Please put your name in the space provided above, and circle the name of your recitation instructor and the time of your recitation.

Do your work for each question within the boundaries of the question. When finished, put your answer to each question in the corresponding answer box that follows the question.

You may use one double-sided page of notes while taking this exam. Calculators are allowed.

Final grades in 6.002 will not be given out by phone or by e-mail. Rather, they should be available through WEBSIS by May 23. You may review and take back your final exam on or after May 23.

Good luck!
Problem 1 – 25%

This problem involves determining the transient response of four networks as they are connected to a load through a switch that closes at \( t = 0 \). In each case, the terminal behavior of the network is described graphically. For each case, graph the terminal current \( i \) and the terminal voltage \( v \) that results during the transient that follows the closure of the switch. Clearly label the axes of both graphs, and clearly label both transients with their analytical expressions.

(1A) The load is an inductor having inductance \( L \) and no initial current.

\[
\begin{align*}
\nu_L &= \nabla - L \frac{di_L}{dt} = -L \frac{di}{dt} \\
\Rightarrow \quad i &= -\frac{\nu}{L} t
\end{align*}
\]
(1B) The load is an inductor having inductance $L$ and no initial current.

\[ i_{ec} = -I \Rightarrow i_{w} = I \]

\[ i_{L}(0^-) = 0 \text{ A} \]

\[ i_L = -\dot{i} = I \left( 1 - e^{-t/(L/R)} \right) u(t) \]

\[ v_L = V = L \frac{d\dot{i}_L}{dt} = L \left[ I \delta(t) - \left( -\frac{1}{L} e^{-\frac{t}{L/R}} - \frac{1}{L/R} \right) \delta(t) \right] \]

\[ = L \left[ \frac{1}{L} e^{-\frac{t}{R/L}} \right] \delta(t) \]

Be careful of $u(t)$ when taking derivative!
\[ i(t) \]

\[ \frac{L}{R} \]

\[ t \]

\[ i = \frac{L}{R} \]

\[ i' = -I_0 (1 - e^{-\frac{t}{L}}) u(t) \]

\[ v(t) \]

\[ R I \]

\[ \frac{L}{R} \]

\[ v = RI e^{-\frac{t}{L}} u(t) \]
The load is an inductor having inductance $L$ and no initial current.

Initially $i_L(t^-) = 0 \Rightarrow i(0) = 0 \Rightarrow$ Box is $V = V$

same case like (1A) until

$$i = -I_x$$

$$\frac{1}{R} = -\frac{I_x + 1}{V} \Rightarrow \frac{V}{R} = -\frac{I_x + 1}{V}$$

$$\Rightarrow \frac{1}{R} - I_x = \frac{V}{R}$$

$$\Rightarrow i = -i_L = -\frac{V}{L} t \text{ when } x = -I_x = \frac{V}{R} - I, \text{ Box change}$$

$$\Rightarrow t = -\frac{L}{V} \left[ \frac{V}{R} - 1 \right] = \frac{IL}{V} - \frac{L}{R} = \frac{L}{RV} (2R - V)$$

Now like (18), except $t \rightarrow t - t_x$ and different starting condition.
The load is a capacitor having capacitance $C$ and no initial voltage.

Initially, $V = V_c = 0$ so box looks like

Normal equivalent:

$$\begin{align*}
i &= \frac{V}{R} \\
\frac{1}{C}i &= -V
\end{align*}$$

$$V_c = IR \left( 1 - e^{-\frac{t}{RC}} \right) u(t).$$

$$i = -i_c = -C \frac{dv_c}{dt} = -C \left\{ IV R \delta(t) - \left[ \frac{IV R e^{\frac{t}{RC}} \delta(t) - \frac{1}{C} e^{\frac{t}{RC}} \mu(t)}{V R} \right] \right\}$$

$$= -C \left\{ \frac{1}{C} e^{\frac{-t}{RC}} \mu(t) \right\} = -I e^{\frac{-t}{RC}} \mu(t).$$

But change at $t = 0$ when $V_c = V$

$$\Rightarrow \quad \frac{V}{IR} = 1 - e^{-\frac{t}{RC}} \Rightarrow e^{-\frac{t}{RC}} = \frac{V}{IR} - 1$$

$$\Rightarrow \quad t^* = -RC \ln \left( \frac{V}{IR} - 1 \right)$$

Now $X = -i_c = C \frac{dv_c}{dt} = 0$

$$I^* = -I \left( 1 - \frac{V}{IR} \right) = -I + \frac{V}{R}$$
Problem 2 – 25%

This problem concerns the second-order circuit shown below. The inductance \( L \) is 25 mH, as indicated. The input \( v_{IN} \) steps from 0 V to 1 V at \( t = 0 \). Prior to the step neither the inductor nor the capacitor store any energy. Three voltage waveforms that correspond to \( v_C, v_L \) and \( v_R \) are also graphed below, although they are not labeled as such. Their common horizontal axis is marked in seconds, and their common vertical axis is marked in volts.
(2A) The three voltage waveforms are labeled "A", "B" and "C". Identify which waveform is which voltage.

<table>
<thead>
<tr>
<th>A: $V_C$</th>
<th>B: $V_R$</th>
<th>C: $V_L$</th>
</tr>
</thead>
</table>

$V_C(\omega) = V_{IN}$ so $A$ is $V_C$

$V_L(0) = V_{IN}$ so $C$ is $V_L$

$\Rightarrow B$ is $V_R$.

(2B) What is the \textit{approximate} value of the capacitance $C$?

$C \approx 1.01 \mu F$

$\omega_d \approx \omega_0$ if high $\alpha$

$\omega_0 = 2\pi f = \frac{2\pi}{T} = \frac{1}{NLC}$

$\Rightarrow C = \frac{1}{L} \left( \frac{1}{2\pi} \right)^2 \ , \ T \approx 1ms$

$\Rightarrow C \approx 1.01 \mu F$
(2C) What is the *approximate* value of the resistance $R$? Hint: $e^{-0.7} \approx 1/2$, and $e^{-1.6} \approx 1/5$.

$$R \approx 17.2 \Omega$$

Time to go from 1 to $\frac{1}{e}$ is $\frac{1}{\alpha}$

where $\alpha = \frac{R}{2L}$ for series circuit

$t_{\text{decay}} = 2.9 \text{ ms} \Rightarrow \alpha = \frac{1}{t_{\text{decay}}} = \frac{R}{2L}$

$
\Rightarrow R \approx \frac{2L}{t_{\text{decay}}} = 17.24 \Omega
$
(2D) Indicate the validity of the following statements by circling “T” for “True” or “F” for “False” as appropriate.

<p>| | | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
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</tbody>
</table>

- The transient oscillatory period increases with increasing $L$.
- The transient oscillatory period increases with increasing $C$.
- The transient rate of decay increases with increasing $R$.
- The transient rate of decay is independent of $C$.
- The transient rate of decay is independent of $L$.
- The quality factor satisfies $Q < 5$.

$$w = \frac{1}{\sqrt{LC}} = \frac{2\pi}{T}$$

$$\alpha = \frac{R}{2L}$$

$$\alpha = \frac{w}{2\alpha} = \frac{(2\pi)}{(1\text{ms})}$$

$$\frac{1}{2.9\text{ms}}$$
Problem 3 – 25%

This problem concerns the op-amp filter shown below; note the definitions of the sinusoidal-steady-state input and output voltages, \( v_{\text{IN}} \) and \( v_{\text{OUT}} \), respectively. Networks \( N_1 \) and \( N_2 \) are single-port networks that contain only capacitors inductors and resistors. Assume that the op-amp is ideal.

\[
v_{\text{IN}} = V_{\text{in}} \cos(\omega t)
\]

\[
v_{\text{OUT}} = V_{\text{out}} \cos(\omega t + \phi)
\]

(3A) Let \( N_1 \) have the frequency-dependent impedance \( Z_1(\omega) = Z_{R1}(\omega) + jZ_{I1}(\omega) \), where \( Z_{R1} \) and \( Z_{I1} \) are the real and imaginary parts of \( Z_1 \), respectively. Similarly, let \( N_2 \) have the frequency-dependent impedance \( Z_2(\omega) = Z_{R2}(\omega) + jZ_{I2}(\omega) \). For this case, find \( V_{\text{out}} \) and \( \phi \) as functions of \( V_{\text{in}} \), \( Z_{R1} \), \( Z_{I1} \), \( Z_{R2} \) and \( Z_{I2} \).

\[
V_{\text{out}} = \frac{\sqrt{Z_{R2} + Z_{I2}}}{\sqrt{Z_{R1} + Z_{I1}}} \cdot V_{\text{in}}
\]

\[
\phi = \tan^{-1}\left(\frac{Z_{I2}}{Z_{R2}}\right) - \tan^{-1}\left(\frac{Z_{I1}}{Z_{R1}}\right) + \pi
\]

\[
V_{\text{out}} = -\frac{Z_2}{Z_1} \cdot V_{\text{in}}
\]

\[
= -\frac{Z_{R2} + jZ_{I2}}{Z_{R1} + jZ_{I1}} \cdot V_{\text{in}}
\]

\[
\Rightarrow \phi = \left[ \alpha_0 - \frac{\beta + \gamma t}{\pi} \right] + \frac{\pi - 1}{\pi}
\]

Be careful!
Suppose that $N_1$ is a single resistor having resistance $R$. Further, suppose that $N_2$ contains one capacitor having capacitance $C$, one inductor having inductance $L$, and one resistor having resistance $\beta R$. For this case, design $N_2$ so that the filter provides the band-pass gain magnitude $|V_{out}/V_{in}|$ defined graphically below. That is, draw the topology of $N_2$, and specify $C$, $L$ and $\beta$ in terms of $R$, and the filter parameters $G$, $\omega_B$ and $\omega_C$. Hint: the function $X^2/(X^2 + Y^2)$ equals $1/2$ when $X = \pm Y$.

\[
\frac{V_{out}}{V_{in}}
\]

\[ G \]

\[
\frac{G}{\sqrt{2}} \quad \omega_B
\]

\[ \omega_C \quad \infty \]

---

**Topology:**

\[
\begin{align*}
C &= \frac{1}{6R\omega_B} \\
L &= \frac{6R\omega_B}{\omega_C^2} \\
\beta &= G
\end{align*}
\]

---

Since $N_1$ is $\frac{Z_1}{R}$, we have

\[
\frac{|V_{out}|}{V_{in}} = \frac{\sqrt{Z_{R^2} + Z_{R^2}}}{R} \quad \text{is} \quad \frac{|Z_2|}{R} \quad \text{so} \quad |Z_2| \quad \text{must}
\]

\[
\text{be the band pass.}
\]

\[
w_C = \frac{1}{NC} \quad , \quad w_B = 2\chi = \frac{1}{\beta RC}
\]

Consequently, want

\[
Z_2 = GR \quad \Rightarrow \beta R = GR \quad \Rightarrow \beta = G
\]

\[
\frac{1}{\chi} = w_C^2 L \quad \Rightarrow \quad w_B = \frac{1}{\beta R} w_C^2 L \quad \Rightarrow \quad L = \frac{\beta R \omega_B}{w_C^2}
\]

\[
\Rightarrow \quad C = \frac{1}{w_C^2 L}
\]
(3C) Repeat the previous part, but design N₂ instead so that the filter provides the band-stop gain magnitude defined graphically below. Again, draw the topology of N₂, and specify C, \( L \) and \( \beta \) in terms of \( R \), and the filter parameters \( G, \omega_B \) and \( \omega_C \).

\[
\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{|V_{\text{out}}|}{|V_{\text{in}}|}
\]

\[
G
\]

\[
\frac{G}{\sqrt{2}}
\]

\[
\omega_C \quad \omega_B
\]

\[
\omega
\]

Topology:

\[
C = \frac{\omega_B}{\omega_C^2 - GR}
\]

\[
L = \frac{GR}{\omega_B}
\]

\[
\beta = G
\]

\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{|V_{\text{out}}|}{|V_{\text{in}}|} \quad \text{again.}
\]

\[
\omega_C = \frac{1}{\sqrt{L/C}}
\]

At low and high f, \( Z_2 = \beta R \), so \( Z_2 = GB \) \( \Rightarrow \beta = G \)

\[
Z_{\text{in}} = \frac{R}{\left( \frac{1}{LC} + sL \right)} = \frac{R}{\left( \frac{s^2LC+1}{sC} \right)}
\]

\[
= \frac{R}{sLC + s^2LC + 1}
\]

\[
\Rightarrow z = \omega_B = \frac{GR}{L}
\]

\[
\Rightarrow \frac{\omega_C^2}{\omega_B^2} = \frac{1}{C}
\]
(3D) Let \( N_1 \) be a single resistor having resistance \( R \). Suppose that for a specific \( N_2 \), \( V_{\text{out}}(\omega) \) and \( \phi(\omega) \) are determined to be the specific functions \( \tilde{V}_{\text{out}}(\omega) \) and \( \tilde{\phi}(\omega) \), respectively. Determine the \( V_{\text{out}}(\omega) \) and \( \phi(\omega) \) that would result if a capacitor having capacitance \( C \) were placed in parallel with the resistor that is \( N_1 \). Express the result in terms of \( R \), \( C \), \( \omega \), \( \tilde{V}_{\text{out}}(\omega) \) and \( \tilde{\phi}(\omega) \).

\[
\begin{align*}
V_{\text{out}} &= \tilde{V}_{\text{out}} N \sqrt{1 + \frac{2 R C}{\omega^2}} \\
\phi &= \tilde{\phi} + \tan^{-1} \left( \frac{\omega R C}{2} \right)
\end{align*}
\]

\[
\begin{align*}
V_{\text{in}} \quad \Rightarrow \quad V_{\text{out}} &= \frac{Z_2}{R} V_{\text{in}} \\
\tilde{V}_{\text{in}} &= -\frac{\sqrt{Z_2^2 + \omega^2 R^2}}{R} V_{\text{in}}
\end{align*}
\]

\[
\begin{align*}
Z_1 &= \frac{R}{1 + j \omega C} = \frac{R}{1 + j \omega R C} \\
&= \frac{R(1 - j \omega R C)}{1 + (\omega R C)^2} \\
&= \frac{R^2 - j \omega R^2 C}{1 + (\omega R C)^2}
\end{align*}
\]

\[
\begin{align*}
\tilde{\phi} &= \tan^{-1} \left( \frac{\omega R C}{Z_2} \right) + \pi \\
\tilde{V}_{\text{out}} &= -\frac{2 R + j \omega R^2 C}{R} \tilde{V}_{\text{in}} \Rightarrow \tilde{\phi}_{\text{out}} &= \pi + \tan^{-1} \left( \frac{\omega R C}{2} \right) - \tan^{-1} \left( -\omega R C \right) \\
&= \tilde{\phi} + \tan^{-1} \left( \omega R C \right)
\end{align*}
\]

Since \( \tan(-\theta) = -\tan(\theta) \).
Problem 4 – 25%

This problem concerns the MOSFET amplifier shown below. The amplifier is constructed with one n-channel MOSFET, one p-channel MOSFET and two resistors. The MOSFET characteristics are also shown below. Note that the operation of each MOSFET in its triode region is simplified; each triode region is compressed onto a quadratic curve. Also note that $K_N$, $K_P$ and $V_{TN}$ are all positive, while $V_{TP}$ is negative.

The equations for the drain current $i_{DN}$ and $i_{DP}$ for an n-channel MOSFET and a p-channel MOSFET, respectively, are given by:

For the n-channel MOSFET:

$$i_{DN} = \begin{cases} 
0 & \text{if } v_{GSN} - V_{TN} \leq 0 \\
\frac{K_N}{2} (v_{GSN} - V_{TN})^2 & \text{if } 0 \leq v_{GSN} - V_{TN} < v_{DSN} \\
\frac{K_N}{2} v_{DSN} & \text{if } 0 \leq v_{DSN} \leq v_{GSN} - V_{TN}
\end{cases}$$

For the p-channel MOSFET:

$$i_{DP} = \begin{cases} 
0 & \text{if } -v_{GSP} + V_{TP} \leq 0 \\
-\frac{K_P}{2} (-v_{GSP} + V_{TP})^2 & \text{if } 0 \leq -v_{GSP} + V_{TP} < -v_{DSP} \\
-\frac{K_P}{2} v_{DSP} & \text{if } 0 \leq -v_{DSP} \leq -v_{GSP} + V_{TP}
\end{cases}$$
(4A) Assume that the n-channel MOSFET operates in its saturation region. Determine \( v_{\text{MID}} \) as a function of \( v_{\text{IN}} \).

\[
v_{\text{MID}} = R_N \frac{k_N}{2} \left( v_{\text{IN}} - v_{\text{TN}} \right)^2
\]

\[
v_{\text{MID}} = R_N \cdot \beta_N \frac{k_N}{2} \left( v_{\text{IN}} - v_{\text{TN}} \right)^2
\]
(4B) The dependence of $v_{\text{MID}}$ on $v_{\text{IN}}$ can be summarized by the figure shown below. Determine the parameters $V_{11}$, $V_{12}$ and $V_{M1}$ in the figure.

\[
V_{11} = V_{7N} \quad V_{12} = \sqrt{\frac{2V_{M1}}{R_{FN}N} + V_{7N}} \quad V_{M1} = V_S - \frac{\sqrt{1 + 2KNPNV_S^2} - 1}{KNPN}
\]

\[
\begin{align*}
F_{N} & \quad v_{\text{IN}} < V_{7N} \quad I_{DN} = 0 \quad \Rightarrow \quad V_{\text{MID}} = 0 \\
\Rightarrow & \quad V_{11} = V_{7N} \\
V_{\text{MID}} & \quad V_{M1} \quad \text{when} \quad V_{\text{IN}} = V_{12} \quad \Rightarrow \quad \text{point when } M_N \text{ enters triode region so } V_{\text{IN}} \text{ does not affect } V_{\text{MID}} \\
& \quad \Rightarrow \quad I_D = \frac{V_{M1}}{R_N} = \frac{K_N}{2} (V_{12} - V_{7N})^2 \\
& \quad \Rightarrow \quad V_{12} = \sqrt{\frac{2V_{M1}}{R_{FN}N} + V_{7N}} \\
\end{align*}
\]

\[
\begin{align*}
\text{C } & \quad V_{12} \quad \Rightarrow \quad I_D = \frac{V_S - V_{DS}}{R_N} = \frac{K_N}{2} V_{DS}^2 \quad \Rightarrow \quad V_{M1} = V_S - V_{DS} \\
\Rightarrow & \quad V_S - V_{DS} = \frac{K_NPN}{2} V_{DS}^2 \\
& \quad 0 = \frac{K_NPN}{2} V_{DS}^2 + V_{DS} + V_S \\
& \quad V_{DS} = -1 \pm \sqrt{1 - 2K_NPNV_S}
\end{align*}
\]
(4C) Assume that the p-channel MOSFET operates in its saturation region. Determine $v_{\text{OUT}}$ as a function of $v_{\text{MID}}$.

\[
v_{\text{OUT}} = \frac{R_p K_p}{2} \left( V_{\text{m, d}} + V_{\eta p} \right)^2
\]

\[
V_{\text{eP}} = -V_{\text{mid}}
\]

\[
V_{\text{P+}} = -x_D R_p R_p
\]

\[
= + \frac{R_p K_p}{2} \left( V_{\text{m, d}} + V_{\eta p} \right)^2
\]
The dependence of $v_{\text{OUT}}$ on $v_{\text{MID}}$ can be summarized by the figure shown below. Determine the parameters $V_{M2}$, $V_{M3}$ and $V_0$ in the figure.

$$V_{M2} = -V_\tau$$
$$V_{M3} = \frac{2V_o}{N} \times \frac{1}{k_p^p} - V_\tau$$
$$V_0 = V_\tau - \frac{\sqrt{1 + 2k_p^pV_\tau} - 1}{k_p^p}$$

By symmetry of NMOS / PMOS

$$\Rightarrow V_{V_N} \Rightarrow -V_\tau$$
$$\Rightarrow V_{M1} \Rightarrow V_0$$

$k_N \rightarrow k_P$, $k_N \rightarrow k_P$
(4E) To achieve a wide operating range for \( v_{\text{OUT}} \) it is desirable for the p-channel MOSFET to enter its triode operating region before the n-channel MOSFET does. Which of the following design constraints is necessary to guarantee this operation? Circle the correct constraints.

<table>
<thead>
<tr>
<th>Constraints:</th>
<th>( V_{M1} &lt; V_{M2} &lt; V_{M3} )</th>
<th>( V_{M1} &lt; V_{M2} &lt; V_{M3} )</th>
<th>( V_{M2} &lt; V_{M3} &lt; V_{M1} )</th>
</tr>
</thead>
</table>

\[ \downarrow V_{M1} \rightarrow \downarrow V_{M2} \quad \text{point where NMOS goes} \]

tried \( \rightarrow \text{want} \quad V_{M2} \quad \text{large} \quad . \]

\[ V_{M3} \quad \text{control} \quad V_{\text{OUT}} \quad \text{PMOS goes} \quad \text{tried} \rightarrow \text{want} \]

\[ V_{M3} < V_{M1} \]

(4F) Assuming that both the n-channel MOSFET and the p-channel MOSFET operate in their saturation regions. Determine \( v_{\text{OUT}} \) as a function of \( v_{\text{IN}} \).

\[ v_{\text{OUT}} = \frac{R_p K_p}{2} \left[ \left( \frac{R_{NK} K_N}{2} \left( V_{IN} - V_{TH} \right)^2 \right) + V_{TP} \right]^2 \]

\[ V_{IN} = \frac{R_p K_p}{2} \left[ \left( \frac{K_N K_P}{2} \left( V_{IN} - V_{TN} \right)^2 \right) + V_{TP} \right]^2 \]

\[ \rightarrow \text{combine \#A and \#C} \]
(4G) Let $v_{IN} = V_{IN} + v_{in}$ and $v_{OUT} = V_{OUT} + v_{out}$ where $V_{IN}$ and $V_{OUT}$, and $v_{in}$ and $v_{out}$, are the large and small-signal components of $v_{IN}$ and $v_{OUT}$, respectively. Assume that both MOSFETs operate in their saturation regions, at bias points established by $V_{IN}$. In this case, determine the linearized small-signal gain of the amplifier, $v_{out}/v_{in}$, evaluated at $V_{IN}$.

$$\frac{v_{out}}{v_{in}} = \frac{r_n}{r_p} \frac{k_n}{k_p} (V_{IN} - V_I) \left[ \frac{r_n k_n}{2} \left( \frac{v_{IN} - V_{IN}}{V_{IN} - V_{IN}} \right)^2 + V_{np} \right].$$

$$\frac{v_{IN}}{v_{IN}} = g_m v_{IN} r_n k_p r_p.$$

$$g_m = k (V_{GS} - V_T)$$
Fixes:
Fall 2022
4. B. in
5. first circuit,
4th circuit

Spring 2020 conflict
2. C → added to graph.
3. A in C
5. C