6.003 (Fall 2007): Problem sets

Homework 1

*Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 12 September 2007**.*

Warmups

1. **Solving differential equations**
   Solve the following differential equation

$$y(t) + 3\frac{dy(t)}{dt} + 2\frac{d^2y(t)}{dt^2} = 1$$

   for $t \geq 0$ assuming the initial conditions

$$y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$$

   and express the solution in closed form.

   [Hint: assume the homogeneous solution has the form $Ae^{s_1 t} + Be^{s_2 t}$.]

2. **Solving difference equations**
   Solve the following difference equation

$$8y[n] - 6y[n-1] + y[n-2] = 1$$

   for $n \geq 0$ assuming the initial conditions

$$y[0] = y[-1] = 0$$

   and express the solution in closed form.

   [Hint: Assume that the homogeneous solution has the form $Az_1^n + Bz_2^n$.]

3. **O&W Problem 1.54 (page 73))**

Problems

4. **Choosing a bank**
   Consider two banks. Bank #1 offers a 6% annual interest rate, but charges a $1 service charge each year, including the year when the account was opened. Bank #2 offers a 5% annual interest rate, and has no annual service charge. Let $x_i[n]$ represent the amount of money you deposit in bank i during year n and $y_i[n]$ represent your balance in bank i. Assume that deposits during year n are credited to the balance in year n but earn no interest until year $n+1$.

   a. Use difference equations to express the relation between deposits and balances for each bank.

   b. Assume that you deposit $100 in each bank in the year 2007 and make no further deposits. Solve your difference equations in part a numerically (using Matlab, Octave, or Python) to determine your balance in each bank during years 0 through 25. Make a plot of these balances. Which bank has the larger balance in the year 2011? Which bank has the larger balance in the year 2031? [Hint: See the Appendix for help with programming.]

5. **Coupled Oscillator**

The following equations characterize a "coupled oscillator":

$$\frac{dy_1(t)}{dt} = y_2(t)$$

and

$$\frac{dy_2(t)}{dt} = -y_1(t).$$

a. Use the forward-Euler method to develop a set of difference equations that approximate this system of differential equations.

b. Solve the difference equations numerically (using Matlab, Octave, or Python) and plot the results. What happens if you choose a step size that is too big? How big is too big?

6. **Making sherry**

Sherry is a wine that is typically aged in a series of barrels called a solera. Each year, a portion (assume half) of the wine in the last barrel is extracted, bottled, and sold. This barrel is then refilled with wine from the next-to-last barrel. The next-to-last barrel is then refilled with wine from the previous barrel, and the process is repeated until the first barrel is refilled with newly crushed juice from this year's grape harvest.

a. To understand the way that the solera system mixes and ages the wine, consider a thought experiment in which $x[n]$ units of a tracer substance (such as deuterium) are added to the new crushed juice in year n. Let $a[n]$, $b[n]$, and $c[n]$ represent the amount of tracer found in the first, second, and third barrels at the end of year n (i.e., after the wine to sell in year n has been drawn out and the crushed joice from year n has been added). Develop difference equations to determine $a[n]$, $b[n]$, and $c[n]$ for $n \geq 0$.

b. Assume that one unit of tracer is added in year 0 and no tracer is added in any other year. Determine the year (or years) during which the amount of tracer in the third barrel is greatest. [Feel free to use a pencil, calculator, or computer to answer this question.]

c. Assume that one unit of tracer is added each year. Determine the steady-state value of tracer in the first barrel, i.e., determine

$$a_{ss} = \lim_{n\to\infty} a[n]$$

[again, pencils, calculators, or computers are welcome].

Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment.

Feel free also to comment on these problems.

Appendix: forward-Euler code

Suppose that we want to solve the following differential equation:

$$y(t) + \frac{dy(t)}{dt} = 1$$

starting with an initial condition $y(0) = 0$. We can use the forward-Euler method to generate the following approximation:

$$y[n+1] = T + (1 - T)y[n]$$

where T is the step size and $y[0] = 0$. Here is Matlab/Octave and Python code that implements this approximation. Octave is a free-software linear-algebra, with a syntax very similar to Matlab, and it is available for most platforms. See **www.octave.org**.

Example Matlab/Octave code

The following Matlab/Octave code evaluates the analytic solution to the differential equation (i.e., $y(t) = 1 - e^{-t}$ for $t > 0$) as well as the numerical approximation. It then plots the two results.

```
t(1) = 0;
y(1) = 0;
a(1) = 1-exp(0.);
T = .1;
for n = 1:99
  t(n+1) = t(n)+T;
  y(n+1) = T+(1-T)*y(n);
  a(n+1) = 1-exp(-t(n+1));
end
plot(t,y,'r',t,a,'b');
```

Example Python code

```
from   scipy import *
import pylab as     p

T = 0.1                                      # time step
t = arange(0, 10.0001, T)
y = zeros(shape(t))                          # Euler-integrated solution
a = 1-exp(-t)                                # discretized analytic solution
for i in range(1,len(t)):
    y[i] = T + (1-T)*y[i-1]
p.plot(t, y, 'r', t, a, 'b')
p.show()
```

Homework 2

*Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 19 September 2007**.*

Warmups

1. Finding unit sample (impulse) responses

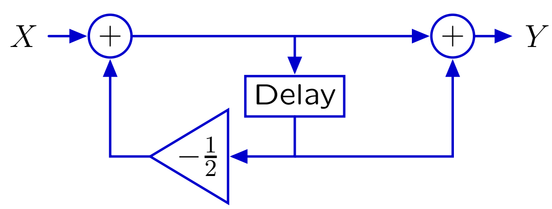
Determine the unit sample (impulse) responses of the systems represented by the following system functionals.

a. $(1 - \mathcal{R})^{-2}$

b. $(1 + \mathcal{R})^3/8$

2. Working with block diagrams

Consider the system represented by the following block diagram.



a. Find a difference equation that characterizes this system.

b. Determine the unit sample response of this system.

3. Finding a system

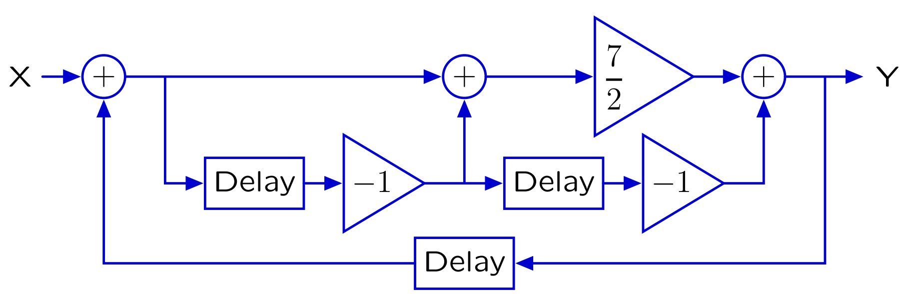
a. Determine a system whose output is $10, 1, 1, 1, 1, \ldots$ when the input is $1, 1, 1, 1, 1, \ldots$. Determine the difference equation and block diagram representations for this system.

b. Determine a system whose output is $1, 1, 1, 1, 1, \ldots$ when the input is $10, 1, 1, 1, 1, \ldots$. Determine the difference equation and block diagram reprsentations for this system.

c. Compare the difference equations in parts a and b. Compare the block diagrams in parts a and b.

4. Finding modes

The following block diagram describes the relation between two discrete-time signals: $x[n]$ and $y[n]$.



a. How many different modes does this system have?

b. Determine closed-form expressions for each mode. [Do not try to determine the amplitudes of the modes. The amplitudes depend on the input X, which is not specified.]

Problems

5. Drug dosing

When you start a medicine, the doctor often says 'Take double the usual dosage for the first dose, then follow the schedule.' In this problem you study this advice to take a so-called loading dose.

A simple model of the body is a tank of blood from which drug exits at a rate proportional to its concentration and into which drug doses instantly arrive. Assume that the patient takes the drug once every 8 hours, and that the concentration of drug in the blood is measured shortly after taking each dose. Draw the block diagram and give the system functional of a discrete-time system with this behavior. Choose the parameter(s) of the system so that, in the absence of new drug, two-thirds of the drug is filtered out (say, by the kidneys) every 8 hours.

Ideally, the level of drug in the blood would instantly go to the desired level and remain steady. Assume that you take the medicine every 8 hours. What dosage schedule will produce the ideal behavior? Interpret your answer in terms of loading doses, whether for or against the idea.

6. Experimental mathematics to debug a black box

You would like to characterize a system but all you know is its unit-sample (impulse) response, which is available as n, $y[n]$ pairs in the Appendix, and is available electronically on our course website as `black_box.tsv`. Use peeling away and educated guessing (see the R04 notes) to find the modes and their amplitudes, and therefore find a closed form for the output signal. What is the system functional and the corresponding difference equation? [Note: Feel free to use a computer, graphing calculator, or paper and pencil.]

7. Making sherry: redux

Reconsider the three-barrel system for "Making Sherry" in Homework #1 (Problem 5).

a. Express the difference equations that describe the three-barrel solera system using system functionals. Sketch a block diagram for the three-barrel solera system.

b. Rewrite the block diagram so that the main body is the cascade of three identical systems. Determine the system functional for this block diagram.

c. Determine the system functional for a ten-barrel solera system analogous to the system functional in part b.

d. Assume that one unit of tracer is added to the freshly crushed grapes during year 0 and no tracer is added in any other year. Write a program to determine how much tracer is in the sherry bottled in year n for a B-barrel solera system, where B is a parameter to your program.

Generally, the amount of tracer that is bottled in year n first increases with n and then decreases. Demonstrate this trend by plotting results for a 20-barrel solera.

Hours

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Appendix

Data for problem 5 (also available as a .tsv file on the course website).

0	1
1	5
2	19
3	65
4	211
5	665
6	2059
7	6305
8	19171
9	58025
10	175099
11	527345
12	1586131
13	4766585
14	14316139
15	42981185
16	129009091
17	387158345
18	1161737179
19	3485735825
20	10458256051
21	31376865305
22	94134790219
23	282412759265
24	847255055011
25	2541798719465
26	7625463267259
27	22876524019505
28	68629840493971
29	205890058352825
30	617671248800299
31	1853015893884545
32	5559051976620931
33	16677164519797385
34	50031510739261339

Homework 3

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on ***Wednesday, 26 September 2007****.*

Warmups

1. Complex numbers

Evaluate the following:

a. i^i

b. $(1 - j\sqrt{3})^{12}$

c. $\mathrm{Re}\left\{e^{5j\theta}\right\}$ in terms of $\sin\theta$ and $\cos\theta$.

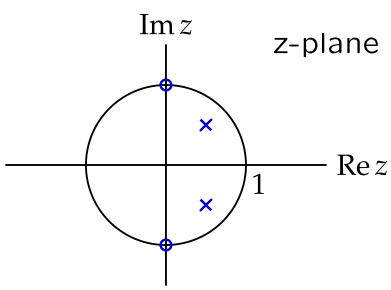
2. Characterizing modes

Determine the number of modes that can be generated by each of the following systems, and characterize each mode as monotonic or oscillatory and as growing, decaying, or neither.

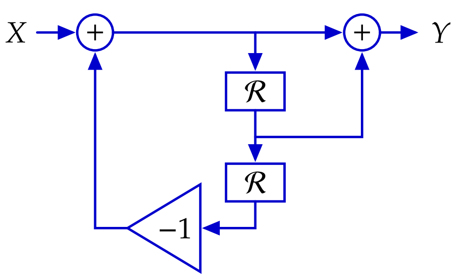
a.

$$\frac{Y}{X} = \frac{1 + \sqrt{2}\mathcal{R} + \mathcal{R}^2}{1 - \sqrt{2}\mathcal{R} + \mathcal{R}^2}.$$

b.



c.



d. Which of parts (a)–(c) was the simplest? Briefly explain why.

3. Lots of poles

a. Factor the system functional $(1 - \mathcal{R}^2)^{-2}$ into pole–zero form. Then give the partial-fractions decomposition, the impulse response of each term in the decomposition, and the sum of the impulse responses. Check your sum by feeding the impulse into a block diagram representing the original system functional.

b. Factor the system functional $(1 + \mathcal{R}^2)^{-2}$ into pole–zero form. Then give the partial-fractions decomposition, the impulse response of each term in the decomposition, and the sum of the impulse responses. Check your sum by feeding the impulse into a block diagram representing the original system functional.

4. Tricky quadratics (adapted from R. D. Middlebrook)

a. Use the quadratic equation and a pocket calculator, Python, or octave/Matlab to find the two roots of

$$x^2 - 10^{10}x + 1 = 0.$$

What went wrong?

b. Now solve it by approximating the square root in the quadratic formula.

c. Now solve it by approximating the equation. Assume that one root is near zero, rewrite the quadratic equation neglecting the least relevant term(s), and then solve for the root.

d. Knowing that root, how can you quickly find the other root?

e. Which route is easier: (1) approximating the equation then solving, or (2) solving via the quadratic formula and then approximating?

5. Exciting particular modes

The system described by the following functional

$$\frac{Y}{X} = \frac{1}{(1 - \frac{1}{2}\mathcal{R})(1 - \frac{1}{3}\mathcal{R})}$$

has modes of the form $\left(\frac{1}{2}\right)^n$ and $\left(\frac{1}{3}\right)^n$. Determine an input sequence $x[n]$ for which the persistent response contains some multiple of the first but not any of the second mode. Assume that the system is initially at rest, and that the first non-zero input sample occurs at $n = 0$.

Problems

6. Comparing forward and backward Euler

In this problem you compare the forward and backward Euler methods of converting a continuous-time system into a discrete-time system. Look again at how to discretize the leaky tank or RC circuit. The differential equation is

$$\tau \dot{V}_{\text{out}} = V_{\text{in}} - V_{\text{out}},$$

where τ is the time constant.

a. Using the forward-Euler approximation with time step T, the discrete-time implementation is

$$y[n] = (1 - \epsilon)y[n-1] + \epsilon x[n-1],$$

where $\epsilon = T/\tau$.

Draw a block diagram, write the system functional, and give a pole–zero diagram. When making the pole–zero diagram, ignore any pure power of $\mathcal{R}$ in the numerator of the system functional (it contributes only a pure delay, which is not interesting).

b. On your pole–zero diagram, show how the poles move as ϵ increases from -1 to 3. For what ranges of ϵ, if any, does the impulse response (i) decay without oscillating, (ii) grow without oscillating, (iii) decay and oscillate, and (iv) grow and oscillate? Write a simulation to confirm your answers. Which of these four ranges would you select for accurately simulating the continuous-time system?

c. For the rest of this problem, assume that ϵ is chosen reasonably. Compute the impulse response, i.e. give the output signal $y[n]$ for $n \geq 0$ when the impulse is the input signal X. Then evaluate $\sum_0^\infty y[n]$.

d. Using the backward-Euler approximation with time step T, the discrete-time implementation is

$$(1+\epsilon)y[n] = y[n-1] + \epsilon x[n],$$

where $\epsilon = T/\tau$.

Draw a block diagram, write the system functional, and give a pole-zero diagram.

e. Compute the impulse response, i.e. give the output signal $y[n]$ for $n \geq 0$ when the impulse is the input signal X. Then evaluate $\sum_0^\infty y[n]$.

f. Give a physical interpretation of $\sum_0^\infty y[n]$. Based on your evaluation of $\sum_0^\infty y[n]$ for forward and backward Euler, which method do you prefer?

7. Periodic system

In this problem you study this variant of the Fibonacci system:

$$y[n] = y[n-1] - y[n-2] + x[n].$$

a. Compute the impulse response and show that it is periodic. What is the period?

b. Draw a block diagram and give the system functional. How many modes does the system have?

c. The system's response to any transient input, not just the impulse, is periodic (you don't need to prove this statement). Therefore what can you conclude about the poles of the system?

d. Mark the poles and zeros of the system on a pole–zero diagram. Make sure the pole locations are consistent with your answers to parts (a) and (c)!

e. Decompose the system functional into partial fractions, and draw a block diagram that corresponds to this decomposition.

8. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

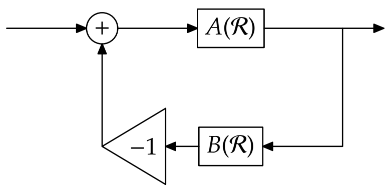
Homework 4
(not collected)

Try all of the following warmups and problems. It will ***not*** *be collected. Solutions will be posted in a few days for you to check yourself.*

Warmups

1. Black's formula

Here is a general feedback block diagram where each block can be itself a system with an interesting functional:



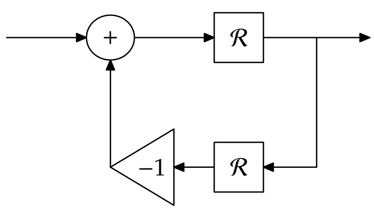
Show that the functional for the whole system is

$$F(\mathcal{R}) = \frac{A(\mathcal{R})}{1 + A(\mathcal{R})B(\mathcal{R})}.$$

This formula is known as *Black's formula* named after Harold Black, one of the inventors of the negative-feedback amplifier.

2. System functionals

a. Here is a feedback-control system:



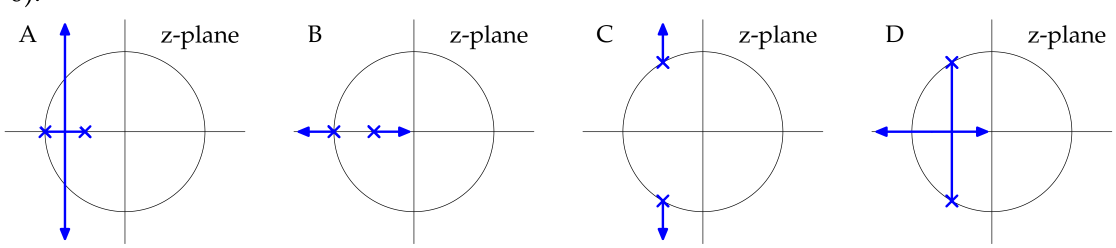
Find the system functional and the poles, showing them on a pole–zero plot.

b. In the preceding diagram, let the gain be $-\beta$ instead of -1. On a pole–zero diagram (the z-plane), sketch how the poles move as β varies from 0 to ∞.

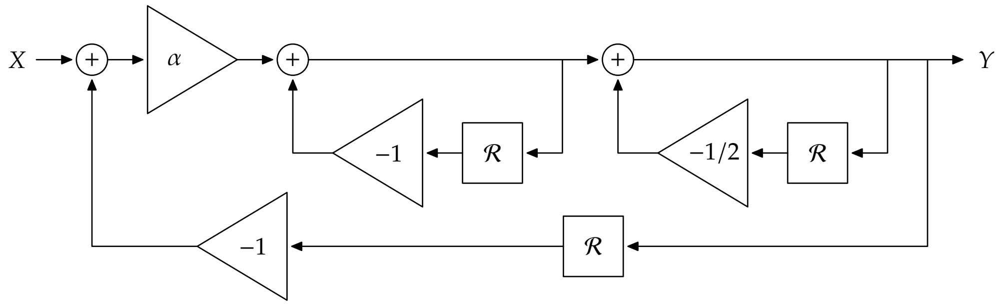
Problems

3. Closed-loop poles

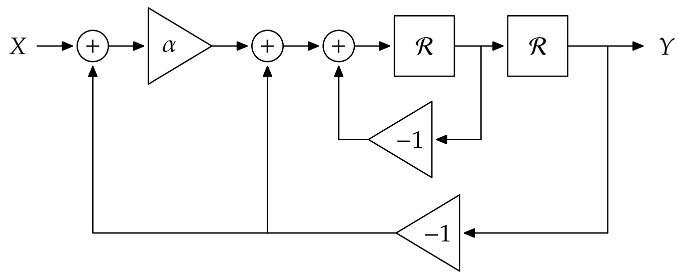
The following plots show the positions of closed-loop poles as functions of the gain α (where $\alpha > 0$):



a. Which if any of the plots A, B, C, or D correspond to the following block diagram?

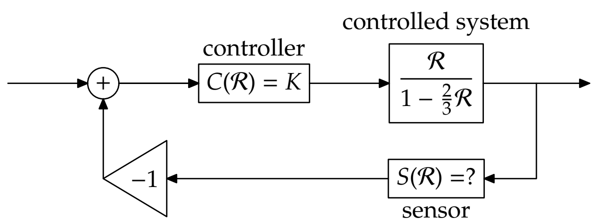


b. Which if any of the plots A, B, C, or D correspond to the following block diagram?



4. Surviving delay

Consider this feedback-control system:



In this problem you simulate to investigate the effect of increasing the delay in the sensor.

Start with the sensor being reasonably fast: $S(\mathcal{R}) = \mathcal{R}$. Let the controller gain K be 1/9. How many poles does the system with feedback have, and where are they? Simulate the system (in Octave/Matlab or Python) to confirm these pole locations.

Now increase the sensor delay by making $S(\mathcal{R}) = \mathcal{R}^2$. Modify your simulation for this case, keeping $K = 1/9$. If your simulation is well written, you should not have much to change to increase the sensor delay. Is the new system stable? If it is, increase the sensor delay until the system becomes unstable, keeping $K = 1/9$. What is the maximum usable sensor delay (i.e. how many powers of $\mathcal{R}$)?

Homework 5

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on ***Wednesday, 10 October 2007****.*

Warmups

1. Impulse responses

Find the impulse responses of the following systems:

1. $\dfrac{1}{1-\mathcal{L}/2}$
2. $\dfrac{\mathcal{L}}{1+3\mathcal{L}}$
3. $\dfrac{1-\mathcal{R}}{1-\mathcal{L}}$

2. Cascade

a. Is the system

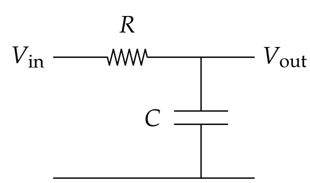
$$\frac{1}{1-0.95\mathcal{R}}\,\frac{1}{1-0.95\mathcal{L}}$$

causal, anticausal, or noncausal?

b. Find the $n = 0$ sample of its impulse response. You can use a computer or, if you are skilled with series, you can do it analytically.

3. RC filter

a. Write the continuous-time system functional (using $\mathcal{A}$) for $V_{\text{out}}/V_{\text{in}}$ in the RC circuit:



Expand the functional as a power series in $\mathcal{A}$.

b. Feed the impulse $\delta(t)$ into it and analytically compute the impulse response of each term in the series. Then add the results to find the impulse response of the system.

c. Give dimensional and extreme-cases arguments to show why the impulse response that you found is reasonable.

4. Find the delta functions

a. Sketch each function for $\epsilon = 1$ and $\epsilon = 1/3$:

1. $e^{-t^2/\epsilon}$
2. $\dfrac{\epsilon}{t^2+\epsilon^2}$
3. $\begin{cases} \dfrac{2}{\epsilon}\left(1-\dfrac{t}{\epsilon}\right) & (0 \le t \le \epsilon); \\ 0 & \text{(otherwise).} \end{cases}$

b. Which functions, if any, become $\delta(t)$ in the limit $\epsilon \to 0$?

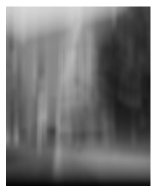
Problems

5. Image reconstruction

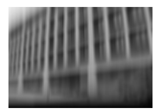
Choose one mangled image from each row, sharpen the image, and identify the building. Here are thumbnails of the images:

a1

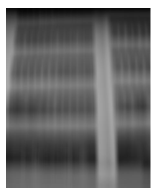
a2



b1



b2



c1

c2

The images are available in machine-readable form (buildings.zip) on the Stellar class site.

6. Leaky tank

The leaky-tank equation is

$$\tau \dot{y} = x - y.$$

where x is the input flow rate, y is the output flow rate, and τ is the time constant.

a. Define $p = 1/\tau$. Then find the system functional in terms of the $\mathcal{A}$ operator, and expand the functional in powers of $\mathcal{A}$.

b. Write a program that finds the system's response to any input signal X by using the series expansion in part (a). Try your program with the following input signals (which are all zero for $t < 0$), choosing two or three interesting values for the time constant τ:

1. $x(t) = 1.$
2. $x(t) = e^{-t}.$

For each input signal and value of τ, explain why your program's answer for the output is reasonable.

7. Hours

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Homework 6

*Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 17 October 2007**.*

Warmups

1. Find the nonsense

You are handed a giant rat's nest of a linear, passive circuit with L's, R's, and C's and asked to analyze it. Which expression(s) below cannot be its system functional V_{out}/V_{in}? The various τ's are time constants made by combining inductor, resistor, and capacitor values.

a. $$\frac{\mathcal{A}/\tau_1}{1+\mathcal{A}/\tau_3^2}$$

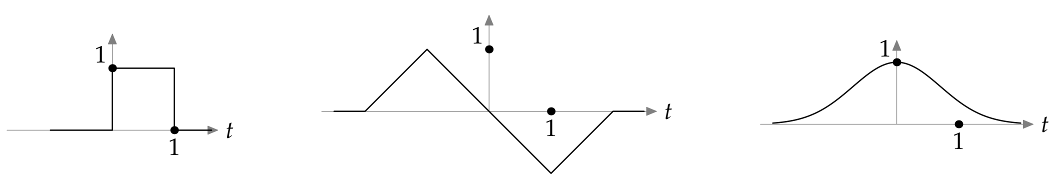
b. $$\frac{\mathcal{A}^2/\tau_3^2}{1+\mathcal{A}/\tau_1+\mathcal{A}^2/\tau_2}$$

c. $$\frac{\mathcal{A}^2}{1+\mathcal{A}/\tau_2+\mathcal{A}^2/(\tau_1\tau_2)+\mathcal{A}^3/(\tau_1\tau_2\tau_3)}$$

d. $$\frac{\tau_1}{\mathcal{A}(1+\mathcal{A}/\tau_2)}$$

2. Practice with the integration operator

a. Sketch the result of applying $\mathcal{A}$ to each of the following signals:

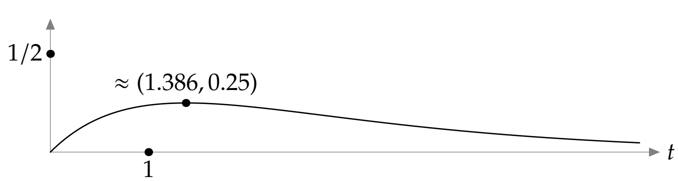


(i) (ii) (iii)

b. Sketch the result of applying $\mathcal{A}-\mathcal{A}^2+\mathcal{A}^3-\mathcal{A}^4+\ldots$ to the unit pulse (the first signal above).

3. Leaky tanks

Here is the impulse response of a cascade of two leaky tanks where the first tank has time constant $\tau_1 = 1$ and the second tank has time constant $\tau_2 = 2$.



a. Sketch the impulse response of a cascade with $\tau_1 = 3$ and $\tau_2 = 6$ and label any local maxima or minima.

b. Sketch the impulse response of a cascade with $\tau_1 = 2$ and $\tau_2 = 1$ and label any local maxima or minima.

Problems

4. Double-pole limit

Here is the system functional for a cascade of two identical leaky tanks or RC circuits:

$$S_1 = \frac{\mathcal{A}^2}{(1+\mathcal{A})^2}.$$

Here is the system functional for a cascade of two almost-identical leaky tanks or RC circuits:

$$S_2 = \frac{0.9\mathcal{A}}{1+0.9\mathcal{A}} \cdot \frac{1.1\mathcal{A}}{1+1.1\mathcal{A}}.$$

a. Find the impulse response of the two systems analytically and sketch the results.

b. Find the impulse response of the two systems numerically.

5. Differential equation

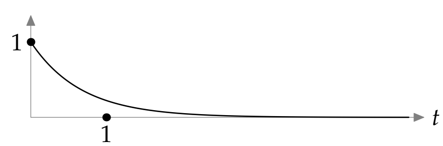
Imagine feeding the unit step function as the input signal X to the differential equation

$$\ddot{y}(t) + \dot{y}(t) + y(t) = x(t).$$

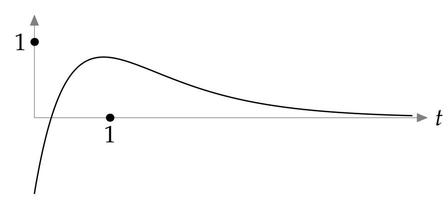
Find the output signal Y analytically in two ways: (1) by using the system functional Y/X and (2) by using 18.03 methods. Always sketch your results!

6. Designing recognizers

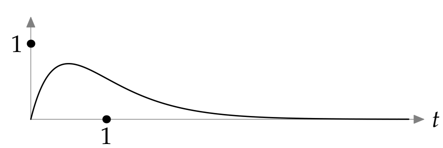
In this problem you design recognizers for particular sound patterns – such as might be used for speech recognition. For each sound pattern (signal) that follows, give the system functional that turns it into the unit step function. You are therefore designing a system that turns on during that sound pattern.



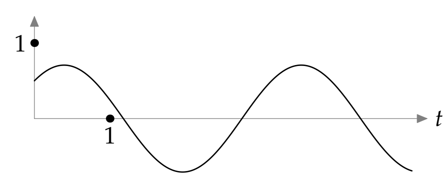
A



B



C



D

Each signal is available on Stellar as a data file with one $(t, f(t))$ pair on each line.

7. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

Homework 7

*Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 24 October 2007**.*

Warmups

1. Ranking frequencies

For a second-order, pole-only system with $Q = 1.5$, order the following frequencies from lowest to highest:

- the oscillation frequency ω_d,
- the natural frequency ω_0,
- the maximum-gain frequency ω_{max}, which is the frequency to which the system is most sensitive (i.e. for which it has the largest gain).

Draw a rough pole–zero diagram for the system, and mark the three frequencies on the positive imaginary axis.

2. Phase

For a second-order system with poles at −1 and −4 (and no zeros), find the frequency at which the phase is −90°, using any method except for the vector method. Then illustrate and confirm that result using the vector method.

3. Multiple representations

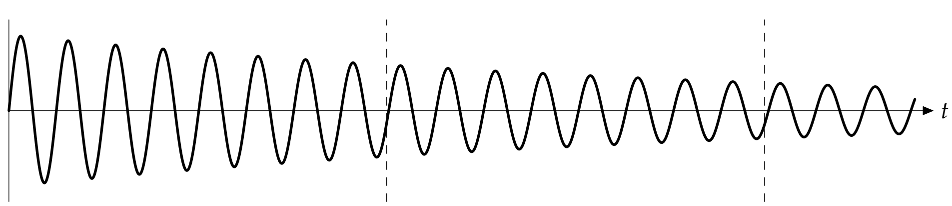
Each part of this question gives partial information about a second-order system without zeros, for example an *LRC* low-pass circuit. For each system, use the information to estimate Q and ω_0 and to sketch the impulse response and pole–zero diagram. If it is possible from the partial information, give scales on the real and imaginary axes of pole–zero diagrams, and give scales on the time axis of the impulse responses. [No need to repeat the information given, e.g. if you are given Q, there's no need to estimate it.]

a. $H(s) = 1/(s^2 + s + 1)$

b. Pole plot (all poles are on the real axis):

c. $Q = 0.5$

d. In the following impulse response, the dashed vertical lines mark one-second intervals.



e. $Q = 5$

4. Maximum gain

For each system, find the maximum-gain frequency ω_{max}, which is the frequency to which the system is most sensitive (has the largest gain).

a. $\dfrac{1}{1+s+s^2}$

b. $\dfrac{s}{1+s+s^2}$

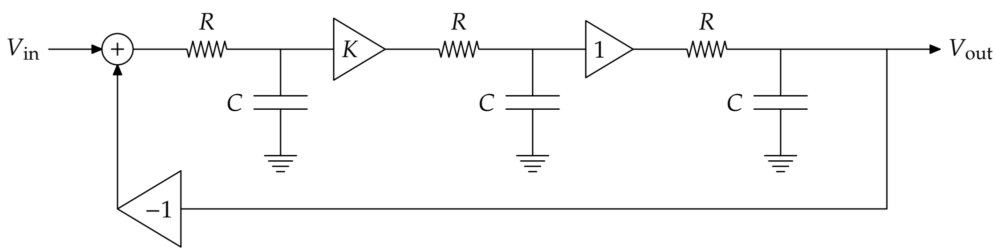
c. $\dfrac{s^2}{1+s+s^2}$

Compare the ω_{max} for these systems and explain qualitatively any similarities or differences.

Problems

5. Feedback

How do you build a reliable oscillator out of analog components? This feedback circuit is a precursor to Hewlett and Packard's founding patent where they solved that hard problem:

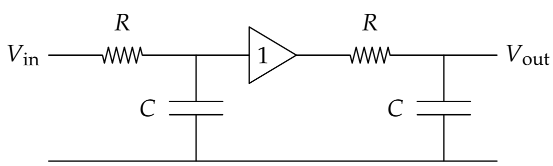


a. With $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$, sketch the pole locations as the gain K varies from 0 to ∞, showing a scale for the real and imaginary axes. Find the K for which the system is barely stable, and label your sketch with that information. What is the system's oscillation period at this K?

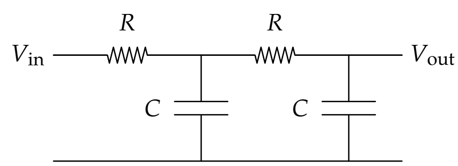
b. How do your results change if each R is increased to $10\,\text{k}\Omega$?

6. Two ways to combine two *RC* circuits

You might have wondered about any differences between cascading two RC circuits with and without an intervening unity-gain buffer. In this question you work out the answer. Here are diagrams of the two possibilities:



with buffer



without buffer

For each system, use the state-variable method to find the differential equation connecting V_{out} and V_{in}. Compare the respective system functions, impulse responses, Q's, and whatever else seems interesting. Discuss qualitatively any similarities and differences.

7. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

Homework 8 [rev.]

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on ***Wednesday, 31 October 2007****.*

Warmups

1. Sketching

a. Sketch these three functions $y(x)$ on one graph (using linear–linear axes): (i) $\ln x$, (ii) $\ln 4x$, and (iii) $\ln x^3$.

b. Sketch the same functions on linear–log axes (the x axis is logarithmic).

c. Sketch $\ln \cosh x$ on the usual (linear–linear) axes.

2. Stepwise sketching

Here is a sequence of functions to sketch on log–log axes. Sketch each function on its own graph, and explain the graphical transformation that takes the preceding sketch to the current one:

a. $1 + x^2$

b. $\sqrt{1 + x^2}$

c. $1/\sqrt{1 + x^2}$

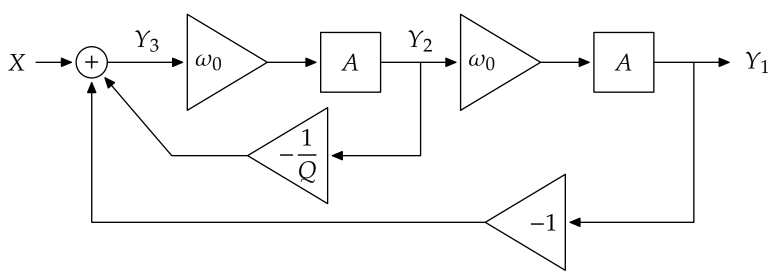
d. $1/\sqrt{1 + 9x^2}$

e. $3x/\sqrt{1 + 9x^2}$

What system function has the magnitude sketch in part (e)?

3. High-Q flavors

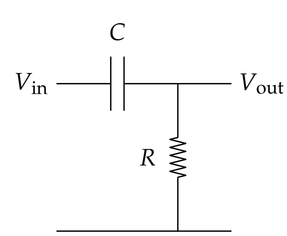
High-Q systems come in three flavors: low-pass, band-pass, and high-pass. Determine the system functionals and system functions for $\frac{Y_1}{X}$, $\frac{Y_2}{X}$, and $\frac{Y_3}{X}$. Determine the flavor of each.



4. Inverted poles

The system function for the CR circuit is

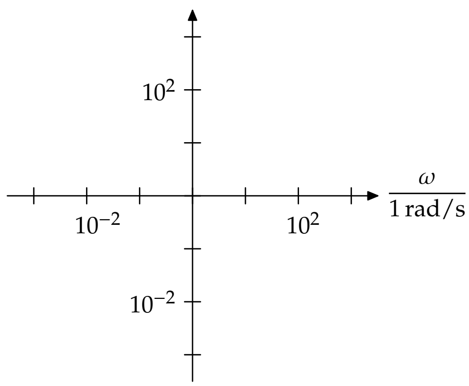
$$H(s) = \frac{\tau s}{1 + \tau s}$$



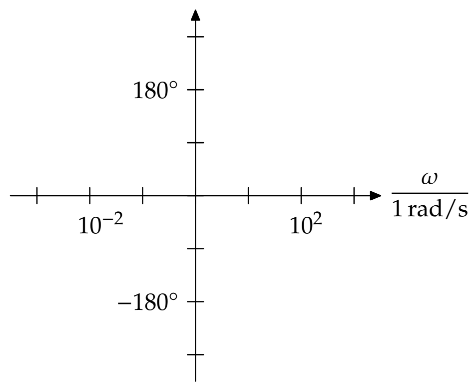
where $\tau = RC$. For this problem use $R = 10\,\mathrm{k}\Omega$ and $C = 10\,\mathrm{nF}$.

a. Sketch its Bode plot (the magnitude and phase) in three steps: (i) sketch the Bode plot for τs, (ii) sketch the Bode plot for $1 + \tau s$, and (iii) sketch the

Bode plot for $\tau s/(1+\tau s)$. For the phase and magnitude, use axes like these (extended in either direction as needed):



magnitude



phase

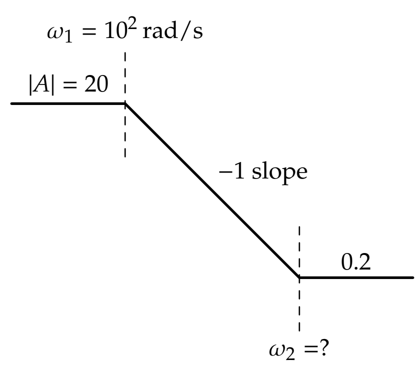
b. An alternative method is to introduce the idea of *inverted poles*. Divide the numerator and denominator by τs to get

$$H(s) = \frac{1}{1 + \dfrac{1}{\tau s}}.$$

Sketch the Bode plot in two steps: (i) sketch the Bode plot for $1/(1+\tau s)$, then (ii) transform it to get the Bode plot for $1/(1+(\tau s)^{-1})$. What transformation did you do, and what does it do pictorially to the Bode plot? Explain why these poles are called inverted poles.

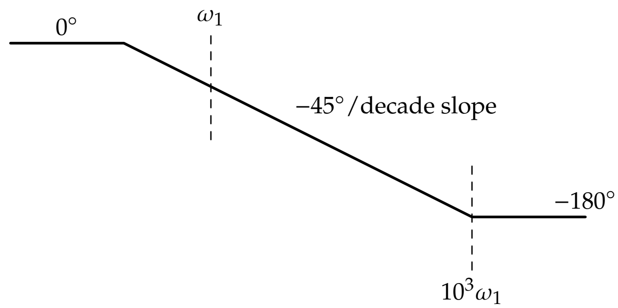
5. Going backwards

You have a mystery system and send test signals into it. You find that its magnitude Bode plot looks like:



What is ω_2 (use units of rad/s)? What $H(s)$ functions are consistent with this plot? Only consider functions that have two singularities (a singularity is a pole or a zero).

Now you return to your raw data and extract this phase plot:

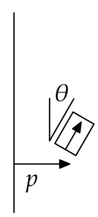


What is $H(s)$?

Problems

6. Robotic Steering

A car is being driven at a constant velocity $v(t) = V$. The goal is to hold the vehicle centered in the lane. Let $p(t)$ represent the deviation of the car's position from the center of the lane and $\theta(t)$ represent the angle that the vehicle makes with the horizontal axis.



If the steering wheel were held at a constant angular position $w(t)$, then the car would move in a circle. Such a trajectory has an angle $\theta(t)$ that increases linearly with time. If $w(t)$ were changed, the radius of the circle, and thus the time rate of change of $\theta(t)$ would change. This suggests a simple model for the relation between the steering wheel position $w(t)$ and $\theta(t)$:

$$\frac{d\theta(t)}{dt} = \alpha w(t)$$

where α is a constant, that is equal here to 1 second^{-1}. The deviation of the position of the car from the center of the lane depends on both the angle $\theta(t)$ and velocity $v(t) = V$. Assume that the following simple model suffices:

$$\frac{dp(t)}{dt} = v(t)\sin(\theta(t)) \approx V\theta(t).$$

In this problem, we consider how this car might be steered automatically, using a feedback system.

a. Assume that we have a sensor that will automatically detect the edges of our lane and determine $p(t)$. We wish to use this position sensor in a feedback system with proportional control, i.e., a constant K times the output $p(t)$ is used control the position of the steering wheel, so that

$$w(t) = Kp(t).$$

1. Determine the response of the system for $t > 0$ when $p(0) = 0$ and $\theta(0) = 1$. Determine the range of behaviors that can result for different values of K, and show responses that are representative of important values of K.

2. What value of K, if any, provides the most acceptable system performance?

b. Assume that we have a sensor that will automatically detect the edges of our lane and determine the time rate of change of $p(t)$. We wish to use this velocity sensor in a feedback system with proportional control, i.e., a constant K times the output $\frac{dp(t)}{dt}$ is used control the position of the steering wheel, so that

$$w(t) = K\frac{dp(t)}{dt}.$$

1. Determine the response of the system for $t > 0$ when $p(0) = 0$ and $\theta(0) = 1$. Determine the range of behaviors that can result for different values of K, and show responses that are representative of important values of K.

2. What value of K, if any, provides the most acceptable system performance.

3. Is the performance of this system better or worse than that in part (a)?

4. Are there any undesirable properties of the performance of this system?

c. Assume that both sensors of $p(t)$ and $\frac{dp(t)}{dt}$ are available so that we can feedback a weighted sum of these signals.

1. Determine the response of the system for $t > 0$ when $p(0) = 0$ and $\theta(0) = 1$. Determine the range of behaviors that can result for different values of weighting factors. Show responses for representative choices of weighting factors.

2. What values of the weighting factors provide the most acceptable system performance?

3. Is the performance of this system better or worse than that in part (a)?

4. Are there any undesirable properties of the performance of this system?

7. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

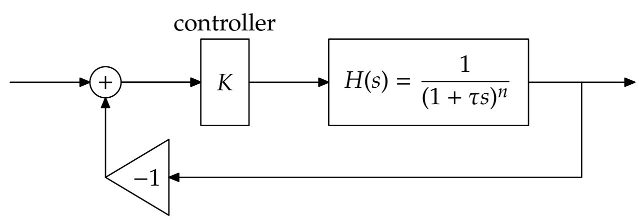
Homework 9
(not collected)

*Try all of the following warmups and problems. It will **not** be collected. Solutions will be posted in a few days for you to check yourself.*

Bode and control

1. Phase oscillator

a. Below is a feedback circuit with two parameters: K, the controller gain; and n, the number of identical RC filter stages in the forward path.

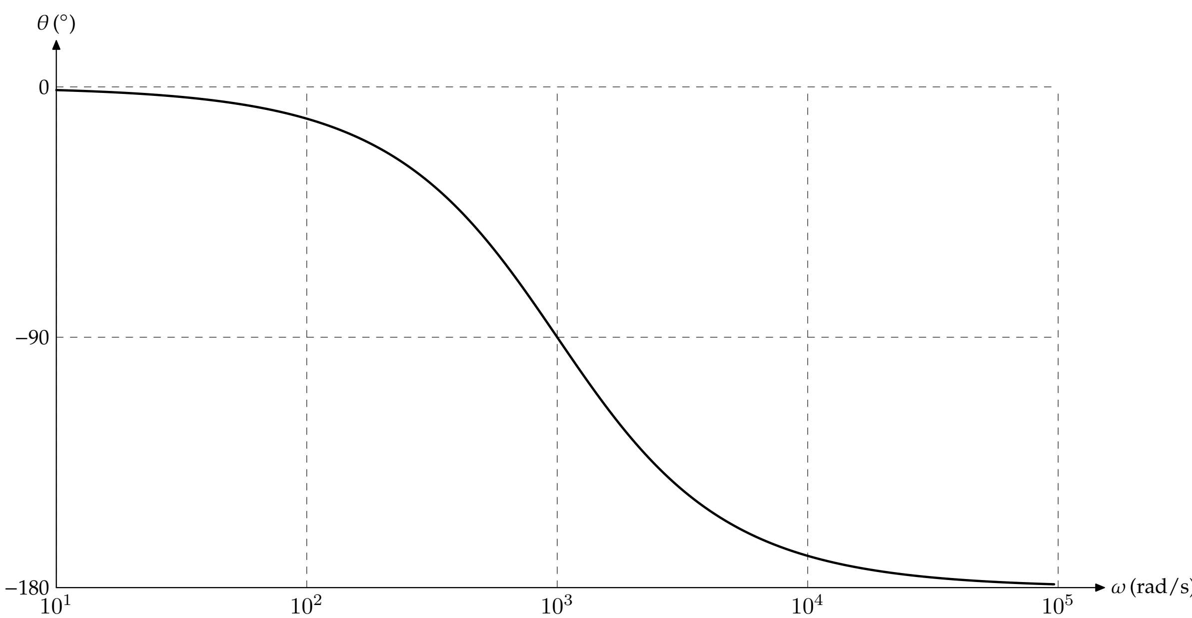


For each positive K, there is a maximum number of stages n_{max} that can be used before the system becomes unstable. Sketch n_{max} as a function of K.

b. How does your sketch change if the time constant τ is doubled?

2. Phase of a second-order system

a. Here is the phase for a system with $Q = 0.5$ and $\omega_0 = 10^3$ rad/s, plotted on the usual Bode linear–logarithmic axes.



1. Draw the three phase asymptotes by matching the slope at $\omega = \omega_0$ in order to make a slanted asymptote that connects the horizontal asymptotes. At what frequencies do

the asymptotes intersect? What is the worst phase error that you make by using these asymptotes instead of the true curve?

2. Draw the three-piece phase asymptote by using the connecting asymptote derived in the Recitation 16 notes (use the corrected version of the notes because the original version lacked a couple factors of 2 in the figure on p. 134). At what frequencies do the asymptotes intersect? What is the worst phase error that you make by using these asymptotes instead of the true curve?

b. Plot the Bode phase accurately for $Q = 5$, perhaps using Octave/Matlab/Python, and then do part (a) for that plot.

Convolution [not on quiz 2]

3. Linearity and time invariance

A linear, time-invariant system can often be represented with a differential equation. Linearity means that the differential equation is linear. What characteristic of the differential equation does time invariance correspond to?

4. Pulses

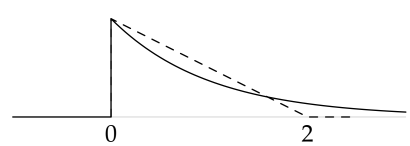
a. By integration compute and sketch $f \star f$ where f is the pulse of unit height and width centered on the origin. Notice that, at its widest, $f \star f$ is twice as wide as f.

b. Use your result from part (a) to find and sketch $f \star f$ where f is the unit pulse that starts at $t = 0$.

c. An alternative method for the calculation in part (b) is to differentiate f to get f', compute $f' \star f'$ by integration, and use that result to compute $f \star f$. So: What is f'? What is $f' \star f'$? And how do you go from $f' \star f'$ to $f \star f$? Sketch f' and $f' \star f'$.

5. Approximating an exponential decay

a. You can approximate the impulse response e^{-t} using a triangle whose vertical side matches the discontinuity in e^{-t}, whose base is along the t axis, and whose area matches the area of the impulse response. Convolve this triangle with itself using integration and compare the result to the convolution of e^{-t} with itself (which is computed in the Recitation 17 notes).



b. Write Octave/Matlab/Python code to compute the convolution of two causal, discrete-time signals. Test your code (and your answer to part (a)!) by asking it to do the convolution in part (a).

Homework 10

*Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 14 November 2007**.*

Warmups

1. **Impulse responses from system functions**
   For the system described by the system function

   $$H(s) = \frac{1 + 2s}{1 + 4s + 3s^2}$$

   how many possible impulse responses $h(t)$ does it have?

   Find the causal $h(t)$.

2. **Laplace transform**
   By integration find the Laplace transform of $h(t) = te^{-t}$ (for $t \geq 0$).

   Once you find it: How else could you have found that answer?

3. Z **transform**
   Find the z-transform of $h[n] = n^2$ (for $n \geq 0$).

Problems

4. **A famous sum**
   The most famous sum in mathematics is

   $$S = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

   first evaluated by the great Euler.

   A 6.003 way to find the sum is to use Parseval's theorem. It connects the time representation of a function with its Fourier representation. The relation is:

   $$\sum_{k=-\infty}^{\infty} |f_k|^2 = f \cdot f,$$

   where f_k is the k^{th} Fourier coefficient, f is the function $f(t)$, and $f \cdot f$ is the dot product as defined in Recitation 18.

   The left side is a dot product of the weight vector with itself, so it is the squared length of f in the Fourier representation. The right side is the dot product of the time representation with itself, so it is the squared length of f in the time representation. Therefore, Parseval's theorem says that lengths are preserved when changing to the Fourier representation.

   So *use Parseval's theorem to find the sum S.* [Many possibilities work.]

5. Overshoot

a. What function $f(t)$ has the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}?$$

You can evaluate the sum analytically or numerically. Either way, guess a closed form for $f(t)$ and then sketch it.

b. Confirm your conjecture for $f(t)$ by integrating $f(t)$ appropriately to find the Fourier coefficients f_n (the weights for the complex exponentials). What happens to the cosine terms?

c. Define the partial sum

$$f_N(t) = \sum_{n=1}^{N} \frac{\sin nt}{n},$$

Plot some $f_N(t)$'s. By what fraction does $f_N(t)$ overshoot $f(t)$ at worst? Does that fraction tend to zero or to a finite value as $N \to \infty$? If it is a finite value, estimate it.

d. Now define the average of the partial sums:

$$F_N(t) = \frac{f_1(t) + f_2(t) + f_3(t) + \cdots + f_N(t)}{N}$$

Plot some $F_N(t)$'s. Compare your plots with those of $f_N(t)$ that you made in the previous part, and qualitatively explain any differences.

6. Find Smiley

Let $x[n]$ represent the sequence $1, -1, -1, 1, -1, 1, 1, 1, -1, 1, -1, 1$. This sequence has a single occurrence of the pattern -1, -1, 1 (starting at $n = 1$ and ending at $n = 3$). One method to automatically locate particular patterns of this type is called "matched filtering." Let $p[n]$ represent the pattern of interest flipped about $n = 0$. Then instances of the pattern can be found by finding the times when $y[n] = (p * x)[n]$ is maximized.

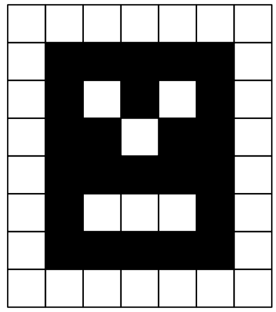
a. Determine a matched filter $p[n]$ that will find occurrences of the sequence: $-1, -1, 1$. Design $p[n]$ so that $(p * x)[n]$ has maxima at points that are centered on the desired pattern, i.e., at $n = 2$ for the sequence above.

b. Write a computer program (see the Appendix for programming hints) to find occurances of -1, -1, 1 within a sequence of 50 randomly generated 1's and -1's. Show (1) a typical random sequence, (2) your code to find the pattern -1, -1, 1, and (3) the answer produced by your code.

The same approach can be used to find patterns in pictures, where the signals are now two-dimensional, and where two-dimensional convolution is defined by

$$y[n, m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} p[k, l] x[n-k, m-l].$$

a. Download the file called **findsmiley.bmp** from the 6.003 web site. The following smiley face appears somewhere in **findsmiley**.



The rows and columns of boxes in this smiley face represent single rows and columns in the **findsmiley** array. White corresponds to a positive array element and black corresponds to a negative array element. Find the row and column of **findsmiley** that corresponds to smiley's nose.

b. One advantage of the matched filter method is that it works even when the signal contains some noise. Make a noisy version of **findsmiley** by adding gaussian distributed noise to the image. Try gaussians with different standard deviations. Determine the largest value of the standard deviation for which your code still finds smiley.

7. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

Appendix: numerical convolution

Matlab/Octave hints

The **conv()** function will will convolve two vectors; the **conv2()** function will convolve two arrays.

You can generate a random sequence of 1's and −1's by first generating a random sequence of n numbers using **randn(1,n)** and then passing that sequence through the **sign** function.

You can find the times when a sequence **y** achieves it's maximum value using the **find(y==max(y))** function.

Try the following exercise to figure out how **conv2()** works.

```
test=[0 0 0 0 0;
      0 0 0 0 0;
      0 0 1 0 0;
      0 0 0 0 0;
      0 0 0 0 0]
figure(1); imagesc(conv2([1 1 1],test)); colormap('gray');
figure(2); imagesc(conv2([1;1;1],test)); colormap('gray');
figure(3); imagesc(conv2([1 0 0;0 1 0;0 1 1],test)); colormap('gray');
```

Use the following command to read **findsmiley.bmp** into the Matlab array **findsmiley**.

```
findsmiley=(double(imread('findsmiley.bmp'))-126.)/128.;
```

One can make a noisy version of **findsmiley** using

```
amplitude = 0.1;
noisy = findsmiley+amplitude*randn(1000,1000);
```

Homework 11

Do all of the following problems, including the question about hours spent on the problem set. Due in recitation on ***Wednesday, 21 November 2007****.*

Warmup problems

1. Practice with transforms

Find the Fourier transforms of these functions of time. For each one, sketch $|f(\omega)|$.

a. $\delta(t)$

b. 1

c. $\sin t$

d. $\cos t$

e. $1 + \cos 2t$

f. $\cos^2 t$

g. e^{-t} (for $t \geq 0$)

It is often easier to try $f(\omega)$ candidates until you find one whose inverse transform is the desired $f(t)$.

2. Practice with inverse transforms

Find and sketch the inverse Fourier transforms of these functions of frequency.

a. $\delta(\omega)$

b. $\delta(\omega - 2) + \delta(\omega + 2)$

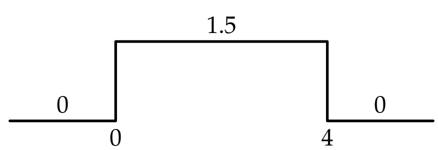
c. 1

3. Multiplication of areas

a. Use Fourier transforms to show that

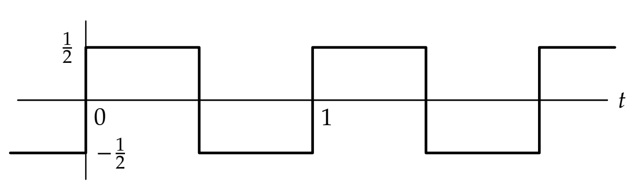
$$\text{Area of } \{f \star g\} = \{\text{Area of } f\} \times \{\text{Area of } g\}.$$

b. Sketch the convolution of this pulse with itself and confirm the area property in part (a):



4. Periodic signal

Find the Fourier transform of the square wave analyzed in Lecture 18:



5. Discrete sinusoids

Consider two DT signals:

$$x_1[n] = \cos\left(\frac{3\pi n}{4}\right) \text{ and}$$
$$x_2[n] = \cos\left(\frac{5\pi n}{4}\right).$$

How many of the following statements are true?

1. $x_1[n]$ is periodic with a period of 8.
2. $x_2[n]$ is periodic with a period of 8.
3. $x_1[n] = x_2[n]$.

Briefly explain.

Medium problems

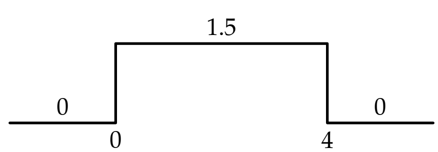
6. Delay

a. What is the Fourier transform of $\mathcal{R}f$ in terms of $f(\omega)$, the Fourier transform of $f(t)$? The $\mathcal{R}$ operator is our old friend that right shifts by one time unit (and it works for CT as well as DT signals).

b. Sketch the inverse transform of

$$f(\omega) = \frac{e^{10j\omega}}{1 - j\omega}.$$

c. Find the transform of this pulse:



Hint: The Lecture 19 notes give the transform of a similar signal.

7. Two-sided exponential decay

a. Sketch the function

$$f(t) = e^{-|t|},$$

where t ranges from $-\infty$ to ∞.

b. Show that its Fourier transform is

$$f(\omega) = \frac{2}{1 + \omega^2}$$

and sketch $f(\omega)$.

c. Find the system function $H(s)$ for the system with $f(t)$ as its impulse response. Mark its poles on a pole–zero diagram, and give the region of convergence.

d. Without doing any more integrals, sketch the Fourier transform of $e^{-|t|/2}$, giving scales on the ω and $f(\omega)$ axes.

Harder problems

8. Filtered impulse train

You send a periodic train of delta functions (with unit period) into an RC filter with $\tau = 1$.

a. What is the DC value (i.e. the average value) of the output signal?

b. Sketch the output signal, and mark the DC value as a horizontal line.

c. What is the peak value of the output signal?

Feel free to simulate, to build a circuit, to work out the quantities analytically, ... Ideally you would use more than one method and check that they agree.

9. Smoothing by convolution

The function $f(t) = e^{-|t|}$ has a discontinuous slope at $t = 0$, which is a rarity in physical systems. To smooth the discontinuity, one can convolve f with itself. The resulting function is a reasonable approximation to the impulse response of a lens, which is why we are asking you to study it in this question.

a. Find and sketch $f \star f$.

b. Find the function $f(t)$ whose Fourier transform is

$$f(\omega) = \left(\frac{2}{1+\omega^2}\right)^2$$

In other words, find the inverse Fourier transform of $f(\omega)$.

There are at least three methods, in order of increasing difficulty: convolution, partial fractions, and contour integration of the inversion formula. Use your *two* favorite methods and confirm that they give the same $f(t)$. If you use partial fractions, you might find the Recitation 11 notes useful for tips on maintaining algebra hygiene.

10. Hours

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Homework 12

Do all of the following problems, including the question about hours spent on the problem set. Due in recitation on ***Wednesday, 28 November 2007****.*

Warmup problems

1. Discrete-time Fourier transforms

Find and sketch the discrete-time Fourier transforms of these signals:

a. the impulse (a.k.a. unit sample) $\delta[n]$

b. $e^{-|n|}$ for all n

c. $n3^{-n}$ for $n \geq 0$

2. Integrals with delta functions

Delta functions often frighten people, especially when they live inside an integral. The cure for (some) fears is familiarity. In this problem you investigate how the delta function behaves inside an integral.

Define the function $\delta_\epsilon(t - t_0)$ as a rectangular pulse of width ϵ, area 1, and centered at t_0. As $\epsilon \to 0$, the function δ_ϵ becomes a delta function.

a. On one graph, sketch $\delta_\epsilon(t - 1)$ for $\epsilon = 0.2$ and $\epsilon = 0.1$.

b. On another graph, sketch $\delta_\epsilon(t + 3)$ for $\epsilon = 0.25$.

c. Let $f(t) = 5 + t$. Sketch $f(t)\delta_{0.2}(t)$. On another graph, sketch $f(t)\delta_{0.2}(t - 2)$.

d. Using the sketches (i.e. without doing integrals), find

$$\int_{-\infty}^{\infty} f(t)\delta_{0.2}(t)\,dt$$

and

$$\int_{-\infty}^{\infty} f(t)\delta_{0.2}(t-2)\,dt.$$

e. Therefore, what are

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t-2)\,dt?$$

Medium problems

3. Mystery convolution
Let $f(t)$ be this single triangle:

$$f(t) = \begin{cases} 2t & \text{for } 0 \le t \le 1/2; \\ 2 - 2t & \text{for } 1/2 \le t \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

a. Sketch $f(t)$.

b. Sketch a function $g(t)$ for which $f \star g = 1$.

c. What are the Fourier transforms of f and g? Use their transforms to explain why $f \star g = 1$.

4. Sampling CT sinusoids
Consider 3 CT signals:

$$x_1(t) = \cos(3000t)\,,$$
$$x_2(t) = \cos(4000t)\,, \text{and}$$
$$x_3(t) = \cos(5000t)\,.$$

Each of these is sampled to generate three DT signals:

$$x_1[n] = x_1(nT)\,,$$
$$x_2[n] = x_2(nT)\,,\ \text{and}$$
$$x_3[n] = x_3(nT)\,,$$

where $T = 0.001$. Which of the DT signals has the highest DT frequency? Which of the DT signals has the lowest DT frequency?

5. Fourier representations of the filtered impulse train
Problem 8 of HW11 asked about feeding a unit-period impulse train (a comb) into an RC circuit with $\tau = 1$. The output signal $y(t)$ is, like the input signal, periodic with period $T = 1$.

a. Find an expression for $y(t)$. *Hint:* Each period is an exponential decay.

b. Find the Fourier transform of $y(t)$. *Hint:* What is the Fourier transform of the impulse response of the RC circuit?

c. From the level of smoothness of $y(t)$, how should its Fourier series coefficients roll off for large k?

d. Find the Fourier coefficients. Does their large-k behavior agree with your prediction in the previous part?

e. Check your answer to the previous part by sampling $y(t)$ with a small spacing and feeding one period to an FFT routine (for example, `fft` in Python or Octave/Matlab).

Harder problems

6. Fourier representations of the pulse

In this problem you connect the four Fourier representations, as we did for you in Lecture 21 and Recitation 21. Lecture 21 used the triangle as the canonical function. Recitation 21 used a triangle convolved with a pulse. You get to use a pulse:

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 1; \\ 0 & \text{otherwise.} \end{cases}$$

a. Find $f(\omega)$, the continuous-time Fourier transform of $f(t)$. State Parseval's theorem for the Fourier transform, and check that it works by applying it to $f(t)$ and $f(\omega)$.

b. Sketch $f_{\text{p}}(t)$, a periodic version of $f(t)$ with period $T = 4$. Find the Fourier transform of $f_{\text{p}}(t)$ and compare it to the Fourier-series coefficients f_k for $f_{\text{p}}(t)$.

State Parseval's theorem for Fourier series.

Applying Parseval's theorem to $f_{\text{p}}(t)$ and f_k and thereby do an interesting sum.

c. Make $f_{\text{s}}(t)$ by sampling $f(t)$ with a sampling interval $\Delta t = 2/3$. [So $f_{\text{s}}(t)$ should be composed of delta functions.] Give the corresponding discrete-time signal $f[n]$.

Find the Fourier transform of $f_{\text{s}}(t)$ and compare it to the discrete-time Fourier transform of $f[n]$.

7. Hours

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Homework 13

Do all of the following problems, including the question about hours spent on the problem set. Due in recitation on ***Wednesday, 05 December 2007****.*

Warmup problems

1. Transform of a Gaussian

What is the Laplace transform of $x(t) = e^{-t^2/2}$? From the Laplace transform, find and sketch the Fourier transform of $x(t)$. Is $x(t)$ bandlimited? Is $x(t)$ infinitely smooth (i.e. does it have derivatives of all orders)?

2. Find the signal

You sample a continuous-time signal $x(t)$ with interval $T = 1$ and find that the samples are periodic. Here is one period:

$$x[n] = \ldots, 2, 1, 0, 1, \ldots,$$

Give two signals $x(t)$ that would produce those samples.

If you know that $x(t)$ contains no frequencies above $\omega = 2$, is $x(t)$ uniquely determined? If so, give $x(t)$. If not, give two possibilities for $x(t)$.

Medium problems

3. Find the sampling rates

You sample the continuous-time signal $x(t) = \cos 10t$ and get a periodic discrete-time sequence. Here is one period:

$$\ldots, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}, \ldots$$

What is the shortest sampling interval T that produces this sequence? What is the next-shortest sampling interval T that produces this sequence?

4. Sampling just a bit too slowly

Use Fourier-transform diagrams to show how sampling $\cos(t - \pi/2)$ using the sampling interval $T = \pi$ destroys information.

5. Bandlimited signals

Sketch each of the following signals. Which are bandlimited? For any signals that are bandlimited, give the band (the region where the frequency representation is nonzero).

a. $\operatorname{sinc} 3t$

b. $e^{-3|t|}$

c. a single triangle: $1 - |t|$ for $|t| \leq 1$ and 0 elsewhere.

d. $(\operatorname{sinc} t)^2$

e. a sawtooth wave: period 2π with $f(t) = t$ for $-\pi < t < \pi$.

f. (hard, optional, not graded!) $(\cos t)/t - (\sin t)/t^2$

6. Finding interpolating polynomials via impulse responses

In this problem you construct an interpolating polynomial from its samples.

By computing the unit-sample response of the interpolation procedure, you can find the interpolating polynomial. We'll do it for you for finding the first-degree polynomial from two samples $x[0]$ and $x[1]$ with sampling interval T.

a. For $T = 1$, the linear polynomial that satisfies $x[0] = 1$ and $x[1] = 0$ is $1 - t$. The linear polynomial that satisfies $x[0] = 0$ and $x[1] = 1$ is t. Therefore the linear polynomial that satisfies $x[0] = a_0$ and $x[1] = a_1$ is

$$a_0 \times (1 - t) + a_1 \times t.$$

Use this result to find the polynomial for an arbitrary sampling interval T.

b. Now you get to use this procedure for quadratic interpolation. First assume $T = 1$. Find the quadratic when $x[-1] = 1$, $x[0] = 0$, and $x[1] = 0$. Then find the quadratic when $x[-1] = 0$, $x[0] = 1$, and $x[1] = 0$. Then find the quadratic when $x[-1] = 0$, $x[0] = 0$, and $x[1] = 1$. Then find the general result if you are given samples $x[-1]$, $x[0]$, and $x[-1]$ with sampling interval T.

c. Sketch $x(t) = e^{-t^2/2}$. Sample $x(t)$ using the interval $T = 1$ from $t = -1 \ldots 1$ and use your result from the previous part to fit an interpolating quadratic through those three points. What is its maximum error in the range $t = -1 \ldots 1$?

Harder problems

7. Interpolation

How could a floating-point processor, or a math library routine, compute mathematical functions? You get to figure out a method. So, in your favorite programming language, write and demonstrate a function to compute $\sin t$ to 15 decimal places, for t between $-\pi$ and π. It can use table lookups (which you can preload with whatever what you want), the four basic operations of arithmetic, and modulo or integer-part.

There are lots of tradeoffs in the design. Do you want a simple algorithm with a huge table? Do you want a complex and subtle algorithm with a small table? There isn't a right answer, so make your own choices and explain your reasons.

Hint: You might want to use the `polyfit` and `polyval` routines in Octave/Matlab or in Python (in the `scipy` module).

8. Hours

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Homework 14

Do all of the following problems. Not graded or collected!

Warmup problems

1. Modulation

What impulse train would shift speech to 1 MHz (a typical AM frequency)?

Why don't we modulate radio signals by multiplying by this impulse train (instead of by the sine wave)?

Medium problems

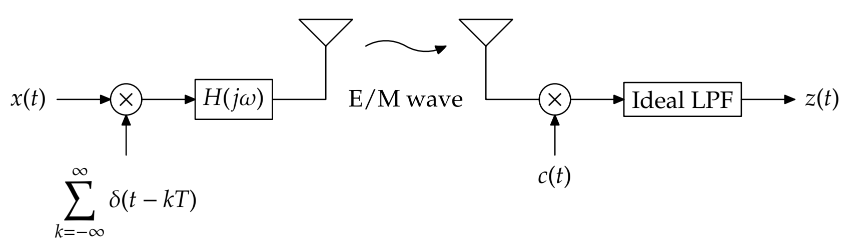
2. How many RC filters

Use a cascade of n identical first-order low-pass filters to implement an antialiasing filter for audio signals. Assume that the sampling rate is 44 kHz, as it is for CDs, and that the corner frequencies of the filters are at 20 kHz. Determine the minimum n so that all frequency components that alias to frequencies between 0 and 20 kHz are attenuated by at least 80 dB (a factor of 10^4 in amplitude). What is the magnitude of the frequency response of the cascade at 20 kHz?

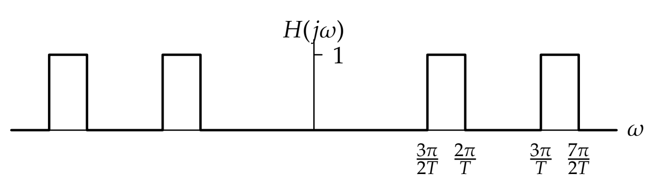
Harder problems

3. Scrambled Transmission

Speech is to be split into and transmitted over multiple frequency bands as follows



where T is a positive, real-valued constant and $H(j\omega)$ is defined below.



a. Let $c(t) = \cos(2\pi t/T)$. If $x(t)$ is bandlimited, is it possible to choose parameters for the ideal lowpass filter so that you can reconstruct $x(t)$ (i.e. make $z(t) = x(t)$)? If so, what is the maximum frequency that can be in $X(j\omega)$? If not, briefly explain why not.

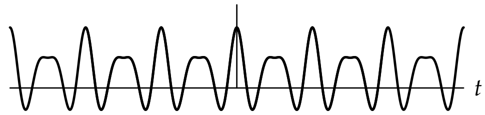
b. Repeat part (a) for $c(t) = \cos(4\pi t/T)$.

c. Repeat part (a) for $c(t) = \cos(2\pi t/T) + \cos(4\pi t/T)$.

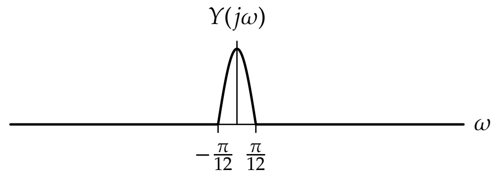
4. Fourier Transforms

Consider two continuous-time signals: $x(t)$ shown below

$$x(t) = 1 + 2\cos\left(\frac{\pi}{3}t\right) + \cos\left(\frac{\pi}{2}t\right)$$



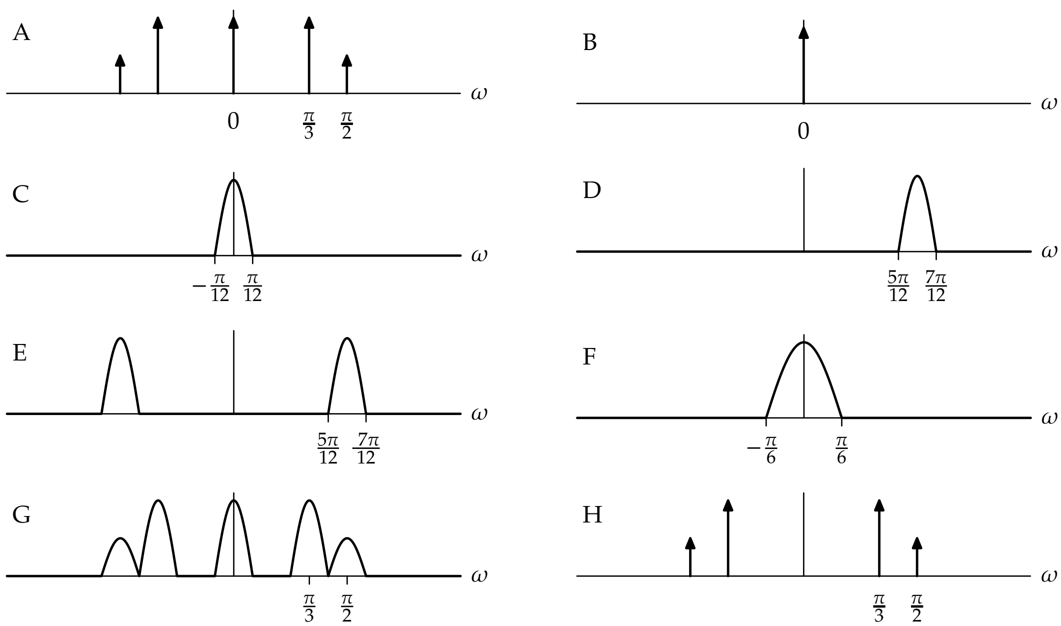
and $y(t)$, whose Fourier transform is shown below. $Y(j\omega)$ is real-valued for all ω.



a. Is it possible to design a linear, time-invariant system whose input is $x(t)$ and whose output is $y(t)$?

b. Is it possible to design a linear, time-invariant system whose input is $y(t)$ and whose output is $x(t)$?

c. Which if any of the following plots shows the magnitude of the Fourier transform of $z_1(t) = x(t)$?



d. Repeat part (c) for $z_2(t) = y(t - \frac{\pi}{2})$.

e. Repeat part (c) for $z_3(t) = x(t)y(t)$.

f. Repeat part (c) for $z_4(t) = x(t) * y(t)$.

g. Sketch the angle of the Fourier transform of $z_5 = y(t - \frac{\pi}{2})$.

h. Sketch the angle of the Fourier transform of $z_6 = y(t)e^{j\frac{\pi}{2}t}$.

i. Sketch the angle of the Fourier transform of $z_7 = \frac{d}{dt}y(t)$.

5. Hours

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