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NOTES FOR 6.002 LECTURE #18, APRIL 15, 2003

THE S-PLANE AND ITS USES

IF A NETWORK IS COMPRISED OF LUMPED ELEMENTS (R, L, C) THE TRANSFER FUNCTION AND THE INPUT IMPEDANCE OR ADMITTANCE WILL ALWAYS HAVE THE FORM OF RATIOS OF POLYNOMIALS IN S (= jω).

IF DRIVEN BY A VOLTAGE:

$\frac{\bar{V}_o}{\bar{V}_i}$, THE VOLTAGE TRANSFER RATIO AND:

$\frac{\bar{I}_i}{\bar{V}_i}$, THE INPUT ADMITTANCE

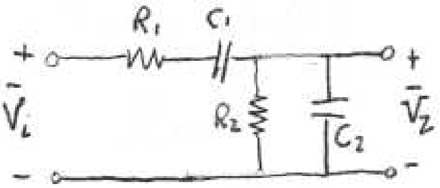
WILL BE RATIOS OF POLYNOMIALS IN S. FOR EXAMPLE:

$$\frac{\bar{V}_o}{\bar{V}_i} = \frac{s^n + k_{n-1}s^{n-1} + \cdots + k_0}{s^m + l_{m-1}s^{m-1} \cdots + l_0}$$

THE NUMERATOR WILL HAVE n ROOTS EACH OF WHICH IS A ZERO - A FREQUENCY AT WHICH THE TRANSFER FUNCTION IS 0.

THE DENOMINATOR WILL HAVE m ROOTS, EACH OF WHICH IS A POLE - A FREQUENCY AT WHICH THE TRANSFER FUNCTION BLOWS UP.

EXAMPLE I (CIRCUIT DISCUSSED IN LECTURE 17)



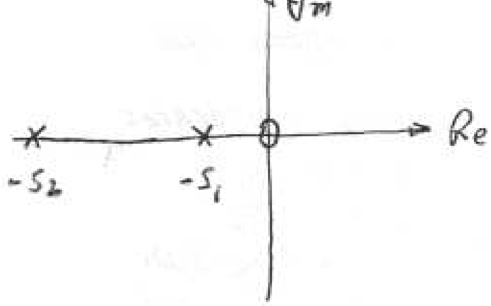
$$\frac{\bar{V}_2}{\bar{V}_1} = \frac{R_2 \| \frac{1}{C_2 s}}{R_2 \| \frac{1}{C_2 s} + R_1 + \frac{1}{C_1 s}} = \frac{s/R_1 C_2}{(s+s_1)(s+s_2)}$$

s_1, s_2 ARE POSITIVE REAL NUMBERS

THE ROOT OF THE NUMERATOR IS $s = 0$

THE ROOTS OF THE DENOMINATOR ARE $s = -s_1$, $s = -s_2$

SHOWN ON THE S-PLANE, THE ROOTS ARE



S-PLANE

O DENOTE A ZERO

X DENOTES A POLE

INTERPRETATION OF POLES AND ZEROS:

$$\bar{V}_2 (s+s_1)(s+s_2) = \bar{V}_1\, s\left(\frac{1}{R_1 C_2}\right)$$

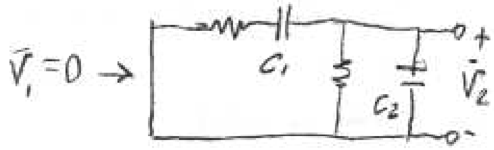
WHEN $s=0$ (THE ZERO) $\bar{V}_2 \equiv 0$ FOR EVERY $\bar{V}_1$

AT THE FREQUENCY OF A ZERO, THE OUTPUT IS ZERO

SUPPOSE $\bar{V}_1 \equiv 0$. UNDER WHAT CONDITIONS CAN $\bar{V}_2$ BE NON-ZERO?

WHEN $s=-s_1$ OR $-s_2$, $\bar{V}_2$ CAN BE NON-ZERO WITH NO INPUT.

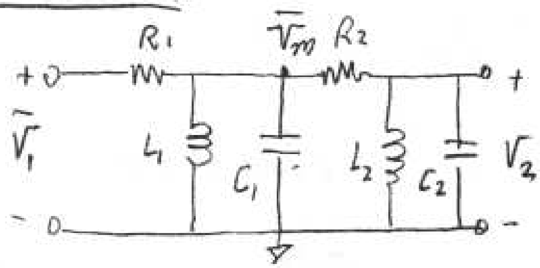
CIRCUIT IS: $\quad v_2(t) = k_1 e^{-s_1 t} + k_2 e^{-s_2 t}$



$-s_1$ AND $-s_2$ ARE THE NATURAL (UNFORCED) FREQUENCIES OF THE NETWORK.

[DROP SOME ENERGY INTO C_1 OR C_2 AND THE CURRENTS AND VOLTAGES EVERYWHERE IN THE NETWORK WILL RESPOND AS SUMS OF $e^{-s_1 t}$ AND $e^{-s_2 t}$.

EXAMPLE II



TO DETERMINE $\bar{V}_2/\bar{V}_1$, DEFINE $\bar{V}_m$ AND WRITE A PAIR OF NODE EQUATIONS:

$$\text{KCL}\begin{cases} \bar{V}_m\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{L_1 s}+C_1 s\right) - \bar{V}_2\frac{1}{R_2} = \bar{V}_1\left(\frac{1}{R_1}\right) \\ -\bar{V}_m\left(\frac{1}{R_2}\right) + \bar{V}_2\left(\frac{1}{R_2}+\frac{1}{L_2 s}+C_2 s\right) = 0 \end{cases}$$

SOLVE THE NODE EQUATIONS FOR $\bar{V}_2/\bar{V}_1$. FOR A SPECIFIC SET OF ELEMENT VALUES, THE SOLUTION IS OF THE FORM:

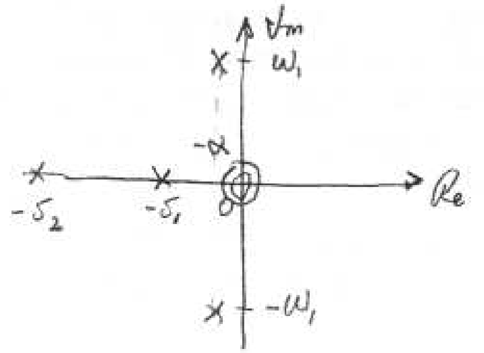
$$(1) \quad \frac{\bar{V}_2}{\bar{V}_1} = K\frac{s^2}{s^4+\alpha s^3+\beta s^2+\gamma s+\delta} = K\frac{s^2}{(s+s_1)(s+s_2)(s+s_3)(s+s_4)}$$

WHERE s_1, AND s_2 ARE REAL NUMBERS AND

s_3 AND s_4 ARE COMPLEX CONJUGATES

ZEROS ARE 0 AND 0, POLES ARE: $\begin{cases} s = -s_1 \\ s = -s_2 \\ s = -\alpha_1 + j\omega_1 \\ s = -\alpha_1 - j\omega_1 \end{cases}$

THE POLE ZERO DIAGRAM IS:



THE BODE DIAGRAM CAN BE CONSTRUCTED FROM THE LOCATIONS OF THE POLES AND ZEROS.

AND THE GENERAL FORM OF THE NATURAL (UNFORCED) TRANSIENT RESPONSE IS:

$$v_2(t) = k_1 e^{-s_1 t} + k_2 e^{-s_2 t} + k_3 e^{-\alpha t} \sin \omega_1 t + k_4 e^{-\alpha t} \cos \omega_1 t$$

THE TRANSFER FUNCTION CAN ALSO BE USED TO DETERMINE THE DIFFERENTIAL EQUATION RELATING v_2 TO v_1.

IN CROSS MULTIPLIED FORM THE VOLTAGE TRANSFER RATIO (1 ON p3) IS:

$$\bar{V}_2 \left(s^4 + \alpha s^3 + \beta s^2 + \gamma s + \delta\right) = K s^2 \bar{V}_1$$

IF THE EXCITATION IS $v_1(t) = \bar{V}_1 e^{st}$, THE RESPONSE IN THE STEADY STATE MUST BE: $v_2(t) = \bar{V}_2 e^{st}$

INASMUCH AS EACH DIFFERENTIATION OF $\bar{V} e^{st}$ YIELD s TIMES THE ORIGINAL FUNCTION, THE DIFFERENTIAL EQUATION MUST BE:

$$\frac{d^4 v_2}{dt^4} + \alpha \frac{d^3 v_2}{dt^3} + \beta \frac{d^2 v_2}{dt^2} + \gamma \frac{d v_2}{dt} + \delta v_2 = K \frac{d^2 v_1}{dt^2}$$

FINALLY, THE S-PLANE DIAGRAM CAN BE USED TO VISUALIZE THE FREQUENCY RESPONSE. TWO APPROACHES:

A) FOR EXAMPLE I DEFINE NUMERATOR AND DENOMINATOR VECTORS:

$$\bar{A}(s) = \frac{1}{R_1 C_2} \frac{(j\omega - 0)}{(j\omega + s_1)(j\omega + s_2)}$$

FOCUS ON CHANGES AS ω INCREASES FROM ZERO.

B) CONSIDER UNIFORM ELASTIC RUBBER SHEET MODEL - SLICE ALONG THE IMAGINARY AXIS.

<u>IN SUMMARY</u>:

1) THE POLE-ZERO DIAGRAM CONTAINS COMPLETE INFORMATION ABOUT THE TIME DOMAIN AND FREQUENCY DOMAIN BEHAVIOR OF A LINEAR NETWORK OR A LINEAR SYSTEM, <u>EXCEPT</u> FOR A SCALE FACTOR

   (E.G IN EXAMPLE I, THE S-PLANE CONTAINS NO INFORMATION ABOUT $1/R_1C_2$)

2) THE POLE-ZERO DIAGRAM CAN BE USED TO CONSTRUCT THE TRANSFER FUNCTION WHICH GOVERNS THE FREQUENCY-DOMAIN BEHAVIOR.

3) IT CAN BE USED TO FORMULATE THE NATURAL (UNFORCED) TRANSIENT RESPONSE.

4) IT CAN BE USED TO FORMULATE THE DIFFERENTIAL EQUATION OF THE CIRCUIT OR SYSTEM.

5) IT CAN BE USED FOR VISUALIZATION OF THE FREQUENCY RESPONSE.