3.11

\[ i(t = 0^-) = I \]

\[ C = 0.1 \mu F \]
\[ L = 40 \text{ mH} \]

A) The natural frequency is the roots of the characteristic equation.

Find differential equation for the circuit:

We know that \( V_L = L \frac{di(t)}{dt} \) and \( V_C = \frac{1}{C} \int i(t) dt \)

Using these equations and KVL in the circuit for \( t > 0 \)

\[ L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt = 0 \]

Differentiate to remove the integral:

\[ L \frac{d^2i(t)}{dt^2} + \frac{1}{C} i(t) = 0 \]

Now plug in \( i(t) = Ae^{st} \):

\[ L s^2 Ae^{st} + \frac{1}{C} Ae^{st} = 0 \]

\[ s^2 + \frac{1}{LC} = 0 \]

Find the roots of this characteristic equation to determine the natural frequency:

\[ s = \pm j \frac{w}{\text{rad/sec}} \]

where \( w = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{\mu F \cdot \text{mH}}} \):

\[ w = 15811.4 \text{ rad/sec} ; f = \frac{w}{2\pi} = 2.5 \text{ kHz} \]

Determine period, first convert \( w \) into frequency:

\[ w = 2\pi f \]

\[ f = \frac{15811.4 \text{ rad/sec}}{2\pi \cdot \text{rad/rev}} = 2516.46 \text{ Hz} \]

The period is \( T = \frac{1}{f} \Rightarrow T = \frac{1}{2516.46} = 3.97 \times 10^{-4} \text{ sec} \)
B.) As seen in part A.) using KVL, the differential equation for current is:
\[ \frac{d^2 i(t)}{dt^2} + \frac{1}{LC} i(t) = 0 \quad \text{for} \quad t > 0 \]

C.) To solve the differential equation:
\[ i(t) = i_n(t) + i_p(t) \quad \text{for this case} \quad i_p(t) = 0 \]

Plugging \( i(t) = Ae^{st} \) to solve for \( i_n(t) \) we get:
\[ \frac{d^2}{dt^2} Ae^{st} + \frac{1}{LC} Ae^{st} = 0 \]
\[ s^2 = -\frac{1}{LC} \]
\[ s = \pm j\omega \quad \text{where} \quad \omega = \sqrt{\frac{1}{LC}} \]

Therefore:
\[ i(t) = Ae^{-j\omega t} + Be^{j\omega t} \]

We can switch this to a more intuitive form:
\[ i(t) = A_1 \cos \omega t + B_1 \sin \omega t \]

Now we use initial conditions to solve for the constants:
\[ i(t=0) = I \quad \Rightarrow \quad I = A_1 \cos 0 + B_1 \sin 0 \]
\[ B_1 = I \]

We can use the voltage to solve for \( A_1 \):
\[ V(t) = \frac{-1}{C} \int i(t) \, dt \]
\[ V(t) = \frac{-I}{wC} \sin \omega t + \frac{A_1}{wC} \cos 2\omega t + K_1 \quad \text{constant of integration} \]
Since there is no impulse into the circuit
\[ v(t,0') = 0 \Rightarrow 0 = \frac{-I}{wC} \sin 0' + \frac{A_1}{wC} \cos 0' + K_1 \]
\[ 0 = \frac{A_1}{wC} + K \Rightarrow A_1 = -K = 0 \]

Therefore the solution is
\[ i(t) = I \cos 2\omega t \quad \text{where} \quad w = \sqrt{L/C} \]
\[ v(t) = \frac{-I}{wC} \sin 2\omega t = -I \sqrt{\frac{C}{L}} \sin 2\omega t \]

or
\[ i(t) = I \cos (15811.4t) \]
\[ v(t) = -632.5I \sin (15811.4t) \]

8.2
\[ i(t=0) = I \]
\[ \begin{array}{c}
\text{C} = 1 \mu F \\
\text{L} = 40 \text{mH} \\
\text{G} = 5 \times 10^4 \text{ mhos}
\end{array} \]

A) By using KCL at the top node, we obtain a differential equation for \( v(t) \)
\[ v(t) \left( G + C \frac{dv(t)}{dt} + \frac{1}{L} \int v(t) dt \right) = 0 \]

Differentiate to remove integral.
\[ G \frac{dv(t)}{dt} + C \frac{d^2v(t)}{dt^2} + \frac{1}{L} v(t) = 0 \]

Arrange into standard form
\[ \frac{d^2v(t)}{dt^2} + \frac{G}{C} \frac{dv(t)}{dt} + \frac{1}{CL} v(t) = 0 \]
B) Plug in \( r(t) = Ae^{st} \) to find the characteristic equation:

\[
Ae^{st} GS + Ae^{st} C s^2 + \frac{1}{L} Ae^{st} = 0
\]

Rearrange:

\[
s^2 + \frac{GS}{C} + \frac{1}{LC} = 0
\]

- Characteristic equation

Now find the roots:

\[
s = \frac{-\frac{GS}{C} \pm \sqrt{\left(\frac{GS}{C}\right)^2 - 4\left(\frac{1}{LC}\right)}}{2}
\]

Plugging values:

\[
s = \frac{-5 \times 10^{-4}}{1 \times 10^{-6}} \pm \sqrt{\frac{5 \times 10^{-4}}{1 \times 10^{-6}} - 4 \left(\frac{1}{(4.1 \times 10^3)(1 \times 10^{-6})}\right)}
\]

\[
s = \frac{-5000 \pm \sqrt{(5000)^2 - 1 \times 10^9}}{2}
\]

\[
s = -2500 \pm j 15612.5
\]

- Natural frequency

C) From part A):

\[
\frac{d^2 v(t)}{dt^2} + \frac{G}{C} \frac{dv(t)}{dt} + \frac{1}{CL} v(t) = 0
\]

In general:

\[
\frac{d^2 y}{dt^2} + 2\alpha \frac{dy}{dt} + \omega_n^2 y = 0
\]

Therefore:

\[
2\alpha = \frac{G}{C}
\]

\[
\alpha = \frac{G}{2C} = 2500
\]

\[
\omega_n^2 = \frac{1}{LC}
\]

\[
\omega_n = \frac{1}{\sqrt{LC}} = 15811.4
\]
\[ Q = \frac{D_0}{\pi \lambda} = \frac{\sqrt{L/L_0}}{\pi} = \frac{\sqrt{L}}{G} = 3.16 \]

\[ Q = \frac{\sqrt{L}}{G} = 3.16 \]

8.3 A)

\[ \begin{align*}
R & \quad V_{\text{cos}(\omega t)} \\
L & \quad V_L \quad ?
\end{align*} \]

Use impedances to solve and to find \( V_L \) use a voltage divider

\[ V_{\text{cos}(\omega t)} = V_{\text{efg}} \]

\[ V_L = \frac{Z_L}{Z_L + Z_R} \tilde{V_i} \]

\[ \tilde{V_L} = \frac{\tilde{L}}{\tilde{R} + \tilde{L}} \]

\[ \tilde{V_i} = \frac{j \omega L}{j \omega L + R} \]

Convert to polar form

\[ \tilde{V_L} = \frac{wL}{\sqrt{R^2 + (j\omega L)^2}} e^{j\phi} \]

where \[ \phi = \frac{\pi}{2} \tan^{-1} \left( \frac{wL}{R} \right) \]

Take real magnitude to get \( V_L \) value

\[ V_L(t) = \frac{wL}{\sqrt{R^2 + (j\omega L)^2}} V_{\text{cos} \left( \omega t + \phi \right)} \]

B)

\[ i(t) = I_{\text{cos}(\omega t)} = I_{\text{efg}} \]

Combine the two parallel impedances

\[ Z_{\text{tot}} = Z_R || Z_C = \frac{1}{\frac{1}{j\omega C} \cdot R} \]

\[ Z_{\text{tot}} = \frac{R}{1 + j\omega RC} \]

\[ Z_{\text{tot}} = \frac{R}{1 + j\omega RC} \rightarrow Z_{\text{tot}} = \frac{R}{1 + j\omega RC} \]
Now we can say
\[ V_C = I Z_{tot} \]
\[ V_C = \frac{R}{1 + \omega RC} I e^{j\omega t} \]
Put this into polar form
\[ V_C = \frac{R}{\sqrt{1 + (\omega RC)^2}} I e^{j\omega t} \]
where \( \phi = -\tan^{-1}\left( \frac{\omega RC}{1} \right) \)
Take real magnitude
\[ V_C = \frac{R}{\sqrt{1 + (\omega RC)^2}} I \cos(\omega t + \phi) \]

8.4

For small \( \omega \), the inductor will look close to a short circuit so \( V_L \) will be very small or close to 0.

From the equation, the \( I \) term in the denominator will dominate and we can ignore the \( \left(\frac{\omega L}{R}\right)^2 \) term. Therefore the magnitude is
\[ JWL \text{ which is approximately zero if } \omega L < \frac{R}{C} \]

B) At high frequencies the inductor has a high impedance, and most of the current will go to the resistor. Therefore the \( V_L \) will equal \( V_L = IR \).
Looking at the equation, the \((\frac{wL}{R})^2\) term dominates the denominator so the magnitude reduces to:

\[ V_L = \frac{I}{\frac{wL}{R}} \Rightarrow V_L = IR \]

C) The impedances of the resistor and the inductor must be equal or the imaginary and real portions of the denominator must be equal.

\[ I = \frac{wL}{R} \]
\[ \omega = \frac{R}{L} \]

D) When the imaginary and real parts are equal, the signal is \(\frac{1}{\sqrt{2}}\) times the magnitude with just the resistor. By plugging into the equation we can get this:

\[ |V_L| = I \frac{(R/L)(L)}{\sqrt{1 + (R/L)(L/R)}} = \frac{IR}{\sqrt{2}} = 0.707\ IR \]

The phase will be \(\phi = 45^\circ\)

Using the equation:

\[ \phi = \tan^{-1}\left(\frac{R/L}{R}\right) = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4} \text{ radians} \]
E) The condition for low frequencies is \( \omega \ll \frac{1}{RC} \).

For low frequencies, the capacitor has a large impedance so there will be a current that is almost zero, and almost all of the voltage across the capacitor.

\[
|V_C(\omega)| = V \quad |i(\omega)| \approx 0
\]

Looking at the equations: For \( i(\omega) \), the \((\omega RC)^2\) term can be ignored and \( |i(\omega)| = \frac{VC}{\omega} \), with small \( \omega \). \( |i(\omega)| \approx 0 \)

For the voltage, the \((\omega RC)^2\) term can be ignored and we have \( |V_C(\omega)| = \frac{V}{1} \) or all of the voltage across the capacitor.

F) The condition for high frequencies is \( \omega \gg \frac{1}{RC} \).

At high frequencies, the opposite is true. The capacitor has a very small impedance, and looks almost like a short. Therefore, the current is limited by the resistor and almost all the voltage falls across the resistor.

\[
|V_C(\omega)| \approx 0 \quad \text{and} \quad |i(\omega)| = \frac{V}{R}
\]

Looking at the equations, we see the \((\omega RC)^2\) term dominates the denominator in both.

\[
|V_C(\omega)| = \frac{V}{\omega RC} \quad \Rightarrow \quad \text{with large } \omega \quad |V_C(\omega)| \approx 0
\]

\[
|i(\omega)| = \frac{VRC}{\omega RC} = \Rightarrow \quad i(\omega) = \frac{V}{R}
\]
G) The voltage divides evenly when the impedances of the C and R are equal or when the imaginary and real parts are equal.

\[ I = \frac{V}{RC} \]  
\[ w = \frac{1}{RC} \]

H) As in G), the magnitude is \( \frac{1}{\sqrt{2}} \) times the total response and the phase is 45°.

\[ |V_C(t)| = \frac{V}{\sqrt{1 + \left(\frac{1}{RC}R\right)^2}} = \frac{V}{\sqrt{2}} = 0.707\, V \]

\[ |x(t)| = \frac{V}{\sqrt{1 + 1}} = \frac{V}{\sqrt{2}} = 0.707\, \frac{V}{R} \]

\[ \psi = \tan^{-1}\left(\frac{1}{RC}R\right) = \tan^{-1}\left(\frac{1}{1}\right) \Rightarrow \phi = 45° \]