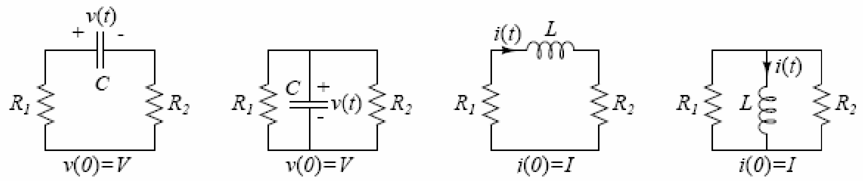
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002: Circuits & Electronics
Problem Set #7 Solution

Exercise 7.1: Each network shown below has a non-zero initial state at $t = 0$, as indicated. Find the network states for $t \geq 0$. Hint: what equivalent resistance is in parallel with each capacitor or inductor, and what decay time results from this combination?



Answer: Each circuit above is a simple first-order system with no forcing term (or input). The expression for the state variable (either $v(t)$ or $i(t)$) is then of the form

$$Ae^{-t/\tau}$$

Because, for each of the 4 circuits given, the capacitor voltage or inductor current at $t = \infty$ is 0. We also know that the the decay time, τ, is CR_{eq} for the capacitor circuits, and L/R_{eq} for inductor circuits. Using these facts, it is easy to find the network states for $t \geq 0$.

$$v(t) = Ve^{-\frac{t}{C(R_1+R_2)}}$$

$$v(t) = Ve^{-\frac{t}{C\frac{R_1R_2}{R_1+R_2}}}$$

$$i(t) = Ie^{-\frac{t(R_1+R_2)}{L}}$$

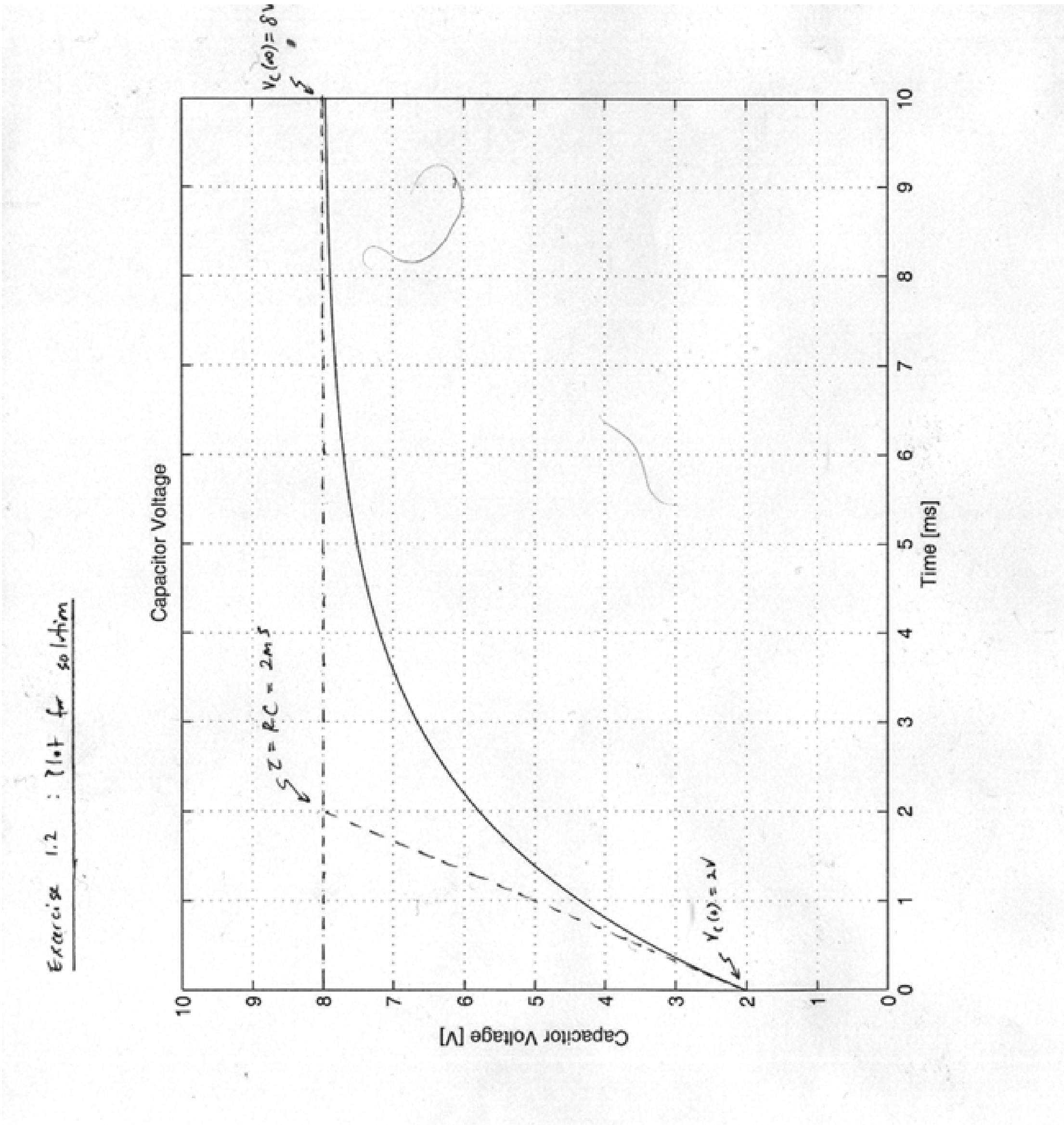
$$i(t) = Ie^{-\frac{t\frac{R_1R_2}{R_1+R_2}}{L}}$$

Exercise 7.2: First, in order to determine V, we need to realize that after a long time, the capacitor becomes a open circuit, meaning no current can flow through the 1kΩ resistor. It then follows that after a long time, $v_c(t) = V - v_r = V$, so the final value that the given graph settles to is the V we are looking for. Hence, V = **8V**. When the switch is closed, $v_c(t)$ cannot change instantaneously since no impulses are applied; therefore $V_{init} = v_c(0) =$ **2V**.

Now we look at the time constant τ = RC of this circuit. Referring to Equation 10.27 on page 680 of the textbook, we recognize that the initial slope of the measured graph will intersect the value of $v_c(\infty)$ = 8V at t = τ. Interpolating this slope (see the attached graph), we obtain τ = 2mS. Therefore, C = **2μF**.

Exercise 7.3: Since the network inductor carries no current prior to t = 0 and no impulses are applied to the system, we must have $i_L(0^-) = i_L(0^+) = 0A$. Let's first examine the circuit at t = ∞. At this point in time, the inductor is simply a short circuit and we have $i_{IN}(\infty)$ = 4mA. Therefore, we can immediately calculate R_1 = 4V/4mA = **1kΩ**. Next, let's examine the circuit at t = 0. At this point in time, the inductor is simply a open circuit and we have $i_{IN}(0)$ = 1mA. Again, we can write 1mA = 4V/(1kΩ+1kΩ+R_2) and solve for R_2. This results in R_2 = **2kΩ**.

Now that we have all the resistor values, we can look into this circuit from the inductor port (i.e. between R_1 and R_2) and calculate the equivalent R_{eq} with v_{IN} turned off. We see that $R_{eq} = R_1$ // (R_2+1kΩ) = 1kΩ // 3kΩ = 750Ω.


Exercise 1.2 : Plot for solution
Capacitor Voltage
Capacitor Voltage [V]
Time [ms]
τ = RC = 2ms
V_C(0) = 2V
V_C(∞) = 8V

From the equation of $i_{IN}(t)$, we observe that $\tau = L/R_{eq} = 1\mu sec$ in this case. Therefore, L = **750μH**.

Problem 7.1: This problem examines the relation between transient responses of linear systems. The network shown below is first driven by a current step at $t = 0$, then driven by a current ramp at $t = 0$, and finally driven by the current step plus the current ramp at $t = 0$. In the first two cases, the inductor has zero initial current, as indicated.

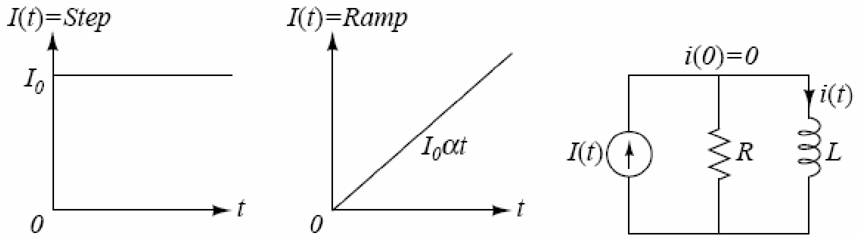
(A) Find the inductor current $i(t)$ in response to the current step shown below. Assume that $i(0) = 0$.

(B) Find the inductor current $i(t)$ in response to the current ramp shown below. Again assume that $i(0) = 0$.

(C) The step input can be constructed from the ramp input according to $I_{\text{Step}}(t) = \frac{1}{\alpha}\frac{d}{dt}I_{\text{Ramp}}(t)$. Show that their respective responses are related in a similar manner.

(D) Would the result from Part C hold if $i(0) \neq 0$? Why or why not?

(E) Finally, find the inductor current $i(t)$ in response to the current step plus the current ramp, that is to $I(t) = I_\circ(1 + \alpha t)$ for $t \geq 0$. This time assume that $i(0) = i_\circ$. Hint: think superposition.



Answer: The differential equation for the circuit can be written as

$$\frac{L}{R}\frac{di}{dt} + i(t) = I(t)$$

(A) Solving the differential equation above for a particular solution and the homogeneous one given the step input we obtain

$$\begin{aligned} i_{\text{P}}(t) &= I_\circ \\ i_{\text{H}}(t) &= -I_\circ\, e^{-\frac{t}{(L/R)}} \end{aligned}$$

Therefore, the current response to the step is given by

$$i(t) = I_\circ\left(1 - \; e^{-\frac{t}{(L/R)}}\right) u_{-1}(t)$$

(B) Guess the particular solution to the differential equation, given the ramp input, to be

$$i_{\text{P}}(t) = At + B$$

Substitution of this solution into the differential equation then yields

$$\frac{L}{R}A + At + B = \alpha I_\circ t$$

Since this equation must hold for all time,

$$\begin{aligned} A &= \alpha I_\circ \\ B &= -\frac{L}{R}\alpha I_\circ \end{aligned}$$

Solving for the homogeneous solution to match the initial condition leads to

$$i_{\mathrm{H}}(t) = \frac{L}{R}\alpha I_\circ\ e^{-\frac{t}{(L/R)}}$$

Therefore, the current response to the ramp is given by

$$i(t) = \left[\alpha I_\circ t - \frac{L}{R}\alpha I_\circ\left(1 - e^{-\frac{t}{(L/R)}}\right)\right] u_{-1}(t)$$

(C) From the solution to Part (B)

$$\begin{aligned} \frac{1}{\alpha}\frac{d}{dt} I_{\mathrm{Ramp}}(t) &= \frac{1}{\alpha}\frac{d}{dt}\left[\alpha I_\circ t - \frac{L}{R}\alpha I_\circ\left(1 - \ e^{-\frac{t}{(L/R)}}\right)\right] u_{-1}(t) \\ &= \left[I_\circ - I_\circ\ e^{-\frac{t}{(L/R)}}\right] u_{-1}(t) + \left[I_\circ t - \frac{L}{R} I_\circ\left(1 - \ e^{-\frac{t}{(L/R)}}\right)\right] u_0(t) \\ &= I_\circ\left(1 - \ e^{-\frac{t}{(L/R)}}\right) u_{-1}(t) = I_{\mathrm{Step}}(t) \end{aligned}$$

(D) The result does not hold if $i(0) \neq 0$. When we differentiate the input, we only differentiate the external source, since the initial condition is an internal source. However, when we differentiate the output, we differentiate both the homogeneous and the particular solutions. Mathematically, the term corresponding to initial condition of $I_{\mathrm{Ramp}}(t)$ gets multiplied by the time constant when differentiated to yield $I_{\mathrm{Step}}(t)$.

(E) Guess the particular solution to the differential equation, given the ramp plus the step input, to be

$$i_{\mathrm{P}}(t) = At + B$$

Then the differential equation reduces to

$$\frac{L}{R}A + At + B = I_\circ + \alpha I_\circ t$$

Thus,

$$\begin{aligned} A &= \alpha I_\circ \\ B &= I_\circ\left(1 - \frac{L}{R}\alpha\right) \\ i_P(t) &= I_\circ\left(1 + \alpha t - \frac{L}{R}\alpha\right) \end{aligned}$$

Note that this particular solution is just the sum of the two particular solutions to each separate input. The same answer could have been found using *superposition* of the answers from Parts (A) and (B), since the input is the superposition of the corresponding step and ramp inputs.

The homogeneous solution does not change for a change in input, so it is still of the form

$$i_H(t) = Ae^{-\frac{t}{(L/R)}}$$

where A is a scaling factor used to satisfy the initial condition.

So, for $t \geq 0$, we have

$$i(t) = I_\circ\left(1 + \alpha t - \frac{L}{R}\alpha\right) + Ae^{-\frac{t}{(L/R)}}$$

Given that $i(0) = i_\circ$ we can write

$$i_\circ = I_\circ\left(1 - \frac{L}{R}\alpha\right) + A$$

Which we can solve for A to find

$$A = i_\circ - I_\circ\left(1 - \frac{L}{R}\alpha\right)$$

The final expression for $i(t)$ for $t \geq 0$ is then

$$i(t) = I_\circ\left(1 + \alpha t - \frac{L}{R}\alpha\right) + \left(i_\circ - I_\circ\left(1 - \frac{L}{R}\alpha\right)\right) e^{-\frac{t}{(L/R)}}$$

Note that the resulting homogeneous solution, like the particular solution, is just the sum of the two homogeneous solutions for the two separate inputs, plus the decaying initial condition. We could have used superposition and just summed the two answers from Parts (A) and (B), and added in the decaying $i_\circ$ term.

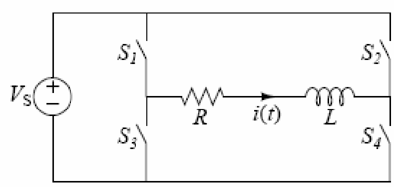
Problem 7.2: The circuit shown below can be used to regulate the current through an inductor. Typical applications include the regulation of currents in motors, solenoids and loud speakers, all of which have inductive windings. We will analyze the circuit assuming that it operates in a cyclic manner with switching period T. During the first part of each period, which lasts for a duration DT, switches S1 and S4 are on while switches S2 and S3 are off. During the second part of each switching period, which lasts for a duration $(1 - D)T$, switches S1 and S4 are off while switches S2 and S3 are on. Note that $0 \leq D \leq 1$.

(A) Assume that D is constant and that the circuit has been operating long enough to reach a cyclic steady state by $t = 0$, at which point a new switching period begins. In terms of the unknown $i(0)$, determine $i(t)$ for $0 \leq t \leq T$.

(B) Use your result from Part (A), and the fact that the circuit operates in a cyclic steady state to determine $i(0)$. Note that with this result, and that from Part (A), $i(t)$ is completely determined.

(C) Find the average value of $i(t)$ over the period $0 \leq t \leq T$. Hint: is it necessary to average the result from Part A, or is there a faster method to find the average?

(D) Suppose that the circuit has been operating with $D \equiv D_1$ for a time long enough to reach a cyclic steady state by $t = 0$. Suppose that D switches to $D = D_2$ at $t = 0$, just as a new switching period begins. In this case, determine $i(t)$ for $t \geq 0$. Hint: can you use your result from Parts (A) and (B) as a particular solution over the interval $0 \leq t$?



Answer:

(A) Over the period $0 \leq t \leq DT$, V_S is applied to the resistor and inductor. Over the period $DT \leq t \leq (1 - D)T$, $-V_S$ is applied to the resistor and inductor.

We know that starting at $t = 0$, the current through the inductor will decay exponentially from its initial value $i(0)$ towards $\frac{V_S}{R}$ with a time constant $\tau = \frac{L}{R}$. This gives

$$i(t, 0 \leq t \leq DT) = \frac{V_S}{R} + \left(i(0) - \frac{V_S}{R}\right) e^{-t\frac{R}{L}}$$

Notice that when $t = 0$, this expression evaluates to $i(0)$, and as $t \to \infty$, $i(t) \to \frac{V_S}{R}$.

For the second part of the switching cycle, when $DT \le t \le T$, the current decays exponentially from its value $i(DT)$ towards $-\frac{V_S}{R}$. Substituting in for $i(DT)$ from the equation above gives

$$\begin{aligned} i(t, DT \le t \le T) &= -\frac{V_S}{R} + \left(\frac{V_S}{R} + i(DT)\right) e^{-(t-DT)\frac{R}{L}} \\ &= -\frac{V_S}{R} + \left(\frac{V_S}{R} + \left[\frac{V_S}{R} + \left(i(0) - \frac{V_S}{R}\right) e^{-DT\frac{R}{L}}\right]\right) e^{-(t-DT)\frac{R}{L}} \\ &= -\frac{V_S}{R} + 2\frac{V_S}{R} e^{-(t-DT)\frac{R}{L}} + \left(i(0) - \frac{V_S}{R}\right) e^{-t\frac{R}{L}} \end{aligned}$$

For the purposes of this solution, let us define $i_L(t; D)$ to be

$$i_L(t; D) = \begin{cases} \frac{V_S}{R} + \left(i(0) - \frac{V_S}{R}\right) e^{-t\frac{R}{L}} & \text{when} \quad nT \le t \le (D+n)T \\ -\frac{V_S}{R} + 2\frac{V_S}{R} e^{-(t-DT)\frac{R}{L}} + \left(i(0) - \frac{V_S}{R}\right) e^{-t\frac{R}{L}} & \text{when} \quad (D+n)T \le t \le (n+1)T \end{cases}$$

Where n is some positive integer that indexes each switching cycle. The n is needed because the piece-wise solution above must apply to all switching cycles for $t \ge 0$, not just the first one.

(B) If the circuit is operating in a cyclic steady state, then $i(T)$ must be equal to $i(0)$. Setting the expression found in Part (A) equal to $i(0)$ and solving for yields

$$\begin{aligned} i(0) &= i(T) \\ i(0) &= -\frac{V_S}{R} + 2\frac{V_S}{R} e^{-(T-DT)\frac{R}{L}} + \left(i(0) - \frac{V_S}{R}\right) e^{-T\frac{R}{L}} \\ \left(1 - e^{-T\frac{R}{L}}\right) i(0) &= \frac{V_S}{R} \left(2e^{(D-1)T\frac{R}{L}} - (1 + e^{-T\frac{R}{L}})\right) \\ i(0) &= \frac{V_S}{R} \left(\frac{2e^{(D-1)T\frac{R}{L}} - (1 + e^{-T\frac{R}{L}})}{1 - e^{-T\frac{R}{L}}}\right) \end{aligned}$$

We can check to see if this answer makes sense by evaluating it at $D = 1$, $D = 0$, and $D = .5$, assuming $T \ll \frac{L}{R}$. When $D = 1$, we expect $i(t)$, and hence $i(0)$, to be $\frac{V_S}{R}$. Likewise, when $D = 0$, we expect $i(0) = -\frac{V_S}{R}$. At $D = .5$ with a fast switching cycle (the period, T, is much shorter than the time constant $\frac{L}{R}$), we should expect $i(0) = 0$. All of these predictions are satisfied by the equation above.

For the purposes of this solution, let us define $i_\circ(D)$ to be

$$i_\circ(D) = \frac{V_S}{R} \left(\frac{2e^{(D-1)T\frac{R}{L}} - (1 + e^{-T\frac{R}{L}})}{1 - e^{-T\frac{R}{L}}}\right)$$

(C) To find the average value of $i(t)$ over the period $0 \le t \le T$ is is not necessary to go through the trouble of averaging the result from Part (A). Consider, for a moment, the average voltage applied across the inductor. We know that $v_L(t) = L\frac{di}{dt}$. Averaging this over the period T requires taking the integral of both sides with respect to t, and dividing the result by T. However, the integral on the right evaluates to $L(i(T) - i(0))$, which we know to be 0, because $i(T) = i(0)$ by definition. The average voltage across an inductor whose terminal current is in cyclic steady state is 0. This means that the average voltage applied to the resistor/inductor pair must appear entirely across the resistor. The average voltage applied is $\frac{V_S DT + -V_S(1-D)T}{T} = V_S(2D - 1)$. The average value of $i(t)$ is then

$$\overline{i(t)} = \frac{V_S(2D - 1)}{R}$$

We expect this to be 0 for $D = 5.$, $\frac{V_S}{R}$ for $D = 1$, and $-\frac{V_S}{R}$ for $D = 0$, which are all true.

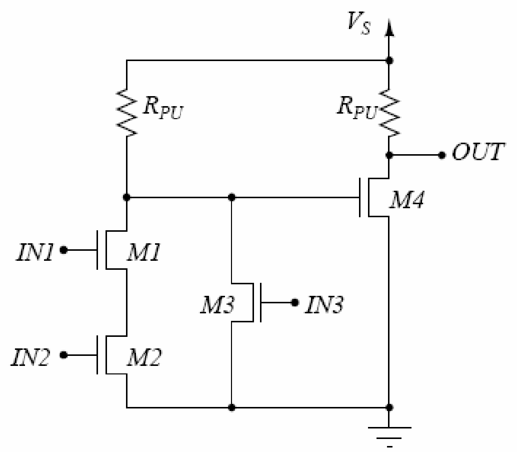
(D) The particular solution for this input, D_2, is just the answer from Part (A) evaluated at $D = D_2$, which is $i_L(t; D_2)$. The homogeneous solution is just an exponential decay times some constant, used to satisfy the initial conditions. In this case, the initial condition is $i(0) = i_\circ(D_1)$. Because $i_L(0; D_2) = i_\circ(D_2)$, the homogeneous solutions is just

$$i_H(t) = (i_\circ(D1) - i_\circ(D2))\, e^{-tR/L}$$

The full solution is the sum of the particular and homogeneous solutions:

$$i(t) = i_L(t; D_2) + (i_\circ(D1) - i_\circ(D2))\, e^{-tR/L}$$

Problem 7.3: Consider the digital logic circuit from Problem 3.1. Model each MOSFET with the switch-resistor model, and let the on-state resistance R_{ON} satisfy $R_{ON} \ll R_{PU}$. Further assume that MOSFET M4 has a gate-to-source capacitance C_{GS}. Given that the inputs IN1, IN2 and IN3 cycle through the combinations 000, 001, 010, 011, 100, 101, 110, 111, determine the average power dissipated by the logic circuit. Assume that each input combination is held for the period T with $T \gg R_{PU}C_{GS}$. Make appropriate simplifications based on the inequalities for R_{ON} and T.



Answer: Because $R_{ON} \ll R_{PU}$, we can approximate $R_{PU} + nR_{ON} \approx R_{PU}$, where n is small ($\leq 2$ for this problem). Also, we can say that the capacitor C_{GS} will fully charge or discharge in each switching period T, because $T \gg R_{PU}C_{GS}$.

First, let's consider the static power dissipated by the pullup resistors R_{PU}. When the gate of M4 is pulled high, the output is low, and only the R_{PU} on the right is dissipating power. When the gate of M4 is low, only the R_{PU} on the left is dissipating power. The circuit must be in one of these two states all the time, so it is constantly dissipating $\frac{V_S^2}{R_{PU}}$ Watts.

Now, let us consider the dynamic power dissipated. Dynamic power comes from the fact that the Gate-Source capacitance, C_{GS} of M4 must be charged and discharged every time the output changes state. If we look at the sequence of inputs in the truth table below, we see that the output changes state 6 times, which means the capacitor charges and discharges 3 times.

IN1	IN2	IN3	OUT
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Every time M4's C_{GS} is charged, $\frac{1}{2}C_{GS}V_S^2$ Joules flow into the capacitor through R_{PU} on the left.

At the same time, an equal amount of energy is lost in R_{PU}. Every time it is discharged, the energy previously stored in C_{GS} is dissipated in the $R_{\mathrm{ON}}s$ of M1, M2, and/or M3. The total energy dissipated, then, is $\frac{1}{2}C_{\mathrm{GS}}V_{\mathrm{S}}^2*$[number of changes] Joules. There are six logic transitions in our case. The power is just the energy dissipated per unit time, which is, for this circuit and combination of inputs $\frac{3}{8T}C_{\mathrm{GS}}V_{\mathrm{S}}^2$ Joules.

The total power dissipation is the sum of the dynamic and static dissipations, which is

$$P_{\mathrm{D}} = V_{\mathrm{S}}^2\left(\frac{1}{R_{\mathrm{PU}}} + \frac{3C_{\mathrm{GS}}}{8T}\right)$$

Problem 7.4:

(A) By defining v_L consistent with the direction of i_L drawn in the figure and writing a KVL equation around the loop, we obtain

$$\begin{aligned} v_C - v_L &= 0 \\ \frac{q_C}{C} + L\frac{d^2q_C}{dt^2} &= 0 \end{aligned}$$

Note carefully the sign change in the second term; this is due to the fact that $i_L = -\frac{dq_C}{dt}$ because when i_L is positive, charge is being drawn out of the capacitor. We now proceed to guess that $q_C(t) = Acos(\omega t + \phi)$ where A, ω, and ϕ are knowns yet to be determined. Plugging this guess into the KVL equation, we obtain

$$\begin{aligned} \frac{A}{C}cos(\omega t + \phi) - A\omega^2 Lcos(\omega t + \phi) &= 0 \\ \omega^2 L &= 1/C \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{LC}} \end{aligned}$$

Since we know that $T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$, we obtain $\mathbf{T = 0.199msec}$.

(B) We can compute part (B) and part (C) without solving the complete differential equation. This is because there are no resistors in the circuit, so the total energy is conserved. We know that at $t = 0$, we have

$$\begin{aligned} W_{tot} &= W_C + W_L \\ &= \frac{1}{2}Cv_C^2 + \frac{1}{2}Li_L^2 \\ &= 50\mu J + 20\mu J = 70\mu J \end{aligned}$$

At the maximum value of v_C, we must have all the energy on the capacitor. Therefore

$$v_{C,max} = \sqrt{\frac{2W_{tot}}{C}} \approx \mathbf{11.83V}$$

(C) Similarly, at the maximum value of i_L, we must have all the energy on the inductor. Therefore

$$i_{L,max} = \sqrt{\frac{2W_{tot}}{L}} \approx \mathbf{374mA}$$

(D) For part (D) and part (E), we will unfortunately have to solve for A and ϕ in $q_C(t)$. We use the two initial conditions to write the following equations at $t = 0$:

$$\begin{aligned} q_C(0) &= Acos(\phi) = Cv_C(0) = 10\mu \\ i_L(0) &= -\frac{dq_C}{dt}\,|_{t=0} = A\omega sin(\phi) = 200mA \end{aligned}$$

Now that we have two equations and two unknowns (ω has already been determined in part A), we solve and obtain $A = 11.832\mu$ and $\phi = 0.564rad$.

We observe that because i_L is positive at $t = 0$, v_C will reach its first maximum positive value at $\omega t + \phi = 2\pi$ since we require $t \geq 0$. Plugging in numbers gives us $\mathbf{t = 180.86}\mu\mathbf{sec}$.

(E) Because i_L is positive at $t = 0$, i_L will reach its first maximum positive value at $\omega t + \phi = \frac{\pi}{2}$. Plugging in numbers gives us $\mathbf{t = 31.84}\mu\mathbf{sec}$.

We can check the answer from part D and part E by recognizing that the two combined gives us $\frac{3}{4}$ of a full cycle. Therefore, $T = \frac{4}{3}(180.86\mu sec - 31.84\mu sec) = 199\mu sec$ which matches exactly with our answer in part A.