MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2002

Final Exam

Name: Solutions **Recitation Section:** ______________

Recitation Instructor: ______________ **Teaching Assistant:** ______________

Enter all your work and your answers directly in the spaces provided on the printed pages. Make sure that your name is on all sheets. Use the backs of the printed pages as scratch paper, but we will only give full credit to answers that you neatly transfer to the spaces on the printed pages. Answers must be derived or explained, not just simply written down. The quiz is closed book, but **calculators and both sides of one $8\frac{1}{2}'' \times 11''$ page of notes are allowed**.

This quiz contains 16 pages including the cover sheet. Make sure that your quiz contains all 16 pages and that you hand in all pages except the last page.

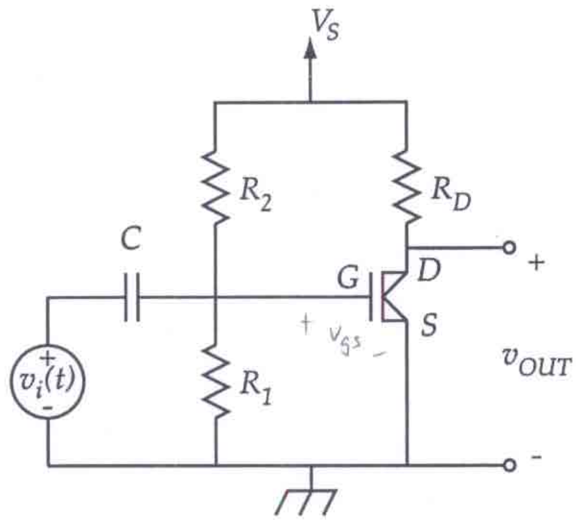
Problem	Points	Grade	Grader
1	20		
2	20		
3	16		
4	20		
5	24		
Total	100		

Name: ______________________________

Problem 1: (*20 points*) The circuit below shows a single-stage amplifier designed around a "MOXFET", a hypothetical transistor similar to a MOSFET, except that

$$i_D = \frac{K}{3}(v_{GS} - V_T)^3$$

For the purposes of this problem, assume that the MOXFET gain parameter $K = 0.625mA/V^3$ and that its threshold voltage is $V_T = 2.2V$.



In the circuit above, $V_S = 10V$, $R_1 = 30k\Omega$, $R_2 = 70k\Omega$, $R_D = 37.5k\Omega$ and $C = 0.15\mu F$. The saturation-region models for the MOXFET are shown below, where (a) is the large-signal (bias) model, and (b) is the small-signal (incremental) model.

(a) (b)

(A) The amplifier output voltage will be of the form $v_{OUT} = V_{OUT} + v_{out}(t)$. Determine V_{OUT} (a numerical value is required).

$$V_{GS} = \frac{R_1}{R_1+R_2} V_S$$

$$V_{OUT} = V_S - i_D R_D$$

$$= V_S - \frac{K}{3}\left[\frac{R_1}{R_1+R_2} V_S - V_T\right]^3 R_D$$

$$= 10 - \frac{(0.625\times10^{-3})}{3}\left[\frac{30}{100}(10) - 2.2\right]^3 (37.5\times10^3)$$

$V_{OUT} =$ 6.0 V

(B) Find an expression for the transconductance g_m in terms of the symbolic MOXFET parameters V_T and K and the operating point V_{GS}. (Do **not** evaluate your answer numerically.)

$$g_m = \left.\frac{\partial i_D}{\partial v_{GS}}\right|_{V_{GS}}$$

$$= K(v_{GS} - V_T)^2 \Big|_{V_{GS}}$$

$$= K(V_{GS} - V_T)^2$$

$g_m =$ $K(V_{GS} - V_T)^2$

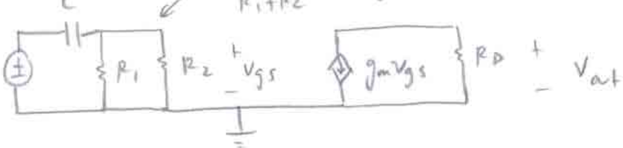
(C) For an input signal of the form $v_i(t) = V_I \cos \omega t$, the complex amplitude of the output signal $v_{out}(t)$ will be of the form

$$\hat{V}_{out} = G_0 \left[\frac{j\omega\tau}{1 + j\omega\tau} \right] V_I$$

In terms of the transconductance g_m and the circuit parameters, find expressions for the high-frequency gain G_0 and the time constant τ. (Do **not** evaluate your answers numerically.)

Small-signal analysis:

$$\frac{R_1 R_2}{R_1 + R_2} = R_{eq}.$$



$$\therefore \quad \hat{V}_{at} = -g_m R_D \frac{R_{eq}}{R_{eq} + \frac{1}{j\omega C}} V_I$$

$$= \underbrace{-g_m R_D}_{G_0} \left[\frac{j\omega C R_{eq}}{1 + j\omega C R_{eq}} \right] V_I$$

$G_0 =$ $-g_m R_D$

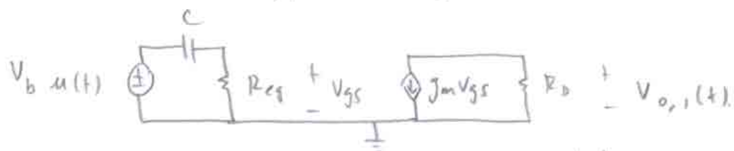
$\tau =$ $C \dfrac{R_1 R_2}{R_1 + R_2}$

(D) The small input signal is now given by $v_i(t) = (V_a \cos \omega t + V_b)u(t)$. The resulting output signal has the form

$$v_{out}(t) = (V_c \cos(\omega t + \phi) + V_d + V_e e^{-\beta t})u(t)$$

Determine V_c, ϕ, V_d, V_e and β in terms of ω, V_a, V_b, τ and G_0, where τ and G_0 are defined in part (C).

Superposition : (1) $v_{i,1}(t) = V_b\, u(t)$.



$\Rightarrow \quad V_{o,1}(t) = -g_m R_D V_b e^{-t/R_{eq}C} u(t).$ for 2nd part of input only.

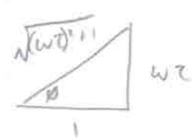
(2) Find $|H(s)|$ and $\measuredangle H(s)$.

$$|H(s)| = \left|\frac{\hat{V}_{out}}{V_I}\right| = G_0 \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$

$$\measuredangle H(s) = \frac{\pi}{2} - \tan^{-1}(\omega\tau)$$

Given $V_i(0) = V_a + V_b$, $V_o(0) = V_c \cos\phi + V_d + V_e$, we need @ $t = 0^+$ $V_c \cos\phi + V_d + V_e = G_0(V_a + V_b)$

$$\Rightarrow V_e = G_0(V_a + V_b) - V_c \cos\phi - \cancel{V_d}^{\,0}$$
$$= G_0(V_a + V_b) - V_c \sin\left(\phi + \tfrac{\pi}{2}\right)$$
$$= G_0(V_a + V_b) - V_c \sin(\pi - \tan^{-1}(\omega\tau))$$
$$= G_0(V_a + V_b) - V_c \sin(\tan^{-1}(\omega\tau))$$



$$= G_0(V_a + V_b) - V_c \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$
$$= G_0(V_a + V_b) - V_a G_0 \frac{(\omega\tau)^2}{1+(\omega\tau)^2}$$

$V_c =$ $V_a G_0 \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$

$\phi =$ $\frac{\pi}{2} - \tan^{-1}(\omega\tau)$

$V_d =$ $\emptyset$

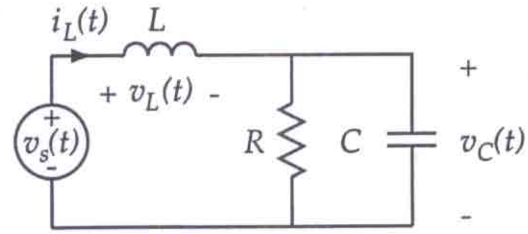
$V_e =$ $G_0\left[(V_a + V_b) - V_a \frac{(\omega\tau)^2}{1+(\omega\tau)^2}\right]$

$\beta =$ $\frac{1}{R_{eq}C} = \frac{1}{\tau}$, $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

You can also solve for V_e using superposition, but it is more difficult in terms of math.

Name: ______________________________

Problem 2: (*20 points*) Consider the RLC circuit shown below.



(A) The voltage source is a unit step at time $t = 0$. Three circuit variables, v_C, v_L, and i_L are presented plotted against time, as shown in the three figures on the next page. Time is measured in milliseconds on the horizontal axes. The vertical axes may represent voltage, measured in Volts, or current, measured in Amperes, as appropriate.

The traces are labeled 1, 2, and 3. (The label is in the upper right corner of each trace.) You are to identify the traces: which trace is which circuit variable? Circle the circuit variable that corresponds to the indicated trace below:

Trace 1:	v_C	v_L	i_L
Trace 2:	v_C	v_L	i_L
Trace 3:	v_C	v_L	i_L

Unit step → continuous $i_L(t)$ and $V_c(t)$

⇒ $i_L(0^-) = i_L(0^+) = 0\,A$ & $V_c(0^-) = V_c(0^+) = 0\,V.$

⇒ $V_L(0^-) = V_s$, assuming $V_s(t) = V u(t)$.

⇒ V_L is trace 3.

$i_c(t) = C \dfrac{dV_c(t)}{dt}$: @ $t = 0$, $i_c(t) = 0$ since $i_L(0^+) = 0\,A$

and $i_R(0^+) = 0\,A$

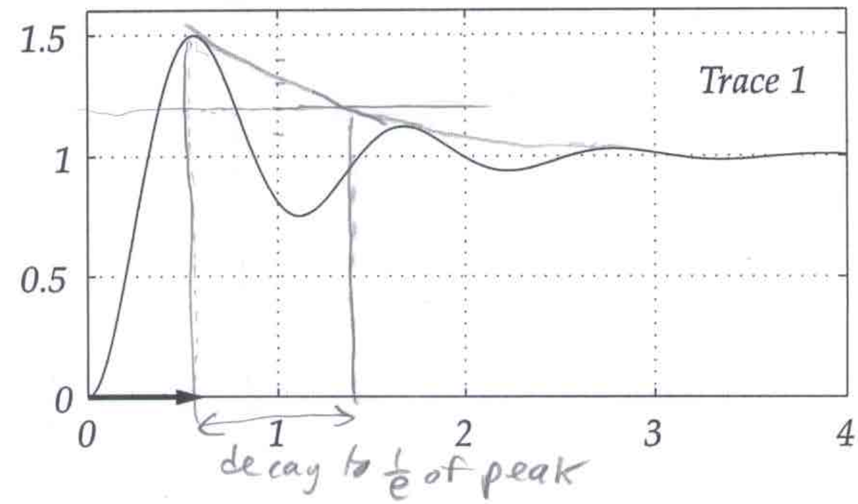
⇒ $\left.\dfrac{dV_c\,t}{dt}\right|_{t=0^+} = 0$

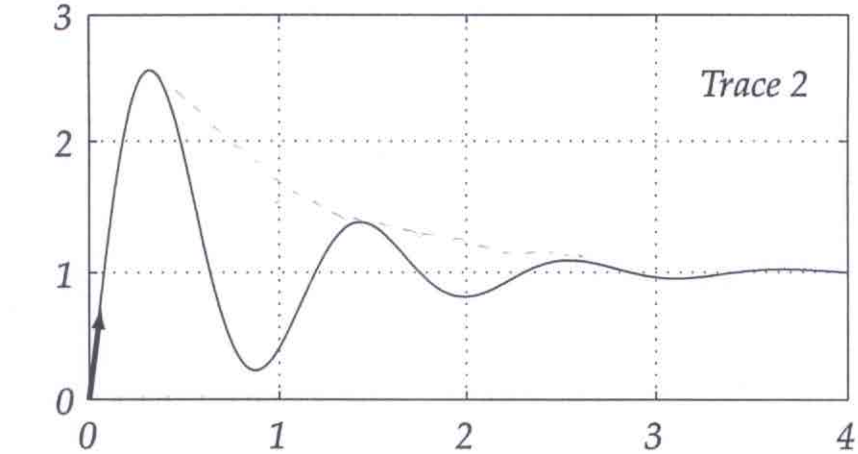
⇒ V_c is trace 1

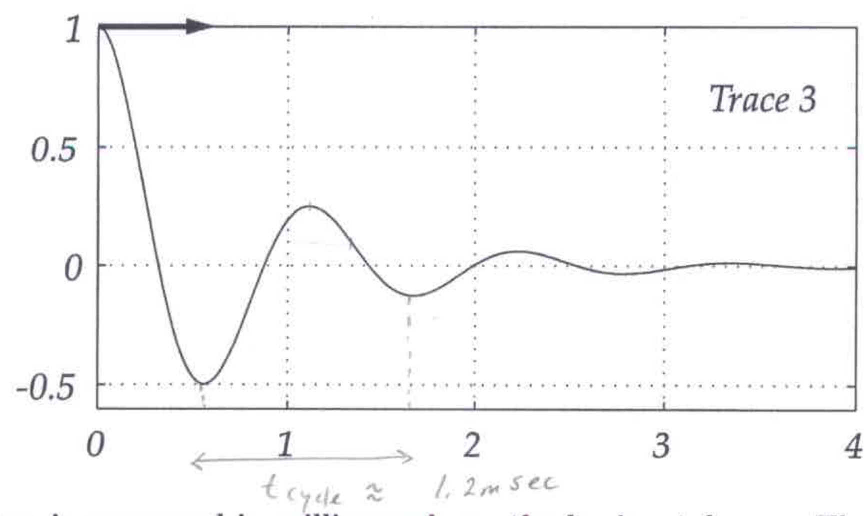
$V_L(t) = L \dfrac{d i_L(t)}{dt}$: @ $t = 0$, $V_L(t) = V_s$ so $\left.\dfrac{d i_L(t)}{dt}\right|_{t=0^+}$

is non-negative.

⇒ i_L is trace 2







Note: Time is measured in milliseconds on the horizontal axes. The vertical axes may represent voltage, measured in Volts, or current, measured in Amperes, as appropriate. The arrows represent the slope of the curve at $t = 0^+$.

(B) Assume the inductance L is $75\mu H$.

(i) What is the approximate numerical value of the capacitance?

Assume $\omega_d \approx \omega_0 = \frac{1}{\sqrt{LC}}$

$\Rightarrow \omega_0^2 L = \frac{1}{C}$

$\Rightarrow C = \frac{1}{(2\pi f)^2 L} = \frac{1}{[\frac{2\pi}{T}]^2 L}$

$= \frac{1}{\left[\frac{2\pi}{1.2\times10^{-3}}\right]^2 (75\mu H)}$

$C =$ $486.34\mu F$ (approx)

(ii) What is the approximate numerical value of the resistance?

This circuit is analogous to parallel RLC circuit since Q is only defined for circuit without source ($V_s(t) = 0$).

$\therefore \alpha = \frac{1}{2RC} \Rightarrow$ response decays to $\frac{1}{e}$ in $\frac{1}{\alpha}$ sec.

Decay takes ≈ 0.9 ms $\Rightarrow R = \frac{1}{2\alpha C} = \frac{t_{decay}}{2C}$

$= \frac{0.9\times10^{-3}}{2(486.34\mu F)}$

$R =$ 0.93Ω (approx)

(iii) Estimate the Q of the circuit. Circle the best answer:

$Q = \frac{\omega_d}{2\alpha} \approx \frac{\omega_0}{2\alpha}$

$= \frac{\left(\frac{2\pi}{1.2\times10^{-3}}\right)}{2(1105.5)} \approx 2.37$

$Q < 1$ $\boxed{1 < Q < 10}$ $10 < Q$

(C) For each question circle the correct completion.

(i) If the resistance R is increased, the Q

$\boxed{\textbf{increases.}}$ **decreases.** **remains the same.**

$Q = \frac{\omega_0}{1/RC} = \omega_0 RC.$

(ii) If the capacitance C is increased, the Q

$\boxed{\textbf{increases.}}$ **decreases.** **remains the same.**

(iii) If the inductance L is increased, the oscillatory period

$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{2\pi}{T}$

$\boxed{\textbf{increases.}}$ **decreases.** **remains the same.**

(iv) If the capacitance C is increased, the oscillatory period

$\boxed{\textbf{increases.}}$ **decreases.** **remains the same.**

(D) After a long time T, v_s returns to zero and the stored energy decays.

(i) If the capacitance C is increased, the time at which the stored energy falls to half its value at time T

$W_C = \frac{1}{2} C V^2$

$\boxed{\textbf{increases}}$ **decreases.** **remains the same.**

Decay of V related to

$\alpha = \frac{1}{2RC}$, $t_{decay} \propto \frac{1}{\alpha} = 2RC$

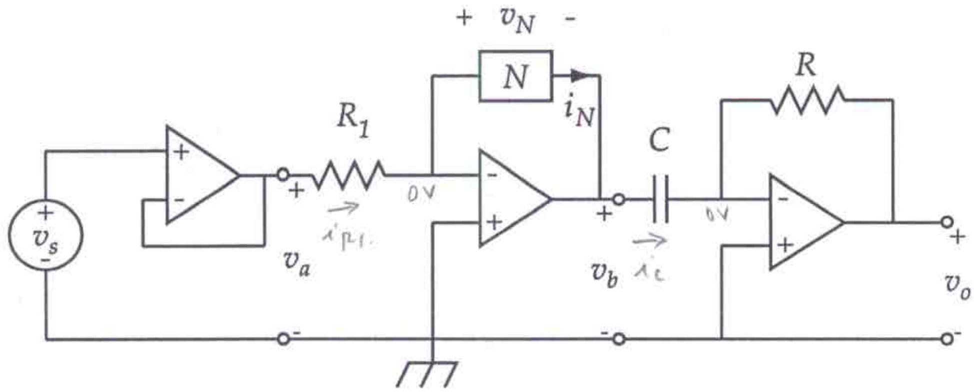
(ii) If the inductance L is increased, the time at which the stored energy falls to half its value at time T

increases. **decreases.** $\boxed{\textbf{remains the same.}}$

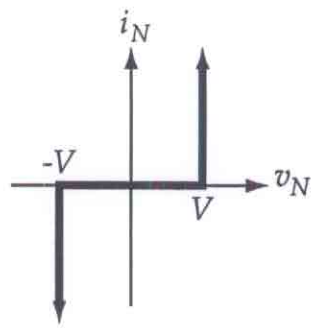
(iii) If the resistance R is increased, the time at which the stored energy falls to half its value at time T

$\boxed{\textbf{increases.}}$ **decreases.** **remains the same.**

Problem 3: (*16 points*)



The source $v_s(t)$ has the waveform shown in the graph on the next page. The nonlinear element N has the i-v relation sketched below. Assume that the op-amps are ideal (i.e., infinite gain, no input current, zero output resistance, no saturation).



On the axes provided, sketch the waveforms of $v_a(t)$, $v_b(t)$ and $v_o(t)$. Follow the time divisions indicated in the graph and label the amplitude of your waveforms. If any of your graphs contains impulses, be sure to indicate their area.

① $v_a = v_s$ since is just current buffer.

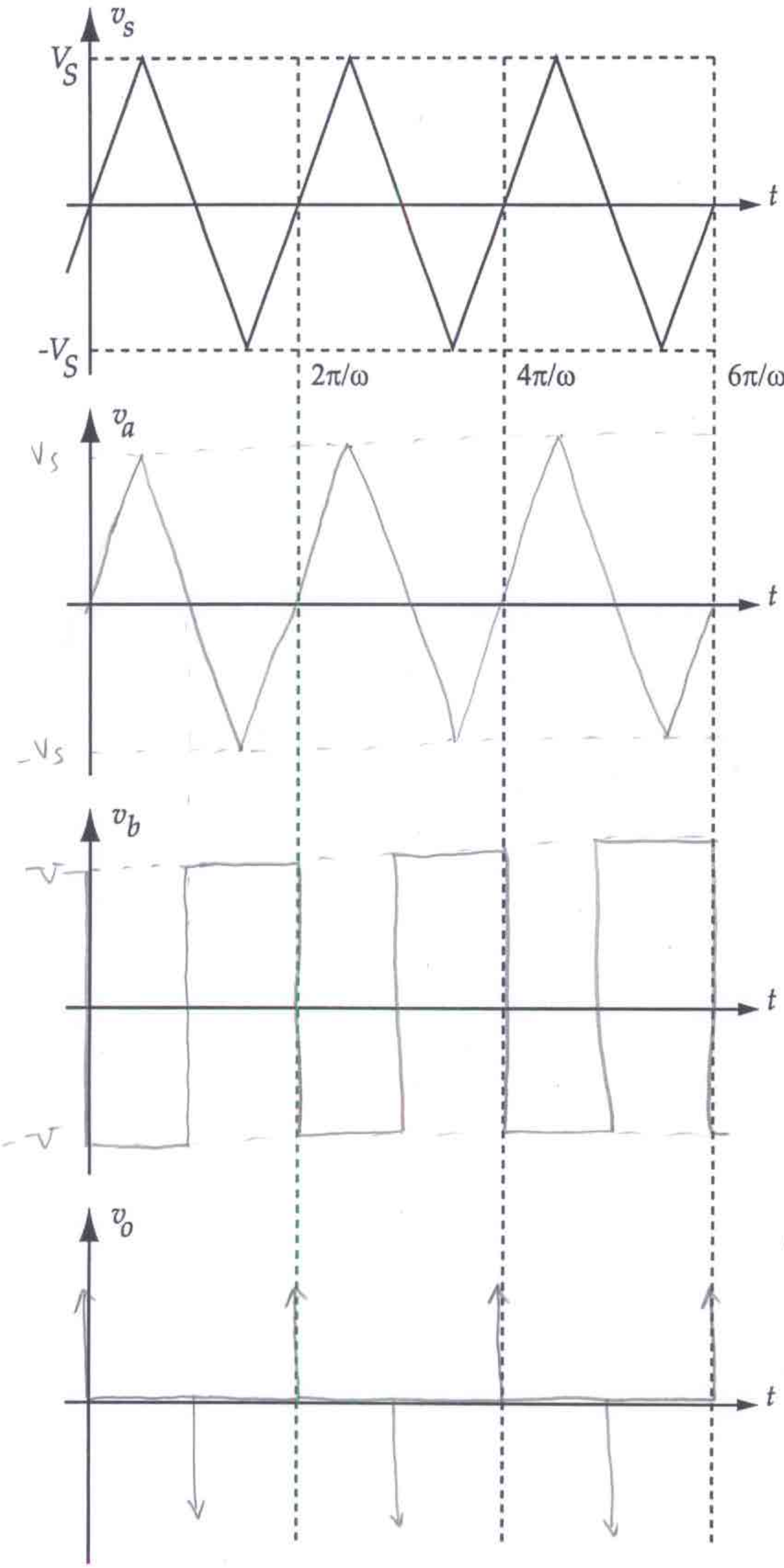
② $i_{R1} = \frac{v_a}{R_1} = i_N \Rightarrow i_N$ has same sign as v_a

$\Rightarrow \begin{cases} v_N = V \text{ when } v_a \text{ +ve} \\ v_N = -V \text{ when } v_a \text{ -ve.} \end{cases}$

③ $v_b = -v_N \Rightarrow \begin{cases} v_b = -V \text{ when } v_a \text{ +ve} \\ v_b = V \text{ when } v_a \text{ -ve.} \end{cases}$

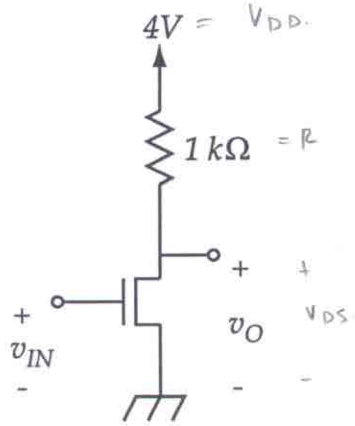
④ $i_c = C\frac{dv_c}{dt} = C\frac{dv_b}{dt} = \frac{-v_o}{R}$.

$\Rightarrow v_o = -RC\frac{dv_b}{dt}$

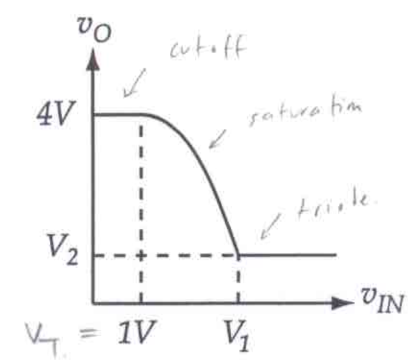


impulse has area $\boxed{2VRC}$

Problem 4: *(20 points)*



(a)



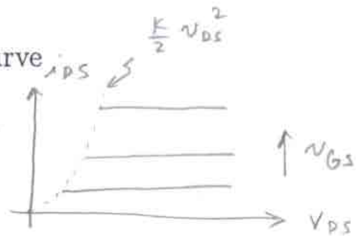
(b)

A simple MOSFET inverter is shown above. The input-output relationship is also shown above **(NOT to scale)**, where the MOSFET has been modeled using the familiar square-law relationship

$$i_{DS} = \frac{K}{2}(v_{GS} - V_T)^2$$

in the saturation region, and the triode region is compressed onto the single curve

$$i_{DS} = \frac{K}{2} v_{DS}^2$$



The MOSFET parameters are $V_T = 1V$ and $K = 1mA/V^2$.

(A) Determine the voltages V_1 and V_2 in the input-output graph for this inverter.

when v_{IN} large, v_o small → enter triode.

Look at triode regim ⇒ $V_{DD} - i_{DS} R = V_{DS}$

⇒ $V_{DD} - \frac{K}{2} v_{DS}^2 R = v_{DS}$.

⇒ $\frac{KR}{2} v_{DS}^2 + v_{DS} - V_{DD} = 0$

⇒ $V_2 = v_{DS}\big|_{triode} = \dfrac{-1 + \sqrt{1 + 2KRV_{DD}}}{KR}$

$= 2V$

Also borders saturation ⇒ $V_{DD} - \frac{KR}{2}(V_1 - V_T)^2 = V_2$

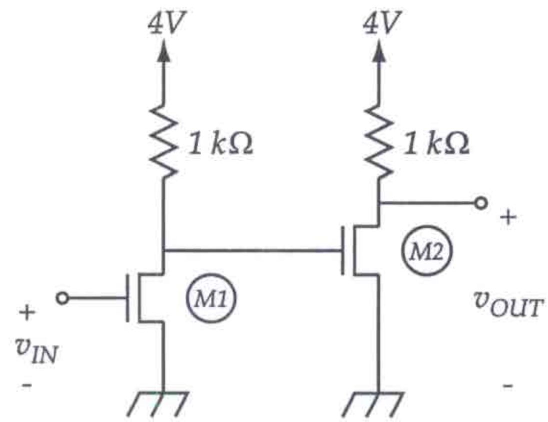
$V_1 =$ ___3,0___ V

⇒ $\sqrt{\frac{2}{KR}(V_{DD} - V_2)} + V_T = V_1$

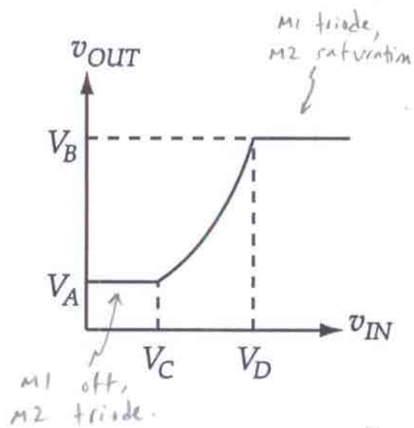
$V_2 =$ ___2.0___ V

⇒ $V_1 = 3V$

Two identical inverters from part (A) are cascaded to form the buffer circuit shown below.



The input-output relationship is qualitatively sketched below **(NOT to scale)**.



① v_{IN} small ⇒ M1 cutoff

⇒ $V_{gs2} = 4V$

⇒ $V_{out} = V_A = 2V$

② v_{in} large ⇒ $V_{out1} = 2V$

⇒ $V_{gs2} = 2V$

Assume M2 saturated, then check assumption:

$V_{out} = V_{DD} - \frac{KR}{2}(V_{gs2} - V_T)^2$

$= 3.5V$

Check: $V_{DS} = 3.5V > V_{GS} - V_T = 1V$

so assumption was correct.

(B) Determine the voltages V_A, V_B, V_C and V_D.

③ V_C is point where M2 transitions from triode → saturation

⇒ $V_{out1} = V_1 = 3V$

Again assume saturation.

$V_{out1} = V_{DD} - \frac{KR}{2}(V_{in} - V_T)^2$

⇒ $\sqrt{\frac{2}{KR}(V_{DD} - V_{out1})} + V_T = V_{in}$

Again, assumption valid since $3V > 2.41V - 1V$.

④ V_D is when $V_{out} = 3.5V$.

⇒ $V_{in2} = V_{out1} = 2V$.

⇒ $V_{in1} = V_D = 3V$

$V_A =$ <u>2.0</u> V

$V_B =$ <u>3.5</u> V

$V_C =$ <u>2.41</u> $= 1 + \sqrt{2}$ V

$V_D =$ <u>3.0</u> V

(C) For a given input voltage each transistor in the buffer circuit will be in one of three regions of operation: cutoff (CO), saturation (SAT) and triode region (TR). As the input voltage v_{IN} is increased from 0 to $4V$, the buffer passes through four states. In the following table, indicate the four states by circling the region of operation of the transistors for each range of input voltage.

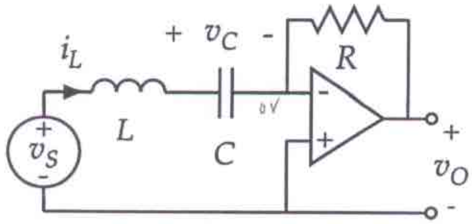
Input Voltage	M1			M2		
$0V < v_{IN} < V_T$	$\boxed{\text{CO}}$	SAT	TR	CO	SAT	$\boxed{\text{TR}}$
$V_T < v_{IN} < V_C$	CO	$\boxed{\text{SAT}}$	TR	CO	SAT	$\boxed{\text{TR}}$
$V_C < v_{IN} < V_D$	CO	$\boxed{\text{SAT}}$	TR	CO	$\boxed{\text{SAT}}$	TR
$V_D < v_{IN} < 4V$	CO	SAT	$\boxed{\text{TR}}$	CO	$\boxed{\text{SAT}}$	TR

$V_T < v_{IN} < V_C$: $v_{GS1} - V_T$ small, so M1 saturated.

$V_C < V_{in} < V_D$: $2{,}41\ V < v_{IN} < 3\ V$

From $V_{in} - V_{out}$ graph, clearly we have M1 saturated. We are in the "curvature" section.

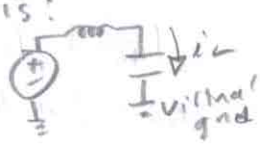
Problem 5: (*24 points*) Match the indicated variable to the form of its response for $t > 0^+$ by circling the appropriate letter corresponding to the choices offered in the last page. *Note: The parameters V_S, I_S, Λ_S, Q_S, V_0 and I_0 are all positive.*



$v_S = V_S u(t)$

$i_L(0^-) = 0 \qquad v_C(0^-) = 0$

Left side of ckt is:



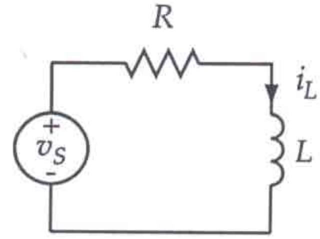
Form of $v_O(t)$:

a b c d e f

g h i j k l

i_L oscillates forever, initial slope positive

∴ v_O oscillates forever, initial slope negative



$v_S = \Lambda_S \delta(t)$

$i_L(0^-) = 0$

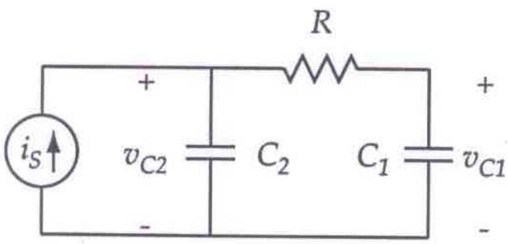
$i_L(0^+) = \frac{\Lambda}{L}$

$i_L(\infty) = 0$

Form of $i_L(t)$:

a b c d e f

g $\boxed{h}$ i j k l



$i_S = Q_S \delta(t)$

$v_{C1}(0^-) = 0 \qquad v_{C2}(0^-) = 0$

$V_{C2}(0^+) = \frac{Q}{C_2}$

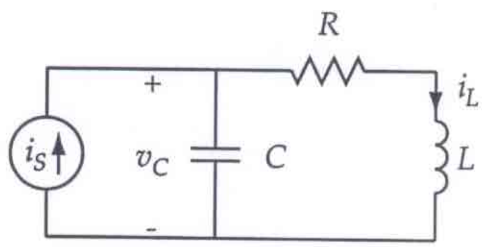
$V_{C1}(0^+) = 0$

$V_{C1}(\infty) = V_{C2}(\infty)$

Form of $v_{C1}(t)$:

a b c d e f

g h i $\boxed{j}$ k l



$i_S = I_S u(t) \qquad \frac{1}{\sqrt{LC}} > \frac{R}{2L}$

$i_L(0^-) = 0 \qquad v_C(0^-) = 0$

$i_L(0^+) = 0$

$i_L(\infty) = I_S$

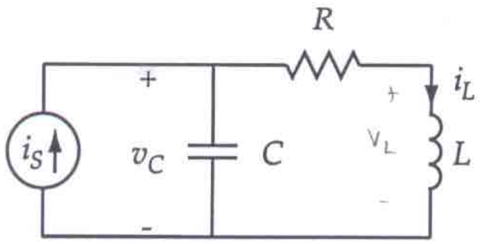
Form of $i_L(t)$:

$\omega_0 > \alpha$, so oscillates

a b c d e f

g h i j k l

initial slope for i_L is zero, since $v_C = 0$ and $v_R = 0$ at time 0^+



$i_S = Q_S \delta(t)$ $\quad \frac{1}{\sqrt{LC}} > \frac{R}{2L}$ $\quad V_C(0^+) = \frac{Q_S}{C}$

$v_C(0^-) = 0$ $\quad i_L(0^-) = 0$ $\quad i_L(0^+) = 0$

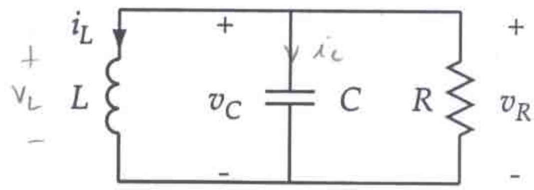
$i_L(\infty) = 0.$

Form of $i_L(t)$: $\quad V_L(0^+) = V_C(0^+) = \frac{Q_S}{C} = L\frac{di_L}{dt}$

$\Rightarrow \left.\frac{di_L}{dt}\right|_{0^+}$ positive

$a \quad b \quad c \quad \boxed{d} \quad e \quad f$

$g \quad h \quad i \quad j \quad k \quad l$



$i_L(0^-) = I_0$ $\quad \frac{1}{\sqrt{LC}} > \frac{1}{2RC}$

$v_C(0^-) = 0$ $\quad V_R = V_C \Rightarrow V_R(0^+) = 0\,V.$

$V_R(\infty) = 0$

Form of $v_R(t)$: $\quad i_C = C\frac{dV_C}{dt}$, @ $t = 0^+$,

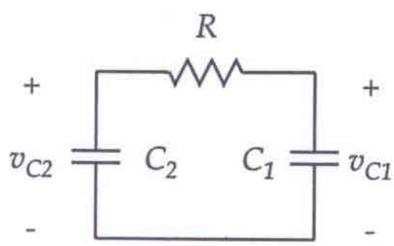
i_C negative

$\Rightarrow \left.\frac{dV_C}{dt}\right|_{0^+}$ neg.

$\Rightarrow \left.\frac{dV_R}{dt}\right|_{0^+}$ neg.

$a \quad b \quad \boxed{c} \quad d \quad e \quad f$

$g \quad h \quad i \quad j \quad k \quad l$



$V_{C1}(\infty) = V_{C2}(\infty)$

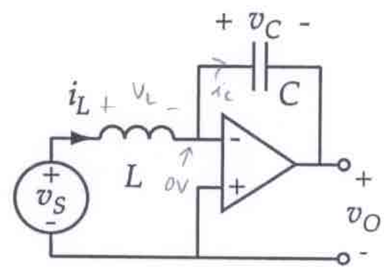
$v_{C1}(0^-) = V_0$

$v_{C2}(0^-) = 0$

Form of $v_{C1}(t)$:

$a \quad b \quad c \quad d \quad e \quad f$

$g \quad h \quad \boxed{i} \quad j \quad k \quad l$



$v_S = V_S u(t)$

$i_L(0^-) = 0$ $\quad v_C(0^-) = 0$

Form of $v_O(t)$:

$a \quad b \quad c \quad d \quad e \quad f$

$g \quad h \quad i \quad j \quad \boxed{k} \quad l$

$i_L = i_C = C\frac{dV_C}{dt}$

$V_L = L\frac{di_L}{dt}$, $\quad V_L$ positive. always

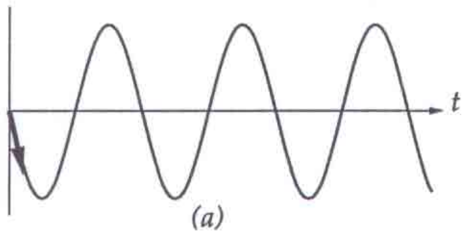
$\Rightarrow \quad i_L = \frac{1}{L}\int V_L\,dt = \frac{V_S T}{L} = i_C.$

$\Rightarrow \quad \frac{dV_C}{dt} = \frac{V_S T}{LC} \quad \Rightarrow \quad \frac{dV_C}{dt}$ positive.

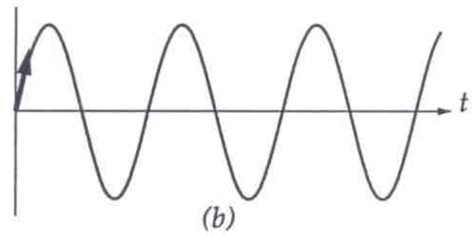
$\Rightarrow \quad V_o$ grows negative unbounded.

Note: You may tear this page off. All writing on this page will be ignored.

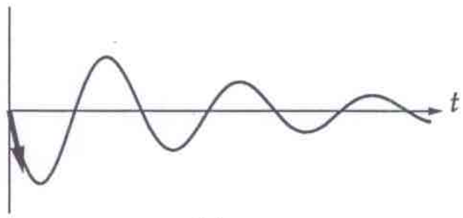
The waveforms may represent current or voltages as a function of time for $t > 0^+$. The arrows represent the slope of the curve at $t = 0^+$.



(a)



(b)



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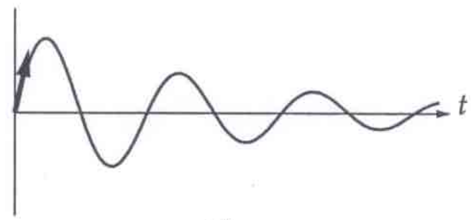

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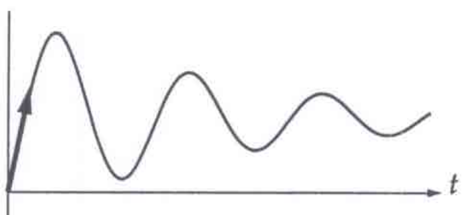

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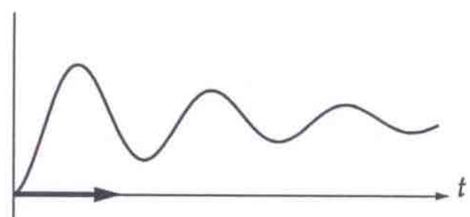

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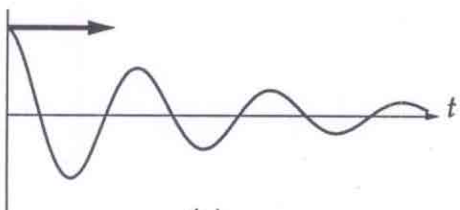
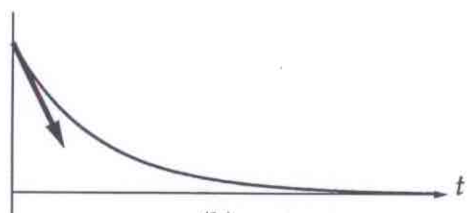


(g)



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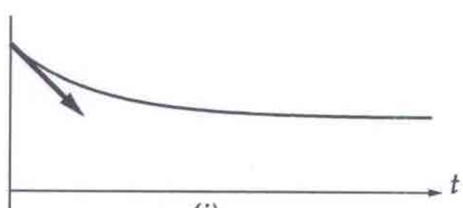
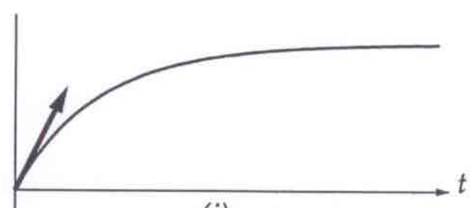
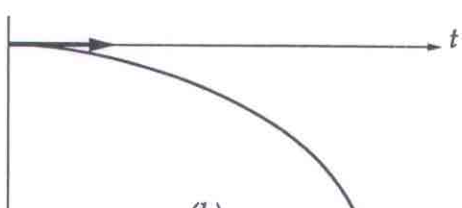


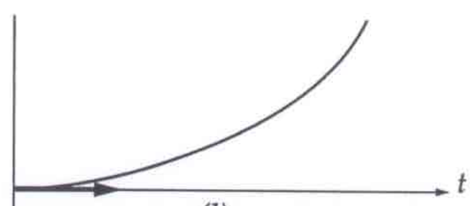
(i)



(j)



(k)



(l)