Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Spring 2002

Final Exam

Name: Solutions

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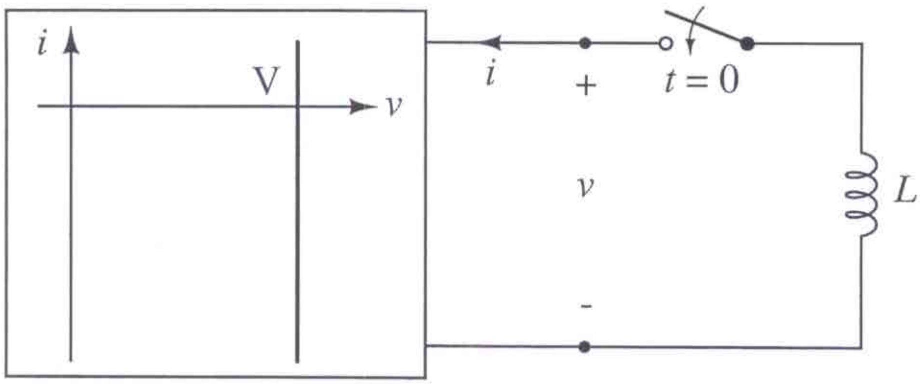
Time:	9	10	10	11	11	12	1	2

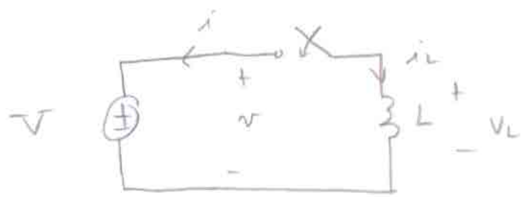
- Please put your name in the space provided above, and circle the name of your recitation instructor and the time of your recitation.
- Do your work for each question within the boundaries of the question. When finished, put your answer to each question in the corresponding answer box that follows the question.
- You may use one double-sided page of notes while taking this exam. Calculators are allowed.
- Final grades in 6.002 will not be given out by phone or by e-mail. Rather, they should be available through WEBSIS by May 23. You may review and take back your final exam on or after May 23.
- Good luck!

Problem 1 – 25%

This problem involves determining the transient response of four networks as they are connected to a load through a switch that closes at $t = 0$. In each case, the terminal behavior of the network is described graphically. For each case, graph the terminal current i and the terminal voltage v that results during the transient that follows the closure of the switch. Clearly label the axes of both graphs, and clearly label both transients with their analytical expressions.

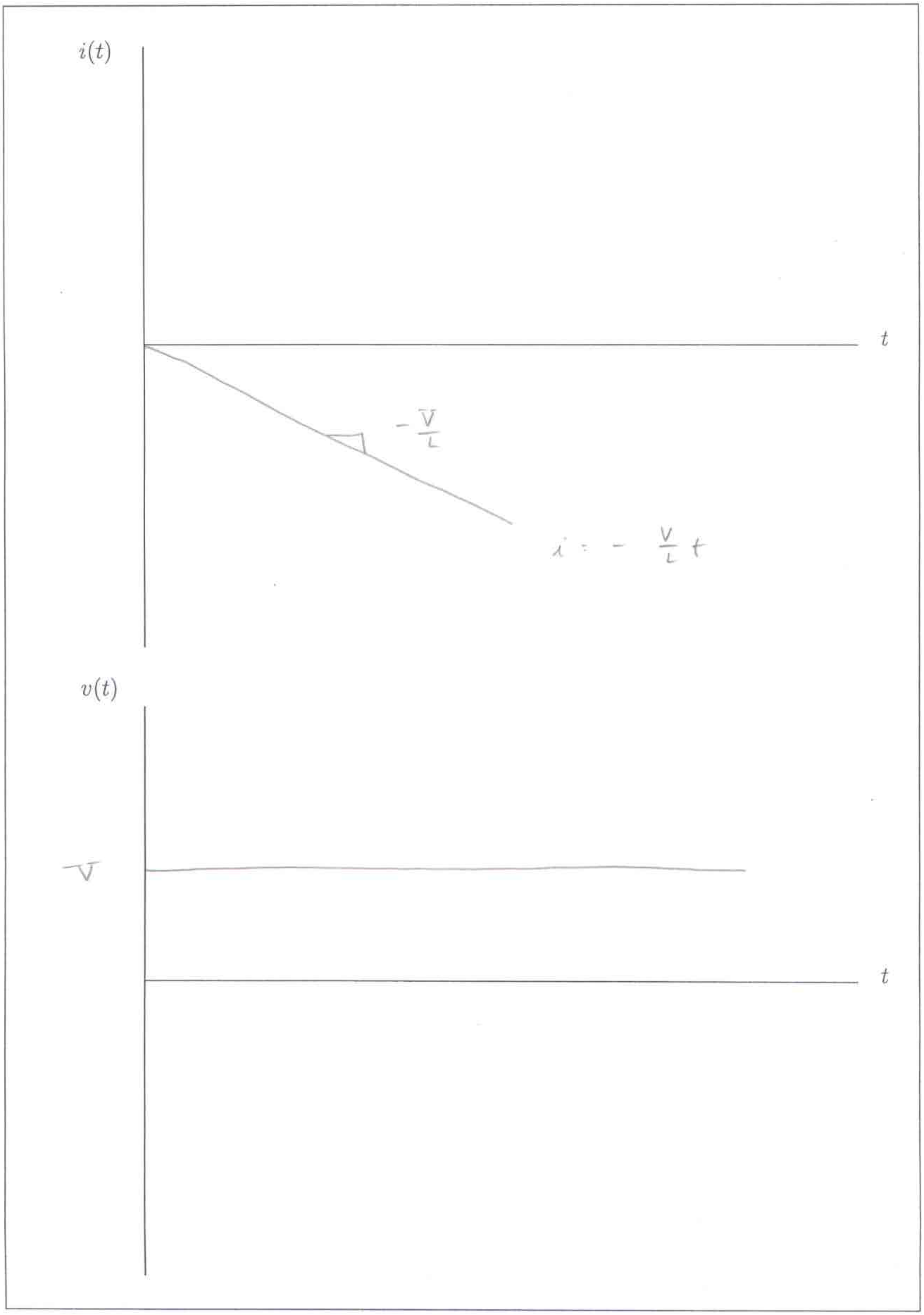
(1A) The load is an inductor having inductance L and no initial current.





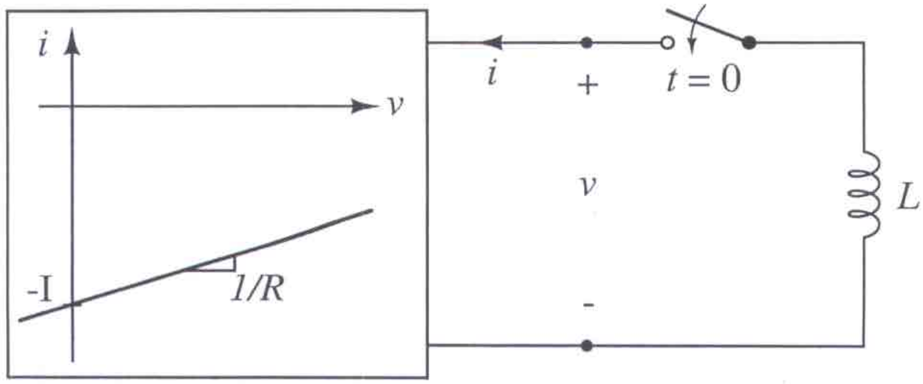
$$V_L = V = L\frac{di_L}{dt} = -L\frac{di}{dt}$$

$$\Rightarrow \quad i = -\frac{V}{L}t$$

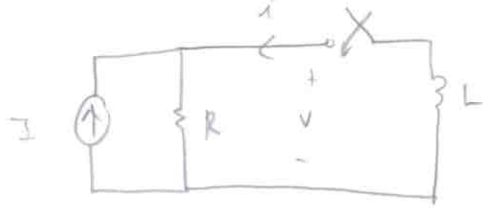


i(t)
t
-V/L
i = - V/L t
v(t)
-V
t

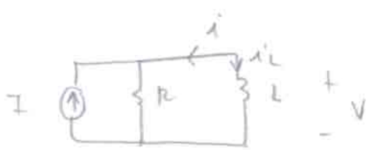
(1B) The load is an inductor having inductance L and no initial current.



$i_{sc} = -I \Rightarrow I_N = I$



$i_L(0^-) = 0$ A.

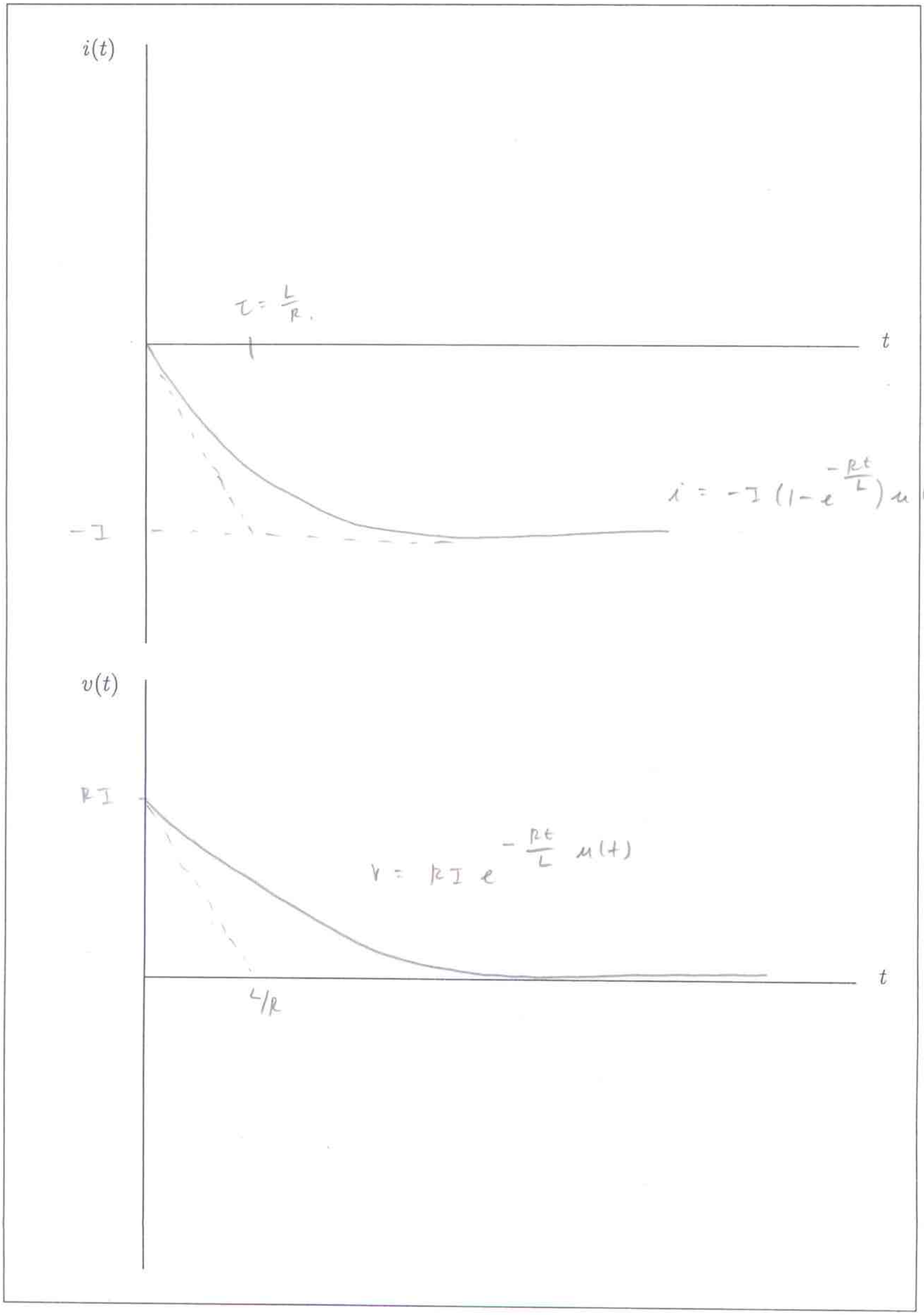


$$i_L = -i = I\left(1 - e^{-t/(L/R)}\right) u(t)$$

$$V_L = V = L \frac{di_L}{dt} = L\left[I\,\delta(t) - \left(\underbrace{I e^{-\frac{t}{L/R}} \delta(t)}_{I\,\delta(t)} - \frac{IR}{L} u(t) e^{-\frac{t}{L/R}} \right) \right]$$

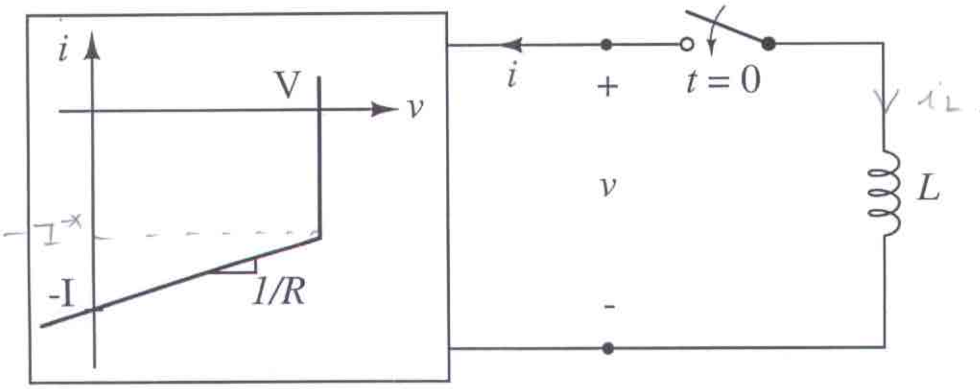
$$= L\left[\frac{IR}{L} e^{-tR/L} \right] u(t)$$

Be careful of $u(t)$ when taking derivative!



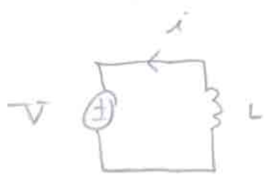
i(t)
τ = L/R
t
−I
i = −I (1 − e^{−Rt/L}) u(t)
v(t)
RI
v = RI e^{−Rt/L} u(t)
t
L/R

(1C) The load is an inductor having inductance L and no initial current.



NL - element.

Initially $i_L(0^-) = 0 \Rightarrow i = 0 \Rightarrow$ Box is $v = V$



same case like (1A) until $i = -I^*$

$$\frac{1}{R} = \frac{-I^* + I}{V} \Rightarrow \frac{V}{R} = -I^* + I$$

$$\Rightarrow -I^* = \frac{V}{R} - I$$

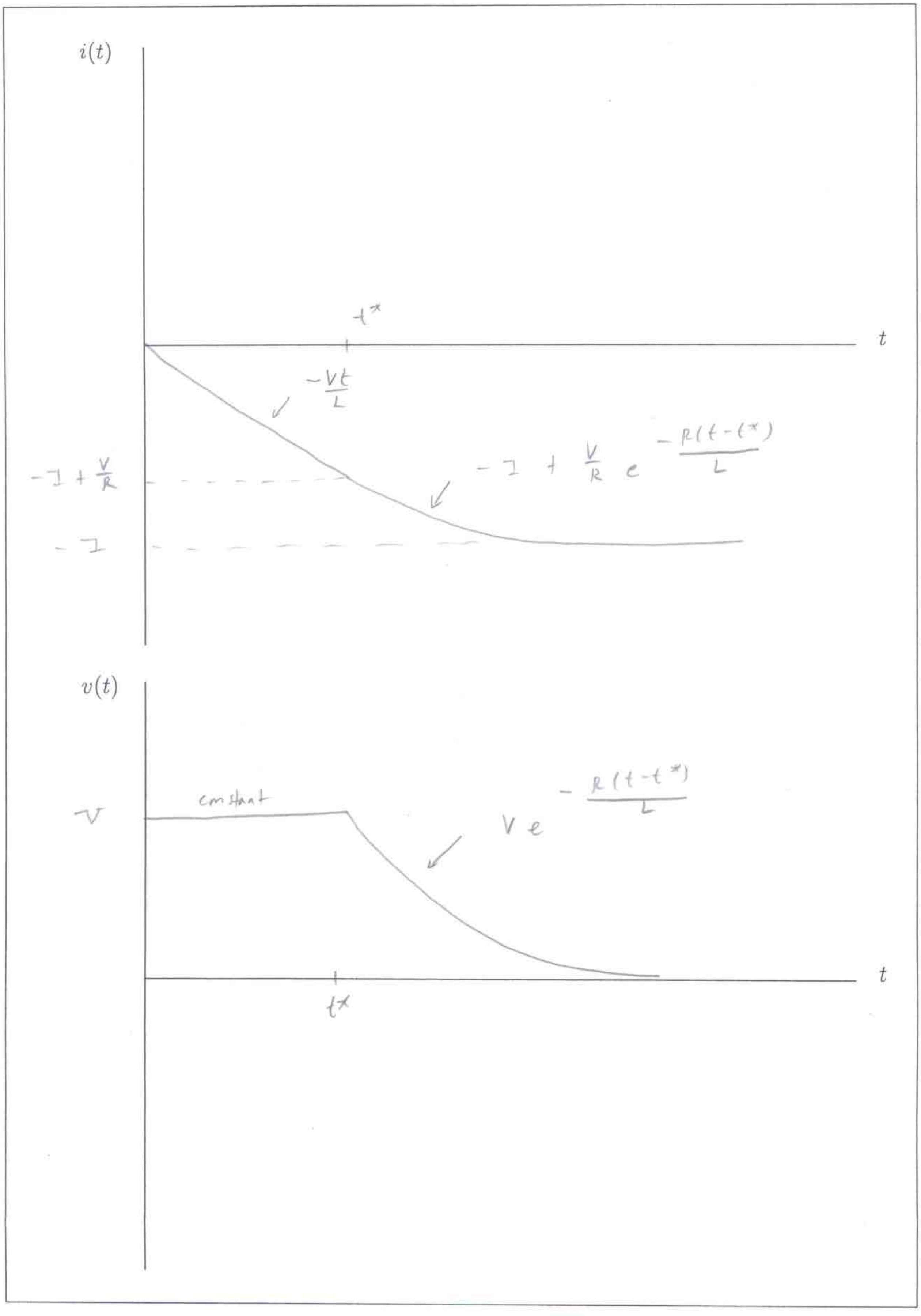
$i = -i_L = -\frac{V}{L}t$ when $i = -I^* = \frac{V}{R} - I$,

Box change $\Rightarrow$ $t^* = -\frac{L}{V}\left[\frac{V}{R} - I\right] = \frac{IL}{V} - \frac{L}{R}$

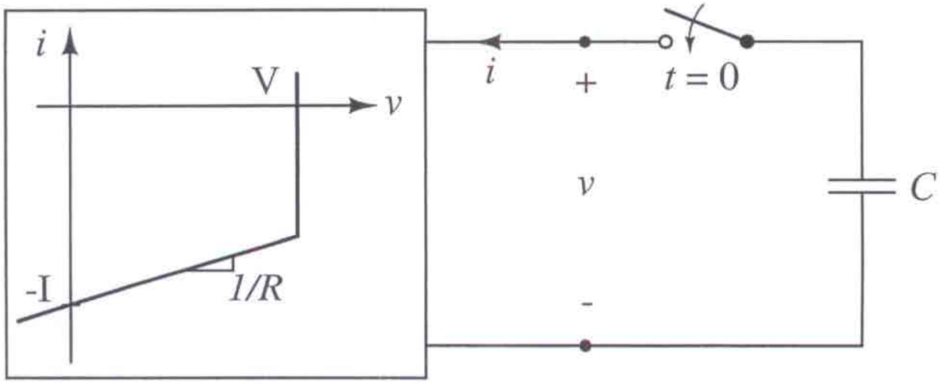
$$= \frac{L}{RV}(IR - V)$$

Now like (1B) except $t \to t - t^*$

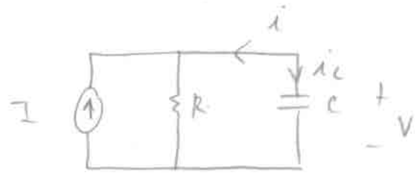
and different starting condition.


i(t)
t
t*
-Vt/L
-I + V/R e^{-R(t-t*)/L}
-I + V/R
-I
v(t)
constant
V
V e^{-R(t-t*)/L}
t
t*

(1D) The load is a capacitor having capacitance C and no initial voltage.



Initially, $V = V_c = 0$ so box looks like Norton equivalent.



$$v_c = IR\left(1 - e^{-\frac{t}{RC}}\right) u(t)$$

$$i = -i_c = -C\frac{dv_c}{dt} = -C\left\{ IR\,\delta(t) - \left[\underbrace{IRe^{-\frac{t}{RC}}\delta(t)}_{IR\,\delta(t)} - \frac{I}{C}e^{-\frac{t}{RC}}u(t) \right] \right\}$$

$$= -C\left\{ \frac{I}{C} e^{-\frac{t}{RC}} u(t) \right\} = -Ie^{-\frac{t}{RC}} u(t).$$

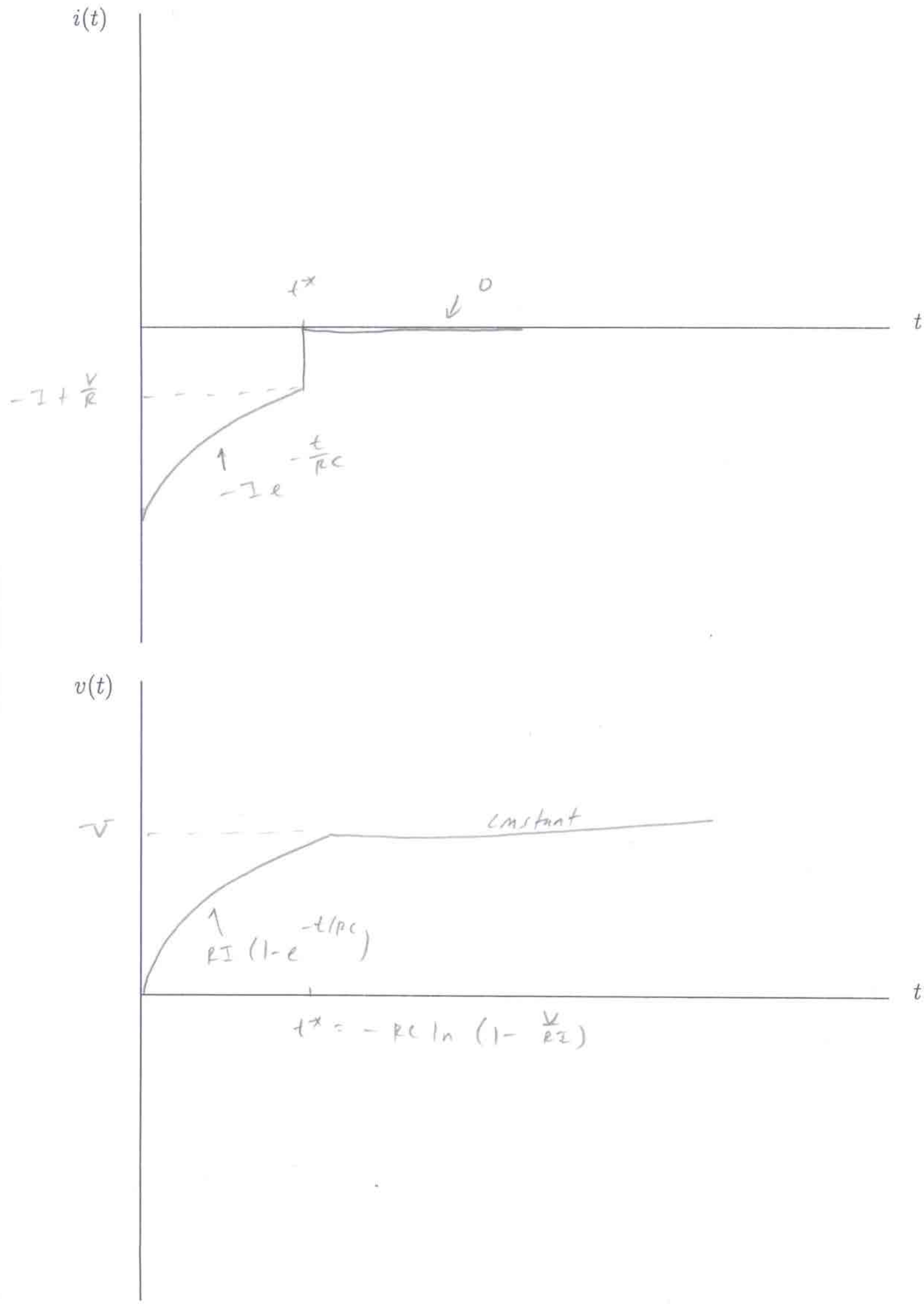
Box change at t^* when $V_c = V$

$$\Rightarrow \frac{V}{IR} = 1 - e^{-\frac{t^*}{RC}} \Rightarrow e^{-\frac{t^*}{RC}} = \frac{IR - V}{IR}$$

$$\Rightarrow t^* = -RC \ln \frac{IR - V}{IR} = -RC \ln\left(1 - \frac{V}{RI}\right)$$

Now $i = -i_c = C\frac{dv_c}{dt} = 0$

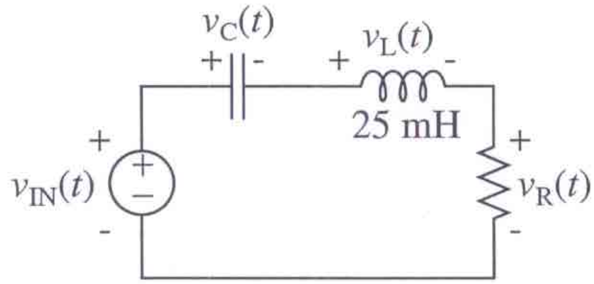
$$I^* = -I\left(1 - \frac{V}{RI}\right) = -I + \frac{V}{R}$$

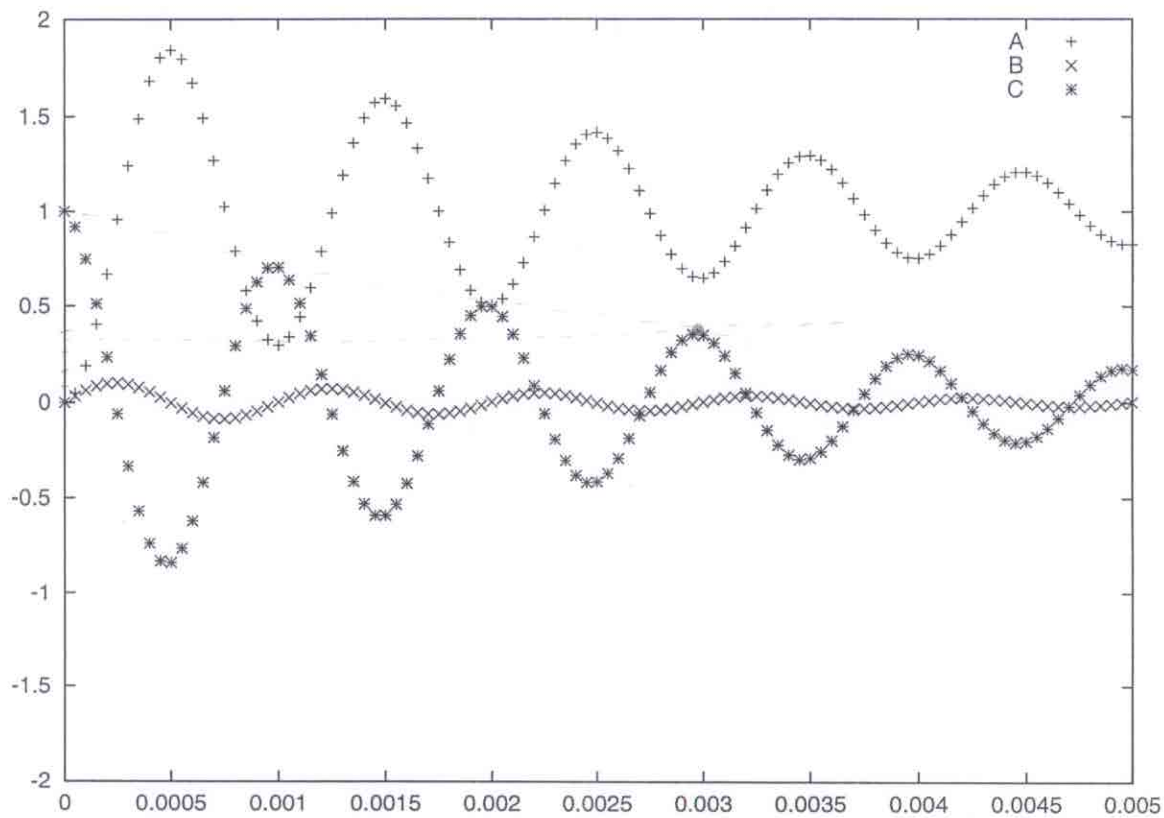


i(t)
t
t*
0
-I + V/R
-I e^{-t/RC}
v(t)
V
constant
RI (1-e^{-t/RC})
t
t* = - RC ln (1- V/RI)

Problem 2 – 25%

This problem concerns the second-order circuit shown below. The inductance L is 25 mH, as indicated. The input v_{IN} steps from 0 V to 1 V at $t = 0$. Prior to the step neither the inductor nor the capacitor store any energy. Three voltage waveforms that correspond to v_{C}, v_{L} and v_{R} are also graphed below, although they are not labeled as such. Their common horizontal axis is marked in seconds, and their common vertical axis is marked in volts.





(2A) The three voltage waveforms are labeled "A", "B" and "C". Identify which waveform is which voltage.

A:	V_C	B:	V_R	C:	V_L

$V_C(\infty) = V_{IN}$ so A is V_C

$V_L(0) = V_{IN}$ so C is V_L

$\Rightarrow$ B is V_R.

(2B) What is the *approximate* value of the capacitance C?

$C \approx$ 1.01 μF

$\omega_d \approx \omega_0$ if high Q

$$\omega_0 = 2\pi f = \frac{2\pi}{T} = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow C = \frac{1}{L}\left(\frac{T}{2\pi}\right)^2 , \quad T \approx 1 mS$$

$$\Rightarrow C \approx 1.01 \mu F$$

(2C) What is the *approximate* value of the resistance R? Hint: $e^{-0.7} \approx 1/2$, and $e^{-1.6} \approx 1/5$.

$R \approx$ 17.2 Ω

Time to go from $1 \to \frac{1}{e}$ is $\frac{1}{\alpha}$

where $\alpha = \frac{R}{2L}$ for series circuit

$t_{decay} \approx 2.9\,ms \Rightarrow \alpha = \frac{1}{t_{decay}} = \frac{R}{2L}$

$\Rightarrow R \approx \frac{2L}{t_{decay}} = 17.24\,\Omega$

(2D) Indicate the validity of the following statements by circling "T" for "True" or "F" for "False" as appropriate.

$\boxed{\text{T}}$	F	The transient oscillatory period increases with increasing L.
$\boxed{\text{T}}$	F	The transient oscillatory period increases with increasing C.
$\boxed{\text{T}}$	F	The transient rate of decay increases with increasing R.
$\boxed{\text{T}}$	F	The transient rate of decay is independent of C.
T	$\boxed{\text{F}}$	The transient rate of decay is independent of L.
T	$\boxed{\text{F}}$	The quality factor satisfies $Q < 5$.

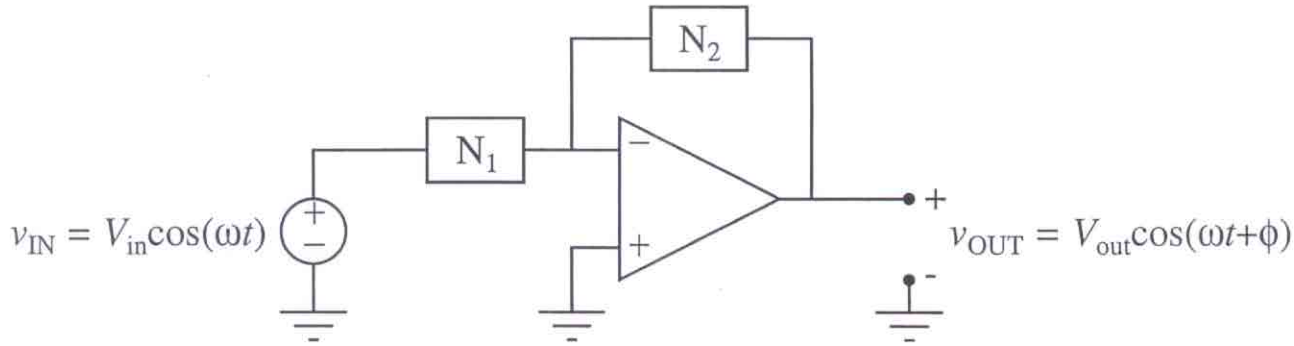
$$\omega = \frac{1}{\sqrt{LC}} = \frac{2\pi}{T}$$

$$\alpha = \frac{R}{2L}$$

$$Q = \frac{\omega}{2\alpha} \approx \frac{\left(\frac{2\pi}{1ms}\right)}{2\,\frac{1}{2.9ms}} \approx 9.11$$

Problem 3 – 25%

This problem concerns the op-amp filter shown below; note the definitions of the sinusoidal-steady-state input and output voltages, v_{IN} and v_{OUT}, respectively. Networks N_1 and N_2 are single-port networks that contain only capacitors inductors and resistors. Assume that the op-amp is ideal.



(3A) Let N_1 have the frequency-dependent impedance $Z_1(\omega) = Z_{R1}(\omega) + jZ_{I1}(\omega)$, where Z_{R1} and Z_{I1} are the real and imaginary parts of Z_1, respectively. Similarly, let N_2 have the frequency-dependent impedance $Z_2(\omega) = Z_{R2}(\omega) + jZ_{I2}(\omega)$. For this case, find V_{out} and ϕ as functions of V_{in}, Z_{R1}, Z_{I1}, Z_{R2} and Z_{I2}.

$$V_{out} = \frac{\sqrt{Z_{R2} + Z_{I2}}}{\sqrt{Z_{R1} + Z_{I1}}} V_{in} \qquad \phi = \tan^{-1}\left(\frac{Z_{I2}}{Z_{R2}}\right) - \tan^{-1}\left(\frac{Z_{I1}}{Z_{R1}}\right) + \pi$$

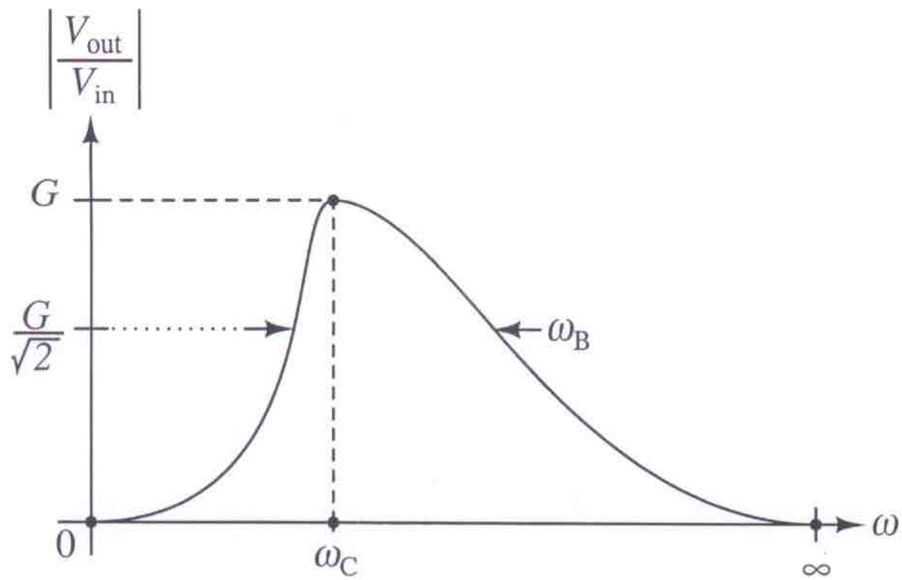
$$V_{out} = -\frac{Z_2}{Z_1} V_{in} = -\frac{Z_{R2} + jZ_{I2}}{Z_{R1} + jZ_{I1}} V_{in}$$

$$= (-1)\frac{Z_{R2} + jZ_{I2}}{Z_{R1} + jZ_{I1}} V_{in}$$

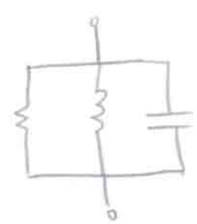
$$\Rightarrow \quad \phi = [\angle \text{top} - \angle \text{bottom}] + \underbrace{\measuredangle -1}_{\pi}$$

Be careful!

(3B) Suppose that N_1 is a single resistor having resistance R. Further, suppose that N_2 contains one capacitor having capacitance C, one inductor having inductance L, and one resistor having resistance βR. For this case, design N_2 so that the filter provides the band-pass gain magnitude $|V_{out}/V_{in}|$ defined graphically below. That is, draw the topology of N_2, and specify C, L and β in terms of R, and the filter parameters G, ω_B and ω_C. Hint: the function $X^2/(X^2+Y^2)$ equals 1/2 when $X = \pm Y$.



Topology:



$C = \frac{1}{GR\omega_B}$

$L = \frac{GR\omega_B}{\omega_c^2}$

$\beta = G$

Since N_1 is $Z_1 = R$, we have

$\left|\frac{V_{out}}{V_{in}}\right| = \frac{\sqrt{Z_{R2}^2 + Z_{I2}^2}}{R}$ $\Big\}$ $\left|\frac{V_{out}}{V_{in}}\right|$ is $\frac{|Z_2|}{R}$, so $|Z_2|$ must provide the band pass.

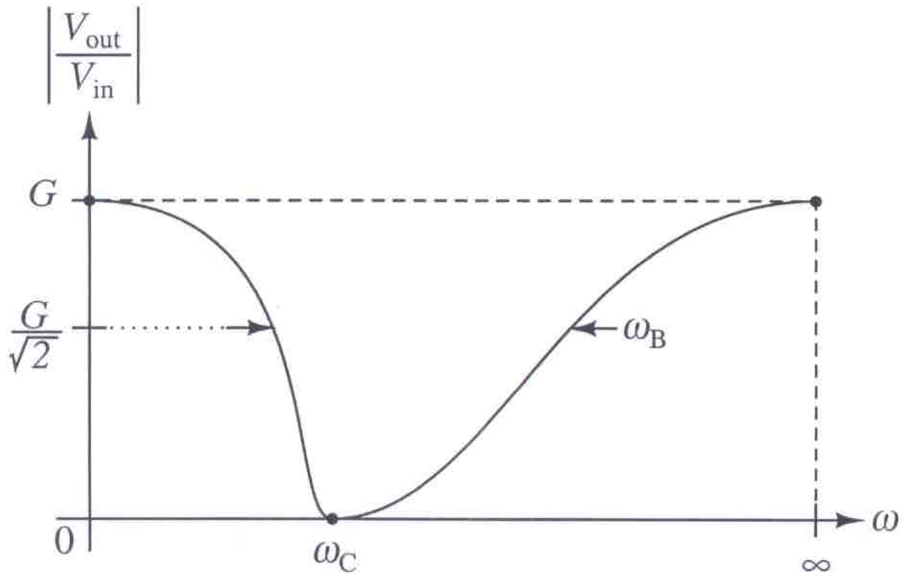
$w_C = \frac{1}{\sqrt{LC}}$, $w_B = 2\alpha = \frac{1}{\beta R C}$

@ resonance, want $Z_2 = GR \Rightarrow \beta R = GR \Rightarrow \beta = G$

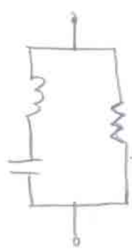
$\Rightarrow \frac{1}{C} = w_c^2 L \Rightarrow w_B = \frac{1}{\beta R} w_c^2 L \Rightarrow L = \frac{\beta R w_B}{w_c^2}$

$\Rightarrow C = \frac{1}{w_c^2 L}$

(3C) Repeat the previous part, but design N_2 instead so that the filter provides the band-stop gain magnitude defined graphically below. Again, draw the topology of N_2, and specify C, L and β in terms of R, and the filter parameters G, ω_B and ω_C.



Topology:



$C = \dfrac{\omega_B}{\omega_c^2\, GR}$

$L = \dfrac{GR}{\omega_B}$

$\beta = G$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{|Z_2|}{R} \quad \text{again.}$$

$$\omega_c = \frac{1}{\sqrt{LC}}$$

At low and high freq, $Z_2 = \beta R$, want $Z_2 = GR \Rightarrow \beta = G$

$$Z_{in} = R \,//\, \left[\frac{1}{sC} + sL\right] = R \,//\, \left[\frac{s^2LC+1}{sC}\right]$$

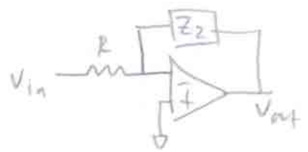
$$= \frac{\frac{R(s^2LC+1)}{sC}}{R + \frac{s^2LC+1}{sC}} = \frac{R(s^2LC+1)}{sRC + s^2LC+1} = \frac{\frac{R}{LC}(s^2LC+1)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\Rightarrow 2\alpha = \omega_B = \frac{\beta R}{L}$$

$$\Rightarrow \frac{\omega_c^2\, GR}{\omega_B} = \frac{1}{C}$$

(3D) Let N_1 be a single resistor having resistance R. Suppose that for a specific N_2, $V_{out}(\omega)$ and $\phi(\omega)$ are determined to be the specific functions $\bar{V}_{out}(\omega)$ and $\bar{\phi}(\omega)$, respectively. Determine the $V_{out}(\omega)$ and $\phi(\omega)$ that would result if a capacitor having capacitance C were placed in parallel with the resistor that is N_1. Express the result in terms of R, C, ω, $\bar{V}_{out}(\omega)$ and $\bar{\phi}(\omega)$.

$$V_{out} = \tilde{V}_{out}\sqrt{1+\omega^2R^2C^2} \qquad \phi = \tilde{\phi} + \tan^{-1}(\omega RC)$$



$$\Rightarrow \quad V_{out} = -\frac{Z_2}{R} V_{in}$$

$$\Rightarrow \quad \tilde{V}_{out} = -\frac{\sqrt{Z_{2R}^2 + Z_{2I}^2}}{R} V_{IN}$$

Now add C . . .

$$Z_1 = \frac{\frac{R}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{R}{1+j\omega RC}$$

$$= \frac{R(1-j\omega RC)}{1+(\omega RC)^2}$$

$$= \frac{R}{1+(\omega RC)^2} - j\frac{\omega R^2C}{1+(\omega RC)^2}$$

$$\tilde{V}_{out,n} = -\frac{\sqrt{Z_{2R}^2+Z_{2I}^2}}{\sqrt{\frac{R^2}{(1+(\omega RC)^2)^2} + \frac{(\omega R^2C)^2}{(1+(\omega RC)^2)^2}}} V_{in}$$

$$= \frac{\tilde{V}_{out}}{\sqrt{\frac{1}{(1+(\omega RC)^2)^2} + \frac{(\omega RC)^2}{(1+(\omega RC)^2)^2}}} = \tilde{V}_{out}\sqrt{(1+(\omega RC)^2)}$$

$$\tilde{\phi} = \tan^{-1}\left(\frac{Z_{2I}}{Z_{2R}}\right) + \pi$$

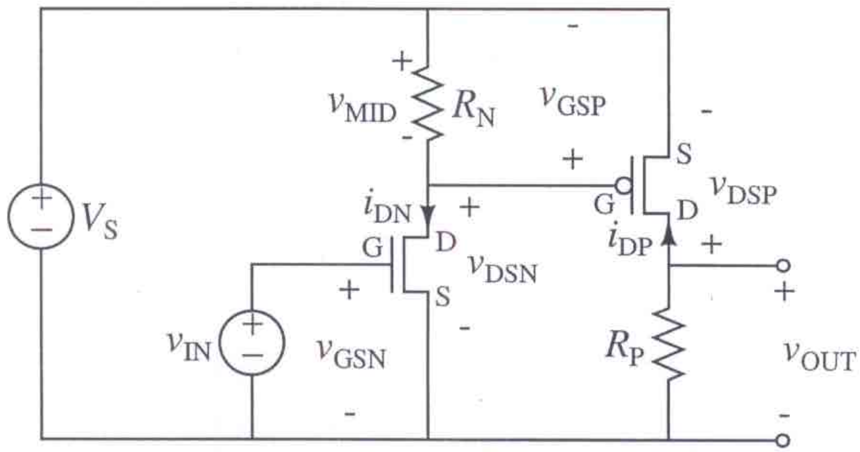
$$\tilde{V}_{out,n} = -\frac{Z_{2R} + jZ_{2I}}{\frac{R}{1+(\omega RC)^2} - j\frac{\omega R^2C}{1+(\omega RC)^2}} \Rightarrow \tilde{\phi}_{,n} = \underbrace{\pi + \tan^{-1}\left(\frac{Z_{2I}}{Z_{2R}}\right)}_{\tilde{\phi}} - \tan^{-1}(-\omega RC)$$

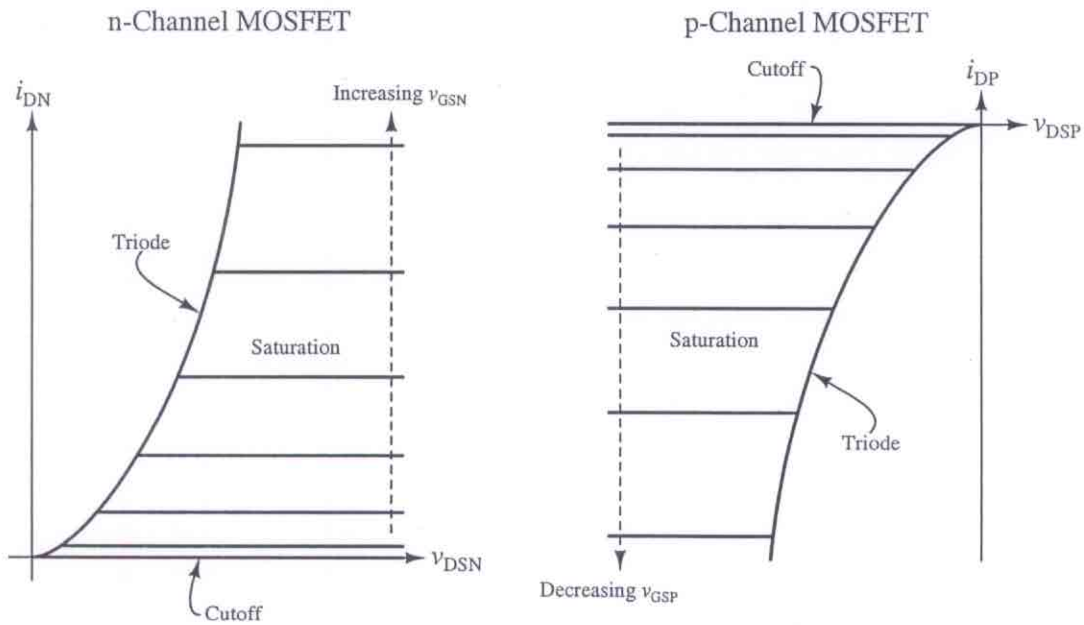
$$= \tilde{\phi} + \tan^{-1}(\omega RC)$$

since $\tan(-\theta) = -\tan(\theta)$

Problem 4 – 25%

This problem concerns the MOSFET amplifier shown below. The amplifier is constructed with one n-channel MOSFET, one p-channel MOSFET and two resistors. The MOSFET characteristics are also shown below. Note that the operation of each MOSFET in its triode region is simplified; each triode region is compressed onto a quadratic curve. Also note that K_N, K_P and V_{TN} are all postive, while V_{TP} is negative.





$$i_{DN} = \begin{cases} 0 & v_{GSN} - V_{TN} \leq 0 \quad \text{(Cutoff)} \\ \frac{K_N}{2}(v_{GSN} - V_{TN})^2 & 0 \leq v_{GSN} - V_{TN} < v_{DSN} \quad \text{(Saturation)} \\ \frac{K_N}{2} v_{DSN}^2 & 0 \leq v_{DSN} \leq v_{GSN} - V_{TN} \quad \text{(Triode)} \end{cases}$$

$$i_{DP} = \begin{cases} 0 & -v_{GSP} + V_{TP} \leq 0 \quad \text{(Cutoff)} \\ -\frac{K_P}{2}(-v_{GSP} + V_{TP})^2 & 0 \leq -v_{GSP} + V_{TP} < -v_{DSP} \quad \text{(Saturation)} \\ -\frac{K_P}{2} v_{DSP}^2 & 0 \leq -v_{DSP} \leq -v_{GSP} + V_{TP} \quad \text{(Triode)} \end{cases}$$

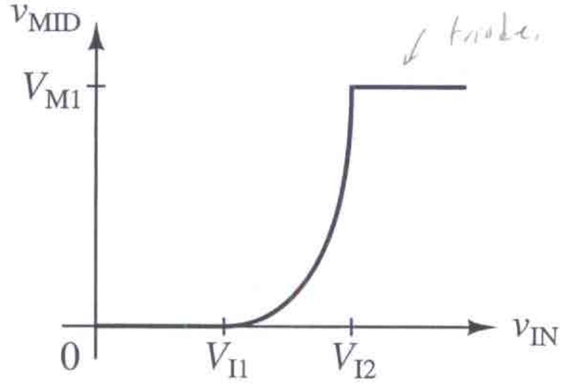
(4A) Assume that the n-channel MOSFET operates in its saturation region. Determine v_{MID} as a function of v_{IN}.

$v_{\text{MID}} = \quad R_N \frac{K_N}{2} (V_{IN} - V_{TN})^2$

$$V_{mid} = R_N \, i_{DN}$$

$$= \frac{R_N K_N}{2} (V_{IN} - V_{TN})^2$$

(4B) The dependence of v_{MID} on v_{IN} can be summarized by the figure shown below. Determine the parameters V_{I1}, V_{I2} and V_{M1} in the figure.



$V_{I1} =$ V_{TN} $\qquad$ $V_{I2} = \sqrt{\dfrac{2V_{M1}}{R_N K_N}} + V_{TN}$ $\qquad$ $V_{M1} = V_S - \dfrac{\sqrt{1+2K_N R_N V_S} - 1}{K_N R_N}$

For $V_{in} < V_{TN}$, $I_{DN} = 0 \Rightarrow V_{mid} = 0$

$\Rightarrow V_{I1} = V_{TN}$

$V_{mid} = V_{M1}$ when $V_{IN} = V_{I2} \Rightarrow$ point when M_N enters triode regim so V_{IN} does not affect V_{mid}

$$\Rightarrow I_D = \frac{V_{M1}}{R_N} = \frac{K_N}{2}(V_{I2} - V_{TN})^2$$

$$\Rightarrow V_{I2} = \sqrt{\frac{2V_{M1}}{R_N K_N}} + V_{TN}$$

@ V_{I2}, $I_D = \dfrac{V_S - V_{DS}}{R_N} = \dfrac{K_N}{2} V_{DS}^2$; $V_{M1} = V_S - V_{DS}$.

$$\Rightarrow V_S - V_{DS} = \frac{K_N R_N}{2} V_{DS}^2$$

$$\Rightarrow 0 = \frac{K_N R_N}{2} V_{DS}^2 + V_{DS} + V_S$$

$$\Rightarrow V_{DS} = \frac{-1 + \sqrt{1 - 2K_N R_N V_S}}{K_N R_N}$$

(4C) Assume that the p-channel MOSFET operates in its saturation region. Determine v_{OUT} as a function of v_{MID}.

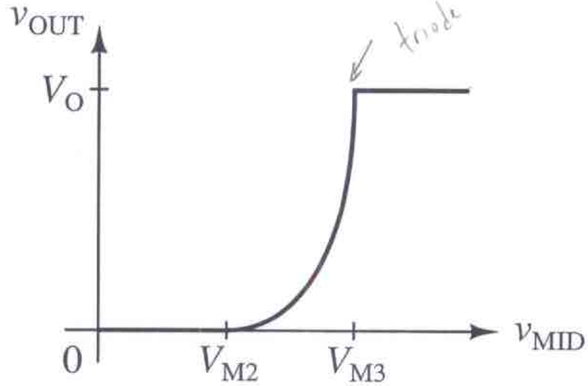
$$v_{OUT} = \frac{R_p K_p}{2} (V_{mid} + V_{Tp})^2$$

$$V_{GSp} = -V_{mid}$$

$$V_{out} = - i_{Dp} R_p$$

$$= + \frac{R_p K_p}{2} (V_{mid} + V_{Tp})^2 .$$

(4D) The dependence of v_{OUT} on v_{MID} can be summarized by the figure shown below. Determine the parameters V_{M2}, V_{M3} and V_{O1} in the figure.



$V_{\mathrm{M2}} = -V_{TP}$ $V_{\mathrm{M3}} = \sqrt{\dfrac{2V_o}{K_p R_P}} - V_{TP}$ $V_{\mathrm{O}} = V_s - \dfrac{\sqrt{1+2K_P R_P V_s}-1}{K_P R_P}$

By symmetry of NMOS / PMOS

$\Rightarrow$ $V_{TN} \Rightarrow -V_{TP}$

$\Rightarrow$ $V_{M1} \Rightarrow V_o$, $R_N \to R_P$, $K_N \to K_P$

(4E) To achieve a wide operating range for v_{OUT} it is desirable for the p-channel MOSFET to enter its triode operating region before the n-channel MOSFET does. Which of the following design constraints is necessary to guarantee this operation? Circle the correct constraints.

Constraints:	$V_{M1} < V_{M2} < V_{M3}$	$V_{M2} < V_{M1} < V_{M3}$	$\boxed{V_{M2} < V_{M3} < V_{M1}}$

$\downarrow V_{M1} \Rightarrow \downarrow V_{I2}$, point where NMOS goes triode $\rightarrow$ want V_{M1} large.

V_{M3} control V when PMOS goes triode $\rightarrow$ want $V_{M3} < V_{M1}$

(4F) Assuming that both the n-channel MOSFET and the p-channel MOSFET operate in their saturation regions. Determine v_{OUT} as a function of v_{IN}.

$v_{OUT} = \frac{R_P K_P}{2}\left[\left(\frac{R_N K_N}{2}(V_{IN} - V_{TN})^2\right) + V_{TP}\right]^2$

$V_{out} = \frac{R_P K_P}{2}\left[\left(\frac{R_N K_N}{2}(V_{IN} - V_{TN})^2\right) + V_{TP}\right]^2$

$\rightarrow$ combine 4A and 4C.

(4G) Let $v_{IN} = V_{IN} + v_{in}$ and $v_{OUT} = V_{OUT} + v_{out}$ where V_{IN} and V_{OUT}, and v_{in} and v_{out}, are the large and small-signal components of v_{IN} and v_{OUT}, respectively. Assume that both MOSFETs operate in their saturation regions, at bias points established by V_{IN}. In this case, determine the linearized small-signal gain of the amplifier, v_{out}/v_{in}, evaluated at V_{IN}.

$$v_{out}/v_{in} = R_N R_P K_N K_P (V_{IN} - V_T)\left[\frac{R_N K_N}{2}(V_{IN} - V_{TN})^2 + V_{TP}\right].$$

$$\frac{v_{out}}{v_{in}} = g_{mn} R_N \, g_{mp} R_P$$

$$g_m = K(V_{GS} - V_T)$$

Fixes:

Fall 2002

1. B. ii

5. first circuit, 4th circuit

$v = L\frac{di}{dt}$

Spring 2000 conflict

2. C → added to graph.

3. A iC

5. C