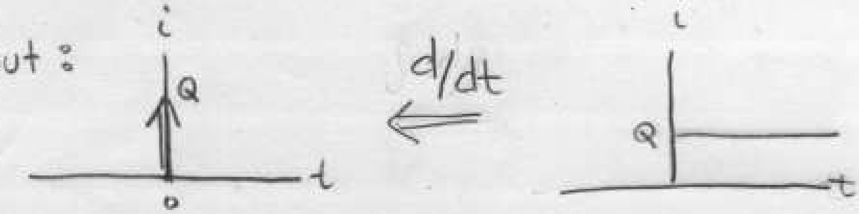
$$\boxed{t\to\infty \quad i=0 \quad v=0}$$

# HW7 SOLUTIONS p.2

(c)

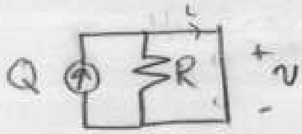



input:



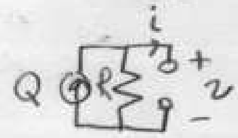

Output of step:

at $t=0^+$ the capacitor looks like a short

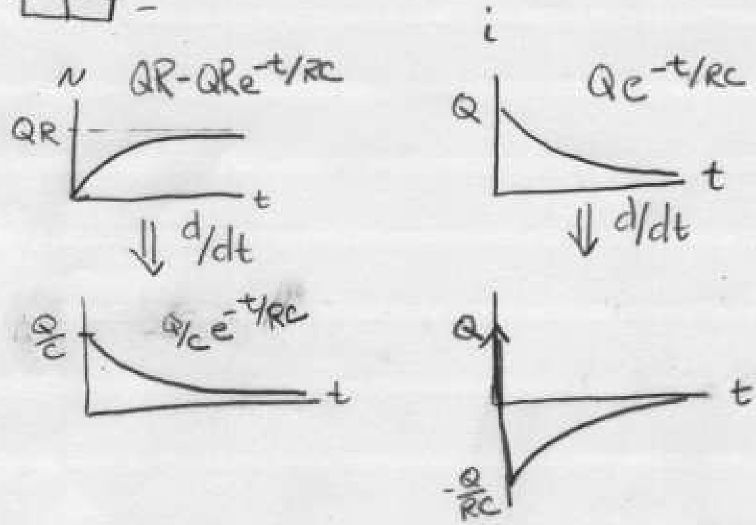



$t=0^+$ $\quad i=Q \quad v=0$

at $t\to\infty$ the capacitor look like an open




$t\to\infty \quad i=0 \quad v=QR$




$\Leftarrow$ IMPULSE RESPONSE $= \frac{d}{dt}$ step RESPONSE

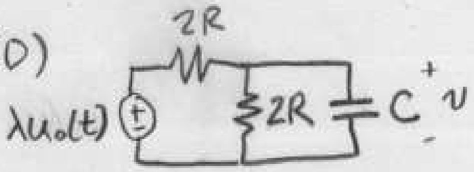
Impulse Response : $Q=CV$, the impulse dumps charge on the capacitor, so $v=Q/c$, which the decays to 0

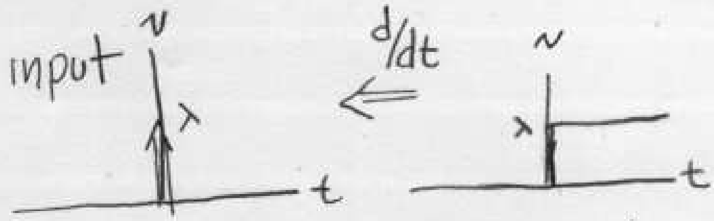
an impulse of Q passes through the cap at $t=0$ (because it acts like a short), the induced voltage then decays as current through R in the negative direction.

$$\boxed{\begin{array}{l} t=0^+ \quad i=-Q/RC, \quad v=Q/c \\ t\to\infty \quad i=0, \quad v=0 \end{array}}$$

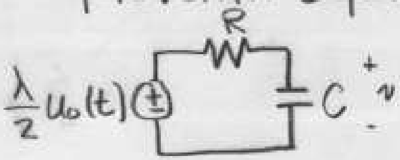
## HW 7 SOLUTIONS p.3

(D)



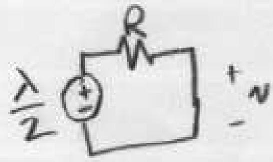




Thévenin Equivalent:



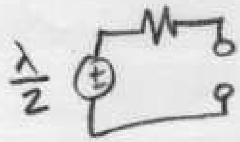

step response:

$t=0^+$ the capacitor looks like a short.

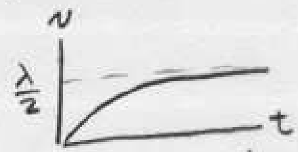



$v=0$

$t\to\infty$ the capacitor looks like an open.




$v=\frac{\lambda}{2}$

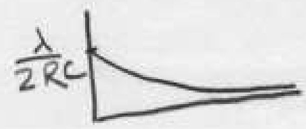



$\frac{\lambda}{2}-\frac{\lambda}{2}e^{-t/RC}$

$\Downarrow$ d/dt

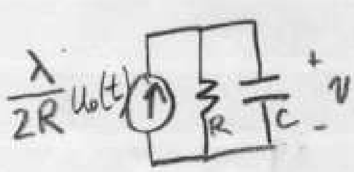
Impulse Response:

$$\boxed{\begin{array}{ll} t=0^+ & v=\frac{\lambda}{2RC} \\ t\to\infty & v=0 \end{array}}$$




$\frac{\lambda}{2RC}e^{-t/RC}$

This is easy to see if we change the Thévenin equivalent to a Norton equivalent.



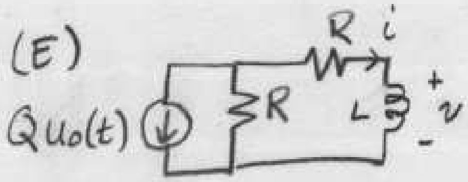

at $t=0^+$ an impulse of charge is dumped on the capacitor.

$$Q=CV$$

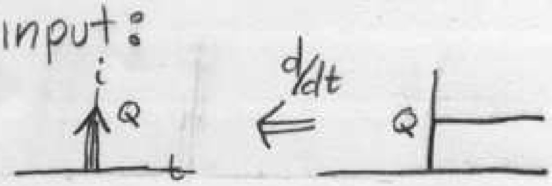
$$V=\frac{Q}{C}=\frac{\lambda}{2RC}$$

# HW 7 SOLUTIONS p.4

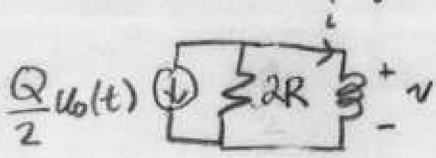
(E)



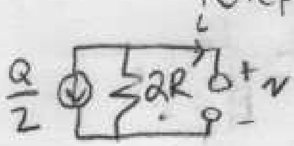

input:




Norton Equivalent:




Step Response:

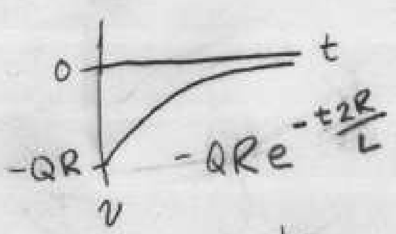


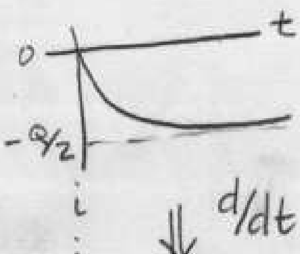

$t=0^+$: the inductor looks like an open circuit, all current flows through the resistor.

$i=0 \quad v = -\frac{Q}{2}\cdot 2R = -QR$


$t \to \infty$: the inductor looks like a short, all current flows through the inductor.

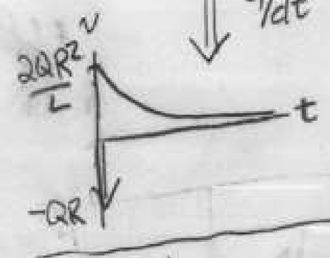
$i = -\frac{Q}{2} \quad v = 0$

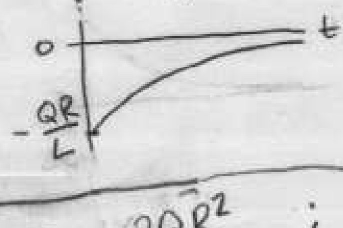



$-\frac{Q}{2} + \frac{Q}{2}e^{-t2R/L}$

IMPULSE RESPONSE:




| | | |
|---|---|---|
| $t=0^+$ | $v = \frac{2QR^2}{L}$ | $i = -\frac{QR}{L}$ |
| $t \to \infty$ | $i=0$ | $v=0$ |

let's look at the ThéVenin EQUIVALENT:


$\lambda = Li$

$t=0$ flux linkage through $L$: $-QR = \lambda$

this induces a current $= -\frac{QR}{L}$

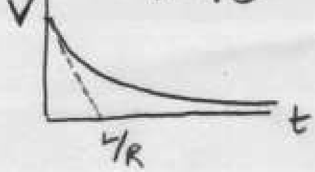
this current induces a voltage across R and decays.

## PROBLEM 7.2

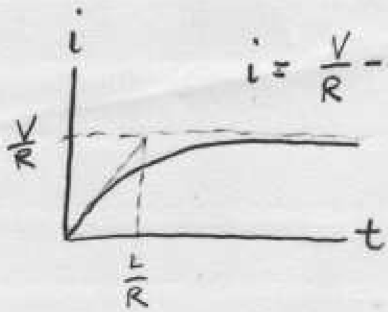
FOR EXPLAINATIONS SEE SOLUTION FOR PROBEM 7.1

(A) $\tau = L/R$

$v = Ve^{-tR/L}$

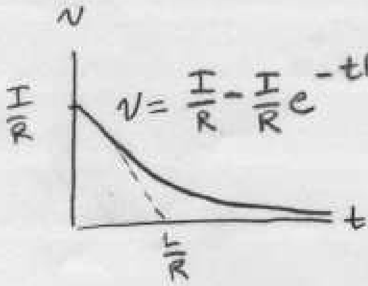


$i = \frac{V}{R} - \frac{V}{R}e^{-tR/L}$

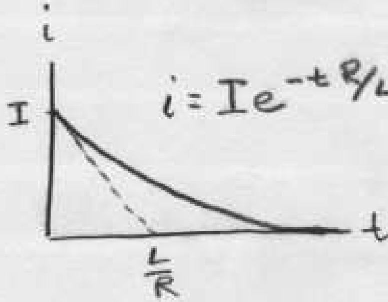


(B) $\tau = L/R$

$v = \frac{I}{R} - \frac{I}{R}e^{-tR/L}$

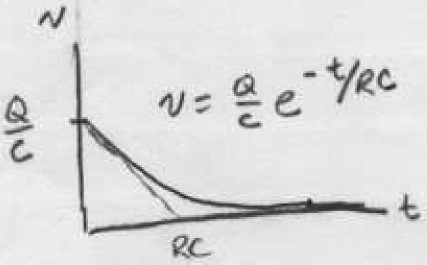


$i = Ie^{-tR/L}$

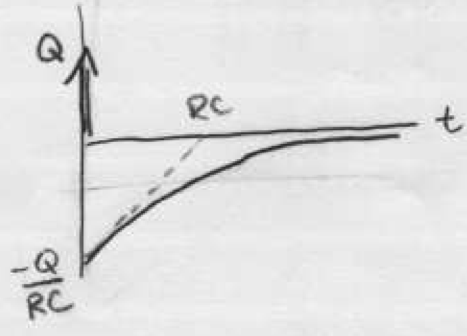


(C) $\tau = RC$

$v = \frac{Q}{C}e^{-t/RC}$

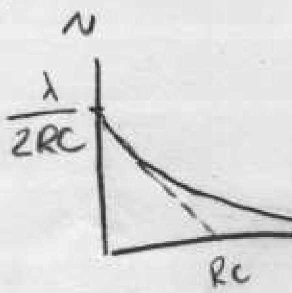


$i = Qu_0(t) - \frac{Q}{RC}e^{-t/RC}$



(D) $\tau = RC$

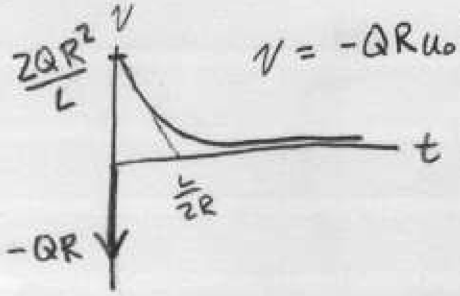
$v = \frac{\lambda}{2RC}e^{-t/RC}$

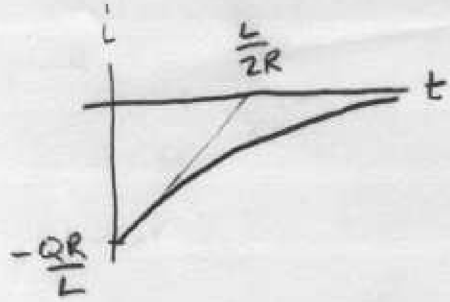


HW 7 SOLUTIONS p.6

(E) $\tau = L/2R$

$$v = -QRu_0(t) + \frac{2QR^2}{L} e^{-\frac{2Rt}{L}}$$




PROBLEM 7.3

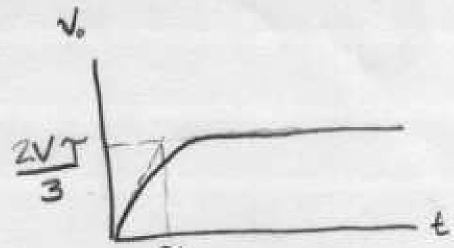
(A) The graph shown is the output of a unit impulse excitation. The integral of this graph is the output of a unit step excitation.

$$v_o(t) = V\int \frac{2}{3} e^{-t/\tau}\, dt \qquad \text{for } t>0$$

$$v_o(t) = -\frac{2V\tau}{3} e^{-t/\tau} + C$$

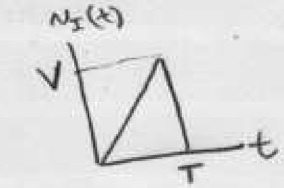
There is no impulse in the output of the impulse response, therefore we can assume that the voltage starts at $v(0)=0$

$$v_o(t) = \frac{2V\tau}{3} - \frac{2V\tau}{3} e^{-t/\tau}$$



HW 7 SOLUTIONS P. 7

(B) input:




$$v_I(t) = \frac{V}{T}u_{-2}(t) - \frac{V}{T}u_{-2}(t-T) - Vu_{-1}(t-T)$$

the input is the sum of 3 inputs,

① a ramp of slope $V/T$ that starts at $t=0$

② a ramp of slope $-V/T$ that starts at $t=T$

③ a down step at $t=T$.

output: sum of 3 individual responses.

① a ramp is the integral of the step.

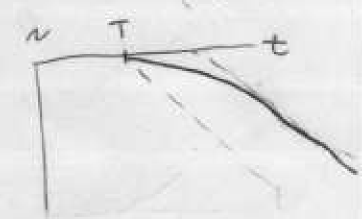
$$v_o = \int\left[\frac{2V\tau}{T3} - \frac{2V\tau}{T3}e^{-t/\tau}\right]dt = \frac{2V\tau}{T3}t + \frac{2V\tau^2}{T3}e^{-t/\tau} + C$$

$v_o(0)=0 \qquad C = -\frac{2V\tau^2}{3T}$

$$\Rightarrow v_o = \frac{2V\tau^2}{T3}e^{-t/\tau} + \frac{2V\tau}{T3}t - \frac{2V\tau^2}{3T}$$

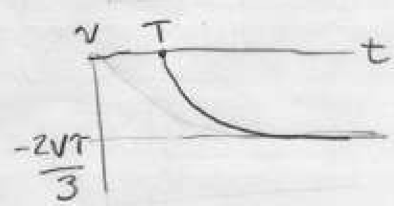

② this is a negative scaled ramp

$$\Rightarrow v_o = u_{-1}(t-T)\left[-\frac{2V\tau^2}{3T}e^{-\frac{(t-T)}{\tau}} - \frac{2V\tau}{3T}(t-T) + \frac{2V\tau^2}{3T}\right]$$




③ this is a negative step

$$\Rightarrow v_o = u_{-1}(t-T)\left[\frac{2V\tau}{3}e^{-\frac{(t-T)}{\tau}} - \frac{2V\tau}{3}\right]$$




Total:

$$\boxed{v_o(t) = \begin{cases} \frac{2V\tau^2}{3T}e^{-t/\tau} + \frac{2V\tau}{3T}t - \frac{2V\tau^2}{3T}, & 0<t<T \\ \frac{2V\tau}{3}e^{-t/\tau}\left[\frac{\tau}{T} + \left(1-\frac{\tau}{T}\right)e^{T/\tau}\right]u_{-1}(t-T), & T<t \end{cases}}$$

See next page for calc. of

$$v_o(t>T) = \frac{2V\tau^2}{3T} e^{t/\tau} + \cancel{\frac{2V\tau}{3T}t} - \cancel{\frac{2V\tau^2}{3T}} +$$

$$-\frac{2V\tau^2}{3T} e^{-\frac{(t-T)}{\tau}} - \cancel{\frac{2V\tau}{3T}t} + \cancel{\frac{2V\tau}{3}} + \cancel{\frac{2V\tau^2}{3T}} +$$

$$\frac{2V\tau}{3} e^{-\frac{(t-T)}{\tau}} - \cancel{\frac{2V\tau}{3}}$$

---

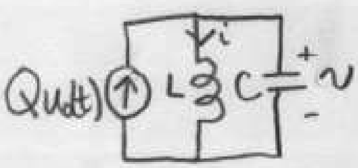
$$= \frac{2V\tau^2}{3T} e^{-t/\tau} - \frac{2V\tau^2}{3T} e^{-t/\tau} e^{T/\tau} + \frac{2V\tau}{3} e^{-t/\tau} e^{T/\tau}$$

$$= \tau \frac{2}{3} V e^{-t/\tau} \left[ \frac{\tau}{T} - \frac{\tau}{T} e^{T/\tau} + e^{T/\tau} \right]$$

$$= \frac{2V\tau}{3} e^{-t/\tau} \left[ \frac{\tau}{T} + \left(1 - \frac{\tau}{T}\right) e^{T/\tau} \right]$$

this is the response after $T = t$
it decays to 0 as $t \to \infty$

## PROBLEM 7.4




(A) The impulse places charge across the capacitor.

$Q = CV$, so $v(0^+) = Q/C$

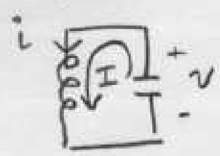
This voltage begins to change the current through the inductor, but cannot change it immediately $v = L\frac{di}{dt}$, so $i(0^+) = 0$

# HW 7 SOLUTIONS P.9

(B) at $t=0^+$ the current through the inductor increases because we have placed a positive voltage across it.

$$\boxed{\frac{di_L}{dt} \Rightarrow (+)}$$

The current induced through the inductor is negative through the capacitor.




$$i = C\frac{dv}{dt}$$

Because the current is negative, the voltage must be decreasing.

$$\boxed{\frac{dv_c}{dt} \Rightarrow (-)}$$

(c) differential equation for $v(t)$:

$$i = -C\frac{dv}{dt} \qquad i = \frac{1}{L}\int V dt$$

$$\frac{1}{L}\int V dt + C\frac{dV}{dt} = 0$$

$$C\frac{d^2v}{dt^2} + \frac{1}{L}V = 0$$

$$V'' + \frac{1}{LC}V = 0$$

choose $v = A\cos(\omega t)$ ← cosine because $v(0) \neq 0$

$$v' = -A\omega\sin(\omega t)$$

$$v'' = -A\omega^2\cos(\omega t)$$

$$-A\omega^2\cos(\omega t) + \frac{A}{LC}\cos(\omega t) = 0 \qquad \omega = \frac{1}{\sqrt{LC}}$$

$$v_o(t) = A\cos\left(\frac{t}{\sqrt{LC}}\right)$$
$$v_o(0) = Q/C$$

$$\boxed{v_o(t) = \frac{Q}{C}\cos\left(\frac{t}{\sqrt{LC}}\right)}$$

$$\boxed{i_o(t) = -C\frac{dv}{dt} = \frac{Q}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)}$$

EQUIVALENTLY, YOU COULD WRITE A DIFFERENTIAN EQN. FOR $i(t)$:

$$v = -\frac{1}{C}\int i\,dt \qquad v = L\frac{di}{dt}$$

$$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$$

$$i'' + \frac{1}{LC}i = 0$$

choose $i = A\sin(\omega t)$ ← sine because $i(0) = 0$

$$i' = A\omega\cos(\omega t)$$

$$i'' = -A\omega^2\sin(\omega t)$$

$$-A\omega^2\sin(\omega t) + \frac{1}{LC}\sin(\omega t) = 0 \qquad \omega = \frac{1}{\sqrt{LC}}$$

$$i(t) = A\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$i(0) = 0$ ← this doesn't help us

$$v(t) = L\frac{di}{dt} = \frac{LA}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$v(0) = Q/C = LA/\sqrt{LC} \qquad A = \frac{Q\sqrt{LC}}{LC} = \frac{Q}{\sqrt{LC}}$$

$$\boxed{i(t) = \frac{Q}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)}$$

$$\boxed{v(t) = \frac{Q}{C}\cos\left(\frac{t}{\sqrt{LC}}\right)}$$