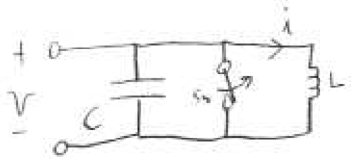
Homework #8 Solutions (S-OSD)

3.11 $i(t=0^+) = I$ $C = 0.1\mu F$

$L = 40mH$



A) The natural frequency is the roots of the characteristic equation.

Find differential equation for the circuit:

We know that $v_L = L\frac{di(t)}{dt}$ and $v_c = \frac{1}{C}\int i(t)dt$

Using these equations and KVL in the circuit for $t>0$

$$L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = 0$$

Differentiate to remove the integral

$$L\frac{d^2 i(t)}{dt^2} + \frac{1}{C} i(t) = 0$$

Now plug in $i(t) = Ae^{st}$:

$$Ls^2 Ae^{st} + \frac{1}{C} Ae^{st} = 0$$

$$s^2 + \frac{1}{LC} = 0$$

Find the roots of this characteristic equation to determine the natural frequency

$\boxed{s = \pm j\omega \frac{rad}{sec}}$ where $\boxed{\begin{array}{l} \omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(.1\mu F)(40mH)}} : \\ \omega = 15811.4 \ rad/sec \ ; \ f = \frac{\omega}{2\pi} = 2.5kHz \end{array}}$

determine period, first convert ω_o into frequency

$$\omega = 2\pi f$$

$$f = \frac{15811.4 \ rad/sec}{2\pi \ rad} = 2516.46 \ Hz$$

The period is $T = \frac{1}{f}$ => $\boxed{T = \frac{1}{2516.46 \ \frac{1}{sec}} = 3.97 \times 10^{-4} \ sec}$

B.) As seen in part A.) using KVL, the differential equation for current is.

$$\frac{d^2 i(t)}{dt^2} + \frac{1}{LC} i(t) = 0 \qquad \text{for} \quad t > 0$$

C.) To solve the diff. eq.:

$i(t) = i_h(t) + i_p(t)$ for this case $i_p(t) = 0$

Plugging $i(t) = Ae^{st}$ to solve for $i_h(t)$ we get:

$$i_h(t) = s^2 \cancel{Ae^{st}} + \frac{1}{LC} \cancel{Ae^{st}} = 0$$

$$s^2 = -\frac{1}{LC}$$

$$s = \pm jw \qquad \text{where} \quad w = \sqrt{\tfrac{1}{LC}}$$

There fore

$$i(t) = Ae^{-jwt} + Be^{jwt}$$

We can switch this to a more intuitive form

$$i(t) = A_1 \sin wt + B_1 \cos wt$$

Now we use initial conditions to solve for the constants

$$i(t=0^+) = I \quad \Rightarrow \quad I = A_1 \cancel{\sin 0}^{\,0} + B_1 \cancel{\cos 0}^{\,1}$$

$$\underline{B_1 = I}$$

We can use the voltage to solve for A_1

$$v(t) = \frac{-1}{C} \int i(t)\,dt$$

$$v(t) = \frac{-I}{wC} \sin wt + \frac{A_1}{wC} \cos wt + K_1$$ constant of integration

Since there is no impulse into the circuit

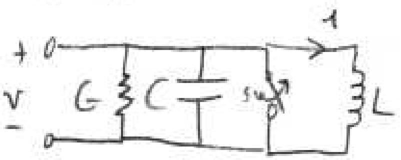
$v(t=0^+) = 0 \Rightarrow \qquad 0 = \frac{-I}{wC}\overset{\to 0}{\sin 0} + \frac{A_1}{wC}\overset{\to 1}{\cos 0} + K_1$

$$0 = \frac{A_1}{wC} + K \Rightarrow A_1 = -K = 0$$

There fore are solution is

$$\boxed{\begin{aligned} &i(t) = I \cos wt \qquad \text{where } w = \sqrt{\tfrac{1}{LC}} \\ &v(t) = \frac{-I}{wC}\sin wt = -I\sqrt{\tfrac{L}{C}}\sin wt \\ &\text{or} \\ &i(t) = I\cos(15811.4\,t) \\ &v(t) = -632.5\, I \sin(15811.4\,t) \end{aligned}}$$

8.21 $i(t=0) = I$



$C = .1\mu F$
$L = 40\,mH$
$G = 5\times 10^{4}$ Mhos

A) By using KCL at the top node, we obtain a differential equation for $v(t)$

$$v(t)G + C\frac{dv(t)}{dt} + \frac{1}{L}\int v(t)\,dt = 0$$

Differentiate to remove integral.

$$G\frac{dv(t)}{dt} + C\frac{d^2v(t)}{dt} + \frac{1}{L}v(t) = 0$$

Arrange into standard form

$$\boxed{\frac{d^2v(t)}{dt} + \frac{G}{C}\frac{dv(t)}{dt} + \frac{1}{CL}v(t) = 0}$$

B) Plug in $v(t) = Ae^{st}$ to find the characteristic equation:

$$\cancel{Ae^{st}}\, Gs + \cancel{Ae^{st}}\, Cs^2 + \frac{1}{L}\cancel{Ae^{st}} = 0$$

Re arrange:

$$\boxed{s^2 + \frac{G}{C}s + \frac{1}{LC} = 0}$$ - characteristic equation

Now find the roots

$$s = \frac{-\frac{G}{C} \pm \sqrt{\left(\frac{G}{C}\right)^2 - 4\frac{1}{LC}}}{2}$$

Plug in values

$$s = \frac{\frac{-5\times10^{-4}}{.1\times10^{-6}} \pm \sqrt{\frac{5\times10^{-4}}{.1\times10^{-6}} - 4\left(\frac{1}{(4\times10^{-3})(.1\times10^{-6})}\right)}}{2}$$

$$s = \frac{-5000 \pm \sqrt{(5000)^2 - 1\times10^{9}}}{2}$$

$$\boxed{s = -2500 \pm j\,15612.5}$$ - natural frequency

C.) From part A.)

$$\frac{d^2v(t)}{dt} + \frac{G}{C}\frac{dv(t)}{dt} + \frac{1}{CL}v(t) = 0$$

In general

$$\frac{d^2y}{dt} + 2\alpha\frac{dy}{dt} + w_0^2 y = 0$$

There fore

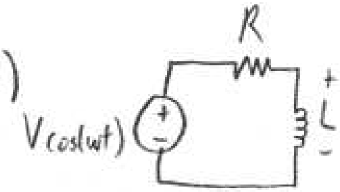
$$2\alpha = \frac{G}{C} \qquad w_0^2 = \frac{1}{LC}$$

$$\boxed{\alpha = \frac{G}{2C} = 2500} \qquad \boxed{w_0 = \sqrt{\frac{1}{LC}} = 15811.4}$$

$$Q = \frac{w_0}{2\alpha} = \frac{\sqrt{\frac{1}{LC}}}{\frac{G}{C}} = \frac{\sqrt{\frac{C}{L}}}{G} = 3.16$$

$$\boxed{Q = \frac{\sqrt{\frac{L}{C}}}{G} = 3.16}$$

8.3 A)



$v_L = ?$

Use impedences to solve and to find v_L use a voltage divider $\quad V\cos(wt) = Ve^{jwt}$

$$\bar{V}_L = \frac{Z_L}{Z_L + Z_R}\bar{V}_i$$

$$\bar{V}_L = \frac{jwL}{jwL + R}$$

Convert to polar form

$$\bar{V}_L = \frac{wL}{\sqrt{R^2 + (wL)^2}} e^{j\phi} \qquad \text{where } \boxed{\phi = \frac{\pi}{2} - \tan^{-1}\left(\frac{wL}{R}\right)}$$

Take real magnitude to get v_L value

$$\boxed{v_L(t) = \frac{wL}{\sqrt{R^2 + (wL)^2}} V\cos(wt + \phi)}$$

B.) $\boxed{i(t) = I\cos(wt)} = Ie^{jwt}$

Combine the two parallel impedances

$$Z_{tot} = Z_R \parallel Z_C = \frac{\frac{1}{jwC} \cdot R}{\frac{1}{jwC} + R}$$

$$Z_{tot} = \frac{\frac{R}{jwC}}{\frac{1 + Rjwc}{jwC}} \Rightarrow Z_{tot} = \frac{R}{1 + jwRC}$$

Now we can say

$$\bar{V}_c = \bar{I}\, Z_{tot}$$

$$\bar{V}_C = \frac{R}{1+jwRC} \bar{I} e^{jwt}$$

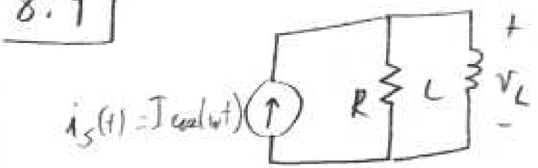
Put this into polar form

$$\bar{V}_C = \frac{R}{\sqrt{1+(wRC)^2}} e^{-j\phi} \bar{I} e^{jwt}$$ where $\boxed{\phi = -\tan^{-1}\left(\frac{wRC}{1}\right)}$

Take real magnitude

$$\boxed{V_C = \frac{R}{\sqrt{1+(wRC)^2}}\, I \cos(wt + \phi)}$$

8.4)



$$v_L(t) = I\frac{wL}{\sqrt{1+\left(\frac{wL}{R}\right)^2}} \cos\left(wt + \frac{\pi}{2} - \tan^{-1}\left(\frac{wL}{R}\right)\right)$$

A) For small w, the inductor will look close to a short circuit so v_L will be very small or close to 0

From the equation, the 1 term in the denominator will dominate and we can ignore the $\left(\frac{wL}{R}\right)^2$ term. Therefore the magnitude is

IwL which is approximately zero if $w \ll \frac{R}{L}$

B) At high frequencies the inductor has a high impedance (∞) and most all of the current will go to the resistor. Therefore (open circ) the v_L will equal $v_L = IR$

Looking at the equation, the $\left(\frac{wL}{R}\right)^2$ term dominates the denominator so the magnitude reduces to:

$$v_L = I \frac{wL}{\frac{wL}{R}} \Rightarrow \underline{v_L = IR}$$

C) The impedances of the resistor and the inductor must be equal or the imaginery and real portions of the denominator must be equal.

$$1 = \frac{wL}{R}$$

$$\boxed{w = \frac{R}{L}}$$

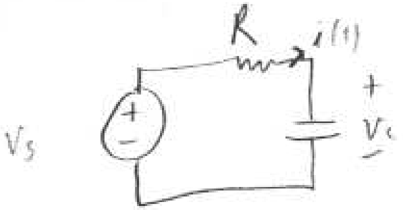
D.) When the imaginery and real parts are equal, the signal is $\frac{1}{\sqrt{2}}$ times the magnitude with just the resistor. By plugging into the equation we can get this

$$\boxed{|v_L| = \frac{I\left(\frac{R}{L}\right)(L)}{\sqrt{1+\left(\frac{R}{L}\right)\left(\frac{L}{R}\right)}} = \frac{IR}{\sqrt{2}} = .707\, IR}$$

The phase will be $\boxed{\phi = 45^\circ}$

Using the equation

$$\phi = \tan^{-1}\left(\frac{R}{L}\cdot\frac{L}{R}\right) \Rightarrow \boxed{\phi = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4} \text{ radians}}$$



$$v_c(t) = V \frac{1}{\sqrt{1+(wRC)^2}} \cos\left(wt + \frac{\pi}{2} - \tan^{-1}(wRC)\right)$$

$$i(t) = \frac{VwC}{\sqrt{1+(wRC)^2}} \cos\left(wt + \frac{\pi}{2} - \tan^{-1}(wRC)\right)$$

E) The condition for low frequencies is $\omega \ll \frac{1}{RC}$.
For low frequencies, the capacitor has a large impedance so there will be a current that is almost zero, and almost all of the voltage across the capacitor.

$\underline{|v_c| = V}$ $\quad$ $\underline{|i(t)| = 0}$

Looking at the equations: For $i(t)$, the $(wRC)^2$ can be ignored and $|i(t)| = \frac{wC}{1}$. With small w, $\underline{|i(t)| = 0}$

For the voltage, the $(wRC)^2$ term can be ignored and we have $|v_c(t)| = \frac{V}{1}$ or all of the voltage across the capacitor

F) The condition for high frequencies is $\omega \gg \frac{1}{RC}$: At high frequencies, the opposite is true. The capacitor has a very small impedance, and looks almost like a short. Therefore, the current is limited by the resistor and almost all the voltage falls across the resistor.

$$\underline{|v_C(t)| \approx 0 \quad \text{and} \quad |i(t)| = \frac{V}{R}}$$

Looking at the equations, we see the $(wRC)^2$ term dominates the denominator in both.

$$|v_c(t)| = \frac{V}{wRC} \Rightarrow \text{with large } w \quad \underline{|v_c(t)| \approx 0}$$

$$|i(t)| = \frac{VwC}{wRC} \Rightarrow \underline{i(t) = \frac{V}{R}}$$

G) The voltage divides evenly when the impedances of the C and R are equal or when the imaginery and real parts are equal.

$$1 = wRC$$

$$\boxed{w = \frac{1}{RC}}$$

H.) As in D.), the magnitude is $\frac{1}{\sqrt{2}}$ times the ~~response~~ total response and the phase is 45°

$$\boxed{|v_C(t)| = \frac{V}{\sqrt{1+(\frac{1}{RC}RC)^2}} = \frac{V}{\sqrt{2}} = .707\,V}$$

$$\boxed{|i(t)| = \frac{V(\frac{1}{RC})C}{\sqrt{1+1}} = \frac{V}{\sqrt{2}R} = .707\,\frac{V}{R}}$$

$$\psi = \tan^{-1}\left(\frac{(\frac{1}{RC})RC}{1}\right) = \tan^{-1}\left(\frac{1}{1}\right) \Rightarrow \boxed{\phi = 45^\circ}$$