6.003 (Fall 2007): Solution sets

Homework 1

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 12 September 2007**.

Warmups

1. **Solving differential equations**

   Solve the following differential equation

   $$y(t) + 3\frac{dy(t)}{dt} + 2\frac{d^2y(t)}{dt^2} = 1$$

   for $t \geq 0$ assuming the initial conditions

   $$y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$$

   and express the solution in closed form.

   [Hint: assume the homogeneous solution has the form $Ae^{s_1t} + Be^{s_2t}$.]

2. **Solving difference equations**

   Solve the following difference equation

   $$8y[n] - 6y[n-1] + y[n-2] = 1$$

   for $n \geq 0$ assuming the initial conditions

   $$y[0] = y[-1] = 0$$

   and express the solution in closed form.

   [Hint: Assume that the homogeneous solution has the form $Az_1^n + Bz_2^n$.]

3. **O&W Problem 1.54 (page 73))**

Problems

4. **Choosing a bank**

   Consider two banks. Bank #1 offers a 6% annual interest rate, but charges a $1 service charge each year, including the year when the account was opened. Bank #2 offers a 5% annual interest rate, and has no annual service charge. Let $x_i[n]$ represent the amount of money you deposit in bank i during year n and $y_i[n]$ represent your balance in bank i. Assume that deposits during year n are credited to the balance in year n but earn no interest until year $n+1$.

   a. Use difference equations to express the relation between deposits and balances for each bank.

   b. Assume that you deposit $100 in each bank in the year 2007 and make no further deposits. Solve your difference equations in part a numerically (using Matlab, Octave, or Python) to determine your balance in each bank during years 0 through 25. Make a plot of these

balances. Which bank has the larger balance in the year 2011? Which bank has the larger balance in the year 2031? [Hint: See the Appendix for help with programming.]

5. **Coupled Oscillator**

   The following equations characterize a "coupled oscillator":

   $$\frac{dy_1(t)}{dt} = y_2(t)$$

   and

   $$\frac{dy_2(t)}{dt} = -y_1(t).$$

   a. Use the forward-Euler method to develop a set of difference equations that approximate this system of differential equations.

   b. Solve the difference equations numerically (using Matlab, Octave, or Python) and plot the results. What happens if you choose a step size that is too big? How big is too big?

6. **Making sherry**

   Sherry is a wine that is typically aged in a series of barrels called a solera. Each year, a portion (assume half) of the wine in the last barrel is extracted, bottled, and sold. This barrel is then refilled with wine from the next-to-last barrel. The next-to-last barrel is then refilled with wine from the previous barrel, and the process is repeated until the first barrel is refilled with newly crushed juice from this year's grape harvest.

   a. To understand the way that the solera system mixes and ages the wine, consider a thought experiment in which $x[n]$ units of a tracer substance (such as deuterium) are added to the new crushed juice in year n. Let $a[n]$, $b[n]$, and $c[n]$ represent the amount of tracer found in the first, second, and third barrels at the end of year n (i.e., after the wine to sell in year n has been drawn out and the crushed joice from year n has been added). Develop difference equations to determine $a[n]$, $b[n]$, and $c[n]$ for $n \geq 0$.

   b. Assume that one unit of tracer is added in year 0 and no tracer is added in any other year. Determine the year (or years) during which the amount of tracer in the third barrel is greatest. [Feel free to use a pencil, calculator, or computer to answer this question.]

   c. Assume that one unit of tracer is added each year. Determine the steady-state value of tracer in the first barrel, i.e., determine

   $$a_{ss} = \lim_{n \to \infty} a[n]$$

   [again, pencils, calculators, or computers are welcome].

Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment.

Feel free also to comment on these problems.

Appendix: forward-Euler code

Suppose that we want to solve the following differential equation:

$$y(t) + \frac{dy(t)}{dt} = 1$$

starting with an initial condition $y(0) = 0$. We can use the forward-Euler method to generate the following approximation:

$$y[n + 1] = T + (1 - T)y[n]$$

where T is the step size and $y[0] = 0$. Here is Matlab/Octave and Python code that implements this approximation. Octave is a free-software linear-algebra, with a syntax very similar to Matlab, and it is available for most platforms. See **www.octave.org**.

Example Matlab/Octave code

The following Matlab/Octave code evaluates the analytic solution to the differential equation (i.e., $y(t) = 1 - e^{-t}$ for $t > 0$) as well as the numerical approximation. It then plots the two results.

```
t(1) = 0;
y(1) = 0;
a(1) = 1-exp(0.);
T = .1;
for n = 1:99
   t(n+1) = t(n)+T;
   y(n+1) = T+(1-T)*y(n);
   a(n+1) = 1-exp(-t(n+1));
end
plot(t,y,'r',t,a,'b');
```

Example Python code

```
from    scipy import *
import pylab as      p

T = 0.1                                 # time step
t = arange(0, 10.0001, T)
y = zeros(shape(t))                     # Euler-integrated solution
a = 1-exp(-t)                           # discretized analytic solution
for i in range(1,len(t)):
    y[i] = T + (1-T)*y[i-1]
p.plot(t, y, 'r', t, a, 'b')
p.show()
```

Homework 2

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 19 September 2007**.

Warmups

1. **Finding unit sample (impulse) responses**

   Determine the unit sample (impulse) responses of the systems represented by the following system functionals.

   a. $(1 - \mathcal{R})^{-2}$

   > This system can be written as the cascade of two accumulators:
   > $$\frac{Y}{X} = \frac{1}{(1 - \mathcal{R})^2} = \left(\frac{1}{1 - \mathcal{R}}\right)\left(\frac{1}{1 - \mathcal{R}}\right).$$
   >
   > In response to a unit impulse, the first accumulator will generate a unit step function, i.e., the output will be one for all time indices $n \geq 0$. In response to the unit step, the second accumulator will count upwards from 1 so the output $y[n]$ will be $n + 1$ for $n \geq 0$.
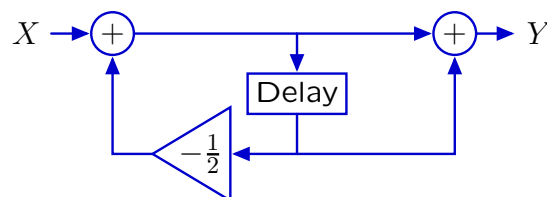
   b. $(1 + \mathcal{R})^3/8$

   > $$\frac{Y}{X} = \frac{(1 + \mathcal{R})^3}{8} = \frac{(1 + 2\mathcal{R} + \mathcal{R}^2)(1 + \mathcal{R})}{8} = \frac{1 + 3\mathcal{R} + 3\mathcal{R}^2 + \mathcal{R}^3}{8}.$$
   >
   > The output will be the sequence $1/8, 3/8, 3/8, 1/8, 0, 0, 0, \ldots$.

2. **Working with block diagrams**

   Consider the system represented by the following block diagram.

   a. Find a difference equation that characterizes this system.

   > Let W represent the input of the delay element. Then $W = X - 0.5\mathcal{R}W$ and $Y = W + \mathcal{R}W$. The overall system functional is then
   > $$\frac{Y}{X} = \frac{Y}{W}\frac{W}{X} = \frac{1 + \mathcal{R}}{1 + 0.5\mathcal{R}}.$$

   b. Determine the unit sample response of this system.

First compute

$$W = \frac{1}{1 + 0.5\mathcal{R}} X = \left(1 - \frac{1}{2}\mathcal{R} + \frac{1}{4}\mathcal{R}^2 - \frac{1}{8}\mathcal{R}^4 + \frac{1}{16}\mathcal{R}^4 - \cdots\right) X.$$

Since X is the unit sample, W is the sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \cdots$. Then $Y = (1 + \mathcal{R})W$ so we add W to a right-shifted version of W:

$$
\begin{array}{llllllll}
W: & 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{16} & \cdots \\
\mathcal{R}W: & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \cdots \\
Y: & 1 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \cdots
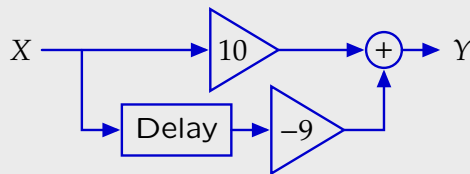\end{array}
$$

3. **Finding a system**

   a. Determine a system whose output is $10, 1, 1, 1, 1, \ldots$ when the input is $1, 1, 1, 1, 1, \ldots$. Determine the difference equation and block diagram representations for this system.

   Notice that $Y = 10X - 9\mathcal{R}X$. This relation suggests the following difference equation

   $$y[n] = 10x[n] - 9x[n - 1]$$

   and block diagram

   b. Determine a system whose output is $1, 1, 1, 1, 1, \ldots$ when the input is $10, 1, 1, 1, 1, \ldots$. Determine the difference equation and block diagram reprsentations for this system.
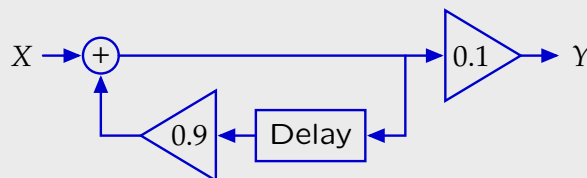
   The difference equation for the inverse relation can be obtained by interchanging y and x in the previous difference equation to get

   $$x[n] = 10y[n] - 9y[n - 1].$$

   So

   $$y[n] = \frac{9y[n - 1] + x[n]}{10},$$

   which has this block diagram

   c. Compare the difference equations in parts a and b. Compare the block diagrams in parts a and b.

> The difference equations for parts a and b have exactly the same structure. The only differ-
> ence is that the roles of x and y are reversed. The block diagrams have similar parts, but the
> topologies are completely different. The first is acyclic and the second is cyclic.

4. **Finding modes**

   The following block diagram describes the relation between two discrete-time signals: $x[n]$
   and $y[n]$.
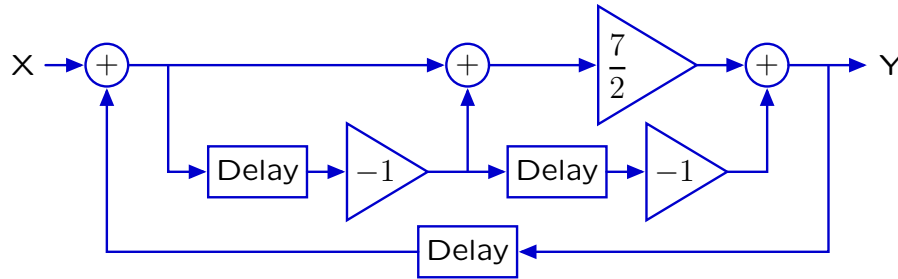
   a. How many different modes does this system have?

   > Let W represent the output of the leftmost adder so that
   >
   > $$W = X + \mathcal{R}Y.$$
   >
   > The output of the middle adder is then $(1 - \mathcal{R})W$. The output of the rightmost adder is
   >
   > $$Y = \frac{7}{2}(1 - \mathcal{R})W + \mathcal{R}^2 W.$$
   >
   > Substituting the former equation for W into the latter yields
   >
   > $$Y = (\frac{7}{2} - \frac{7}{2}\mathcal{R} + \mathcal{R}^2)(X + \mathcal{R}Y).$$
   >
   > Solving for the functional, we find
   >
   > $$\frac{Y}{X} = \frac{\frac{7}{2} - \frac{7}{2}\mathcal{R} + \mathcal{R}^2}{1 - \frac{7}{2}\mathcal{R} + \frac{7}{2}\mathcal{R}^2 - \mathcal{R}^3}.$$
   >
   > The denominator of this functional can be factored to give
   >
   > $$\frac{Y}{X} = \frac{\frac{7}{2} - \frac{7}{2}\mathcal{R} + \mathcal{R}^2}{(1 - \frac{1}{2}\mathcal{R})(1 - \mathcal{R})(1 - 2\mathcal{R})}.$$
   >
   > The system has three modes, one corresponding to each factor in the denominator.

   b. Determine closed-form expressions for each mode. [Do not try to determine the ampli-
      tudes of the modes. The amplitudes depend on the input X, which is not specified.]

      > The roots of the denominator (above) are $\frac{1}{2}$, 1, and 2. Therefore the three modes have the forms
      > $\frac{1}{2}^n$, 1^n, and 2^n for $n \geq 0$.

Problems

5. **Drug dosing**

When you start a medicine, the doctor often says 'Take double the usual dosage for the first dose, then follow the schedule.' In this problem you study this advice to take a so-called loading dose.

A simple model of the body is a tank of blood from which drug exits at a rate proportional to its concentration and into which drug doses instantly arrive. Assume that the patient takes the drug once every 8 hours, and that the concentration of drug in the blood is measured shortly after taking each dose. Draw the block diagram and give the system functional of a discrete-time system with this behavior. Choose the parameter(s) of the system so that, in the absence of new drug, two-thirds of the drug is filtered out (say, by the kidneys) every 8 hours.

Ideally, the level of drug in the blood would instantly go to the desired level and remain steady. Assume that you take the medicine every 8 hours. What dosage schedule will produce the ideal behavior? Interpret your answer in terms of loading doses, whether for or against the idea.
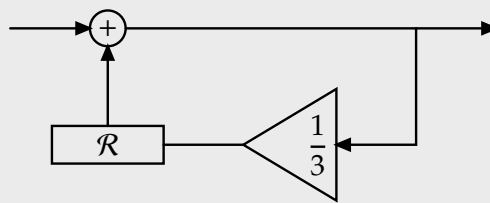

One-third of the drug remains in the bloodstream after 8 hours (one dosing interval). Therefore

$$y[n] = \frac{1}{3}y[n-1] + x[n]$$

and the corresponding functional is

$$\frac{Y}{X} = \frac{1}{1 - \frac{1}{3}\mathcal{R}}$$

The block diagram is

We want an input $x[n]$ that produces an output $y[n] = 1$ for $n \geq 0$. Notice that we can just run the difference equation backwards using the desired output:

$$x[n] = y[n] - \frac{1}{3}y[n-1]$$

$$x[0] = y[0] - \frac{1}{3}y[-1] = 1 - 0 = 1 \quad \text{(no drug in body at } n = -1\text{)}$$

$$x[1] = y[1] - \frac{1}{3}y[0] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$x[2] = y[2] - \frac{1}{3}y[1] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$x[3] = y[3] - \frac{1}{3}y[2] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\cdots$$

Thus the desired dosing is to take one unit initially and $\frac{2}{3}$ of a unit for all subsequent doses.

This schedule is an example of using a loading dose. The first dose is somewhat higher – here 1.5 times higher – than the steady dosage, in order to jump-start the drug's level to its steady-state value.

This discrete-time analysis ignores the behavior between sampling intervals, i.e. between the 8-hour dosage intervals. We will return to this problem when we study continuous-time systems to see whether or how much the drug's level in the blood overshoots the ideal of a step function.

6. **Experimental mathematics to debug a black box**

You would like to characterize a system but all you know is its unit-sample (impulse) response, which is available as n, $y[n]$ pairs in the Appendix, and is available electronically on our course website as `black_box.tsv`. Use peeling away and educated guessing (see the R04 notes) to find the modes and their amplitudes, and therefore find a closed form for the output signal. What is the system functional and the corresponding difference equation? [Note: Feel free to use a computer, graphing calculator, or paper and pencil.]

The following Python code reads the values from "black_box.tsv" and computes successive ratios $y[n+1]/y[n]$.

```
from scipy import *

y = zeros(100)
for line in open("black_box.tsv","r"):
    (n,fn) = line.split()
    y[int(n)] = float(fn)
y = y[0:int(n)+1]                        # chop of unused part of y
print y[1:] / y[:-1]                     # y[n+1]/y[n]
```

The ratios converge to 3 so subtract out 3^n. Continue the preceding Python code by adding these lines:

```
N = arange(len(y))                       # N = [0, 1, 2, ... ,max_n]
print y - 3.0**N                         # try taking out 3^n
```

The numbers still grow quickly. We could be off by a multiplicative constant, since the 3^n mode could have any amplitude. To find the amplitude, divide out 3^n:

```
print y / 3.0**N                         # try dividing out 3^n
```

The result is 3. So subtract out $3 \cdot 3^n = 3^{n+1}$ and compute the successive ratios of that signal:

```
z = y - 3.0**(N+1)                       # take out first mode
print z
print z[1:] / z[:-1]                      # z[n+1]/z[n]
```

Now we get a ratio of 2. The first few samples of Z are -2, -4, and -8. So $z[n] = -2^{n+1}$, which means that Y is given by

$$y[n] = 3^{n+1} - 2^{n+1}$$

The combination of partial fractions that generates these modes is

$$\frac{3}{1 - 3\mathcal{R}} - \frac{2}{1 - 2\mathcal{R}}.$$

Adding the fractions gives this form of the system functional

$$\frac{Y}{X} = \frac{1}{1 - 5\mathcal{R} + 6\mathcal{R}^2}.$$

The corresponding difference equation is

$$y[n] = 5y[n-1] - 6y[n-2] + x[n].$$

Let's check whether this difference equation generates the input data when X is the unit sample (impulse). Indeed, it gives the output signal $1, 5, 19, 65, \ldots$.

7. **Making sherry: redux**

   Reconsider the three-barrel system for "Making Sherry" in Homework #1 (Problem 5).

   a. Express the difference equations that describe the three-barrel solera system using system functionals. Sketch a block diagram for the three-barrel solera system.

   The difference equations for the three-barrel sherry system can be expressed in operator notion as

   $$Y = 0.5\mathcal{R}C$$

   $$(1 - 0.5\mathcal{R})C = 0.5\mathcal{R}B$$

   $$(1 - 0.5\mathcal{R})B = 0.5\mathcal{R}A$$

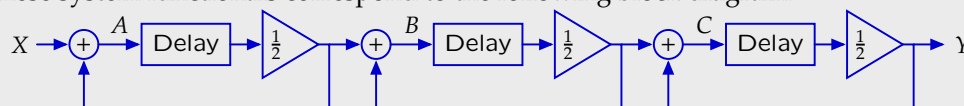 $$(1 - 0.5\mathcal{R})A = X$$

   Therefore

   $$\frac{Y}{C} = 0.5\mathcal{R}$$

   $$\frac{C}{B} = \frac{0.5\mathcal{R}}{1 - 0.5\mathcal{R}}$$

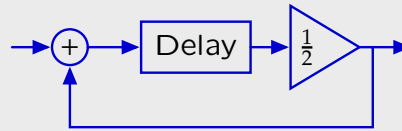 $$\frac{B}{A} = \frac{0.5\mathcal{R}}{1 - 0.5\mathcal{R}}$$

   $$\frac{A}{X} = \frac{1}{1 - 0.5\mathcal{R}}$$

   These system functionals correspond to the following block diagram.

   b. Rewrite the block diagram so that the main body is the cascade of three identical systems. Determine the system functional for this block diagram.

The system in part a is a cascade of three of the following systems.

The system functional for this block diagram

$$\frac{Y}{X} = \left(\frac{0.5\mathcal{R}}{1 - 0.5\mathcal{R}}\right)^3.$$

c. Determine the system functional for a ten-barrel solera system analogous to the system functional in part b.

The system functional for a ten-barrel solera system is

$$\frac{Y}{X} = \left(\frac{0.5\mathcal{R}}{1 - 0.5\mathcal{R}}\right)^{10}.$$

d. Assume that one unit of tracer is added to the freshly crushed grapes during year 0 and no tracer is added in any other year. Write a program to determine how much tracer is in the sherry bottled in year n for a B-barrel solera system, where B is a parameter to your program.

```
# compute impulse response of a many-barrel sherry-making system

from scipy import *
import pylab as p
from sys import argv

# use 20 barrels, unless overridden on the cmd line
try    : n = int(argv[1])
except : n = 20

# signals[k] is the input to barrel k+1.  In particular, signals[0] is
# the input signal.  signals[-1], which is the last element of the
# signals vector, is the output signal, which is the amount of tracer
# bottled.  At every time step the signals vector is updated.
signals       = zeros(n+1)            # +1 to include the input signal
signals[0]    = 1                     # 1 unit of tracer is input at time 0
bottledtracer = []                    # for storing output signal vs time

# indicies to slice the signals vector.  It's 0,0,1,2,3, ..., n-1.
indicies = [0] + range(n)
while len(bottledtracer) < 3*n:         # reaches a max before 3n years
    bottledtracer.append(signals[-1])   # save output sample for plotting
    signals = (signals + signals[indicies])/2.0
    signals[0] = 0 # no more tracer after year 0
```
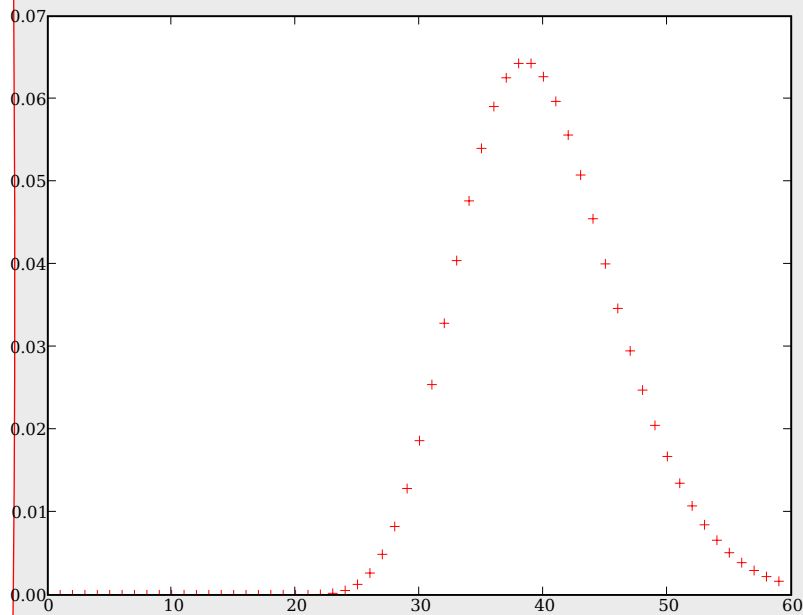
```
p.plot(bottledtracer,'r+')
p.show()
```

Generally, the amount of tracer that is bottled in year *n* first increases with *n* and then decreases. Demonstrate this trend by plotting results for a 20-barrel solera.

The following plot shows the result of running the previous program.

The output is zero for many years (why?) and is a maximum in years 38 and 39.

Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

Appendix

Data for problem 5 (also available as a .tsv file on the course website).

```
 0  1
 1  5
 2  19
 3  65
 4  211
 5  665
 6  2059
 7  6305
 8  19171
 9  58025
10  175099
11  527345
12  1586131
13  4766585
14  14316139
15  42981185
16  129009091
17  387158345
18  1161737179
19  3485735825
20  10458256051
21  31376865305
22  94134790219
23  282412759265
24  847255055011
25  2541798719465
26  7625463267259
27  22876524019505
28  68629840493971
29  205890058352825
30  617671248800299
31  1853015893884545
32  5559051976620931
33  16677164519797385
34  50031510739261339
```

Homework 3

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 26 September 2007**.

Warmups

1. **Complex numbers**

   Evaluate the following:

   a. i^i

   > Euler's formula says that $i = e^{i\pi/2}$, so
   >
   > $$i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}.$$

   b. $(1 - j\sqrt{3})^{12}$

   > The magnitude of $z = 1 - j\sqrt{3}$ is 2, and its phase angle is $-60°$. So the magnitude of z^{12} is $2^{12} = 4096$ and its phase is $-12 \times 60°$, which is 0. So $z^{12} = 4096$.

   c. $\mathrm{Re}\left\{e^{5j\theta}\right\}$ in terms of $\sin\theta$ and $\cos\theta$.

   > From Euler's formula,
   >
   > $$e^{5j\theta} = \cos 5\theta + j\sin 5\theta.$$
   >
   > Its real part is $\cos 5\theta$. But we need to write it in terms of $\sin\theta$ and $\cos\theta$. So we better start with
   >
   > $$e^{j\theta} = \cos\theta + j\sin\theta.$$
   >
   > So
   >
   > $$e^{5j\theta} = \left(e^{j\theta}\right)^5 = (\cos\theta + j\sin\theta.)^5.$$
   >
   > The real part of the expansion is the odd-numbered terms, since those have only even powers of j. In order, they are
   >
   > $$\cos^5\theta,$$
   > $$-10\cos^3\theta\sin^2\theta,$$
   > $$5\cos\theta\sin^4\theta.$$
   >
   > So the real part, which is also $\cos 5\theta$, is
   >
   > $$\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

2. **Characterizing modes**

Determine the number of modes that can be generated by each of the following systems, and characterize each mode as monotonic or oscillatory and as growing, decaying, or neither.

a.

$$\frac{Y}{X} = \frac{1 + \sqrt{2}\mathcal{R} + \mathcal{R}^2}{1 - \sqrt{2}\mathcal{R} + \mathcal{R}^2}.$$
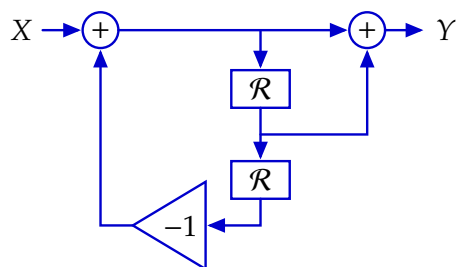
The denominator is second order in $\mathcal{R}$, so the system has two modes. The denominator factors as $(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})$. Since the product $p_1 p_2$ is 1 whereas the sum $p_1 + p_2$ is $\sqrt{2}$, both poles must be complex. Since their product is 1 and they are complex conjugates, the poles lie on the unit circle. Thus both poles are oscillatory and neither growing nor decaying.

b.
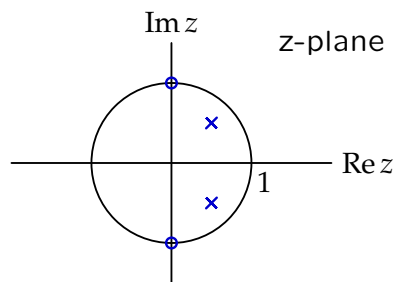
The system has two poles and therefore two modes. Both poles are inside the unit circle but off the positive real axis, so both are decaying and oscillatory.

c.

The first loop is a feedback loop with functional $1/(1 + \mathcal{R}^2)$. The second loop is a feedforward loop with functional $1 + \mathcal{R}$, so the system functional is $(1 + \mathcal{R})/(1 + \mathcal{R}^2)$. This second-order denominator produces two poles and hence two modes. The factored form is $(1 + j\mathcal{R})(1 - j\mathcal{R})$ so the poles are at $\pm j$, on the unit circle. The corresponding modes are oscillatory but neither growing nor decaying.

d. Which of parts (a)–(c) was the simplest? Briefly explain why.

> The pole–zero plot in part (b) was the simplest because one can tell at a glance what the behavior of the poles are, just by looking at their location relative to the unit circle and to the positive real axis.

3. **Lots of poles**

   a. Factor the system functional $(1 - \mathcal{R}^2)^{-2}$ into pole–zero form. Then give the partial-fractions decomposition, the impulse response of each term in the decomposition, and the sum of the impulse responses. Check your sum by feeding the impulse into a block diagram representing the original system functional.

   > Pole-zero form:
   >
   > $$(1 - \mathcal{R}^2)^{-2} = \frac{1}{(1 - \mathcal{R}^2)^2} = \frac{1}{(1 - \mathcal{R})(1 - \mathcal{R})(1 + \mathcal{R})(1 + \mathcal{R})}$$
   >
   > Partial fraction form:
   >
   > $$(1 - \mathcal{R}^2)^{-2} = \frac{1}{(1 - \mathcal{R}^2)^2} = \frac{0.25}{1 - \mathcal{R}} + \frac{0.25}{(1 - \mathcal{R})^2} + \frac{0.25}{1 + \mathcal{R}} + \frac{0.25}{(1 + \mathcal{R})^2}$$
   >
   > mode 0: $0.25(1)^n$
   > mode 1: $0.25(n + 1)(1)^n$
   > mode 2: $0.25(-1)^n$
   > mode 3: $0.25(n + 1)(-1)^n$
   > sum: $0.25(n + 2)(1 + (-1)^n)$
   >
   > Using block diagram. Impulse response of $\frac{1}{1-\mathcal{R}^2}$ is $1, 0, 1, 0, 1, 0, \dots$. Feed this into $\frac{1}{1-\mathcal{R}^2}$ to get $1, 0, 2, 0, 3, 0, \dots$. This is the same as the "sum" above.

   b. Factor the system functional $(1 + \mathcal{R}^2)^{-2}$ into pole–zero form. Then give the partial-fractions decomposition, the impulse response of each term in the decomposition, and the sum of the impulse responses. Check your sum by feeding the impulse into a block diagram representing the original system functional.

   > Pole-zero form:
   >
   > $$(1 + \mathcal{R}^2)^{-2} = \frac{1}{(1 + \mathcal{R}^2)^2} = \frac{1}{(1 + j\mathcal{R})(1 + j\mathcal{R})(1 - j\mathcal{R})(1 - j\mathcal{R})}$$
   >
   > Partial fraction form:
   >
   > $$(1 + \mathcal{R}^2)^{-2} = \frac{1}{(1 + \mathcal{R}^2)^2} = \frac{0.25}{1 + j\mathcal{R}} + \frac{0.25}{(1 + j\mathcal{R})^2} + \frac{0.25}{1 - j\mathcal{R}} + \frac{0.25}{(1 - j\mathcal{R})^2}$$
   >
   > mode 0: $0.25(-j)^n$
   > mode 1: $0.25(n + 1)(-j)^n$
   > mode 2: $0.25(j)^n$
   > mode 3: $0.25(n + 1)(j)^n$
   > sum: $0.5(n + 2)\cos(\pi n/2)$
   >
   > Using block diagram. Impulse response of $\frac{1}{1+\mathcal{R}^2}$ is $1, 0, -1, 0, 1, 0, -1, 0, \dots$. Feed this into $\frac{1}{1+\mathcal{R}^2}$ to get $1, 0, -2, 0, 3, 0, -4, 0, \dots$. This is the same as the "sum" above.

4. **Tricky quadratics (adapted from R. D. Middlebrook)**

   a. Use the quadratic equation and a pocket calculator, Python, or octave/Matlab to find the two roots of

   $$x^2 - 10^{10}x + 1 = 0.$$

   What went wrong?

   > Ask Python for the two roots with
   >
   > ```
   > from math import *
   > print (10**10-sqrt(10**20-4))/2, print (10**10+sqrt(10**20-4))/2
   > ```
   >
   > and you get 0 and 10^{10}. But 0 is not a root of the quadratic. What went wrong is that the root near zero depends, if you use the quadratic formula, on a small difference between large numbers. Here the large numbers are 10^{10} and $\sqrt{10^{20}-4}$, which are almost equal. Subtracting wipes out many digits of accuracy.

   b. Now solve it by approximating the square root in the quadratic formula.

   > The square root is $\sqrt{b^2 - 4}$, where $b = -10^{10}$ since the other coefficients of the quadratic are $a = 1$ and $c = 1$. To approximate the quadratic, take out the big part, which is $\sqrt{b^2}$:
   >
   > $$\sqrt{b^2 - 4} = \sqrt{b^2}\sqrt{1 - 4/b^2}.$$
   >
   > For $|b| \gg 1$, the square root is approximately $1 - 2/b^2$. Assume that square roots mean 'positive square root' and let the $\pm$ in the quadratic formula take care of the other possibility. Then the roots are
   >
   > $$\frac{-b \pm |b|(1 - 2/b^2)}{2} = -b\frac{1 \mp (1 - 2/b^2)}{2}.$$
   >
   > The two roots are then $-b$ and $-1/b$.

   c. Now solve it by approximating the equation. Assume that one root is near zero, rewrite the quadratic equation neglecting the least relevant term(s), and then solve for the root.

   > If one root x is near zero, then $x^2 \ll 1$, so the equation is approximately
   >
   > $$-10^{10}x + 1 = 0$$
   >
   > or
   >
   > $$x = 10^{-10}.$$

   d. Knowing that root, how can you quickly find the other root?

> The product of the roots is the constant coefficient, which is 1, so the other root must be 10^{10}.

e. Which route is easier: (1) approximating the equation then solving, or (2) solving via the quadratic formula and then approximating?

> In general, and in this example, it is much faster to approxiate first and then solve, rather than to solve then approximate. Why save baggage (small terms) that you will throw away at the end of a trip?

5. **Exciting particular modes**

   The system described by the following functional

   $$\frac{Y}{X} = \frac{1}{(1 - \frac{1}{2}\mathcal{R})(1 - \frac{1}{3}\mathcal{R})}$$

   has modes of the form $\left(\frac{1}{2}\right)^n$ and $\left(\frac{1}{3}\right)^n$. Determine an input sequence $x[n]$ for which the persistent response contains some multiple of the first but not any of the second mode. Assume that the system is initially at rest, and that the first non-zero input sample occurs at $n = 0$.

   > The $(1/2)^n$ mode is the impulse response of the system $1/(1 - \mathcal{R}/2)$. So cascade $1 - \mathcal{R}/3$ with the full system
   >
   > $$\frac{1}{(1 - \mathcal{R}/2)(1 - \mathcal{R}/3)},$$
   >
   > to get the system $1/(1 - \mathcal{R}/2)$. Therefore, feeding the impulse response of $1 - \mathcal{R}/3$ to the full system produces the pure mode that we want. The impulse response of $1 - \mathcal{R}/3$ is the signal $1, 1/3, 0, 0, 0, \ldots$.

Problems

6. **Comparing forward and backward Euler**

   In this problem you compare the forward and backward Euler methods of converting a continuous-time system into a discrete-time system. Look again at how to discretize the leaky tank or *RC* circuit. The differential equation is

   $$\tau \dot{V}_{\text{out}} = V_{\text{in}} - V_{\text{out}},$$

   where τ is the time constant.

   a. Using the forward-Euler approximation with time step T, the discrete-time implementation is

   $$y[n] = (1 - \epsilon)y[n - 1] + \epsilon x[n - 1],$$

   where $\epsilon = T/\tau$.

Draw a block diagram, write the system functional, and give a pole–zero diagram. When making the pole–zero diagram, ignore any pure power of $\mathcal{R}$ in the numerator of the system functional (it contributes only a pure delay, which is not interesting).

> The system functional is
>
> $$\frac{\epsilon\mathcal{R}}{1 - (1 - \epsilon)\mathcal{R}}.$$
>
> It has a pole at $z = 1 - \epsilon$.

b. On your pole–zero diagram, show how the poles move as ϵ increases from -1 to 3. For what ranges of ϵ, if any, does the impulse response (i) decay without oscillating, (ii) grow without oscillating, (iii) decay and oscillate, and (iv) grow and oscillate? Write a simulation to confirm your answers. Which of these four ranges would you select for accurately simulating the continuous-time system?

> The pole moves backward along the real axis from $z = 2$ when $\epsilon = -1$ to $z = -2$ when $\epsilon = 3$. The impulse response decays without oscillating when $0 < z < 1$, so when $0 < \epsilon < 1$. It grows without oscillating when $z > 1$, so when $\epsilon < 0$. It decays and oscillates when $-1 < z < 0$, so when $1 < \epsilon < 2$. It grows and oscillates when $z < -1$ so when $\epsilon > 2$.
>
> For the behavior for various ϵ, see the notes for Recitation 1.
>
> The original, continuous-time system decays without oscillating when fed an impulse, so the discrete-time simulation should do the same, if possible. Therefore, choose $0 < \epsilon < 1$. In practice, choose ϵ as close to zero as possible since the continuous-time limit is $\epsilon \to 0$.

c. For the rest of this problem, assume that ϵ is chosen reasonably. Compute the impulse response, i.e. give the output signal $y[n]$ for $n \geq 0$ when the impulse is the input signal X. Then evaluate $\sum_0^\infty y[n]$.

> The $\epsilon\mathcal{R}$ in the numerator multiplies the usual mode output $(1 - \epsilon)^n$ by ϵ and shifts it right one step. So the impulse response is
>
> $$0, \epsilon, \epsilon(1 - \epsilon), \epsilon(1 - \epsilon)^2, \epsilon(1 - \epsilon)^3, \ldots.$$
>
> It is a geometric sequence with sum
>
> $$\frac{\epsilon}{1 - (1 - \epsilon)} = 1.$$

d. Using the backward-Euler approximation with time step T, the discrete-time implementation is

$$(1 + \epsilon)y[n] = y[n - 1] + \epsilon x[n],$$

where $\epsilon = T/\tau$.

Draw a block diagram, write the system functional, and give a pole-zero diagram.

e. Compute the impulse response, i.e. give the output signal $y[n]$ for $n \geq 0$ when the impulse is the input signal X. Then evaluate $\sum_0^\infty y[n]$.

The sum is 1.

f. Give a physical interpretation of $\sum_0^\infty y[n]$. Based on your evaluation of $\sum_0^\infty y[n]$ for forward and backward Euler, which method do you prefer?

The sum is an integral of the rate, so it is the amount of water that exits the tank in this discrete-time approximation. One unit of water enters the tank because for the impulse, $\sum x[n] = 1$. Backward and forward Euler both have this property.

7. **Periodic system**

In this problem you study this variant of the Fibonacci system:

$$y[n] = y[n-1] - y[n-2] + x[n].$$

a. Compute the impulse response and show that it is periodic. What is the period?

The period is 6.

b. Draw a block diagram and give the system functional. How many modes does the system have?

The system functional is

$$\frac{1}{1 - \mathcal{R} + \mathcal{R}^2}.$$

The denominator is second degree in $\mathcal{R}$, so the system has two modes.

c. The system's response to any transient input, not just the impulse, is periodic (you don't need to prove this statement). Therefore what can you conclude about the poles of the system?

They lie on the unit circle and have angles that are a rational fractions of a full circle.

d. Mark the poles and zeros of the system on a pole–zero diagram. Make sure the pole locations are consistent with your answers to parts (a) and (c)!

> The poles are on the unit circle at $(1 \pm j\sqrt{3})/2$. Each is one-sixth of a full circle, so the period of the response should be 6, which it is.

e. Decompose the system functional into partial fractions, and draw a block diagram that corresponds to this decomposition.

$$\frac{1}{1 - \mathcal{R} + \mathcal{R}^2} = \frac{1}{j2 \sin \frac{\pi}{3}} \left(\frac{e^{j\pi/3}}{1 - e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1 - e^{-j\pi/3}\mathcal{R}} \right)$$
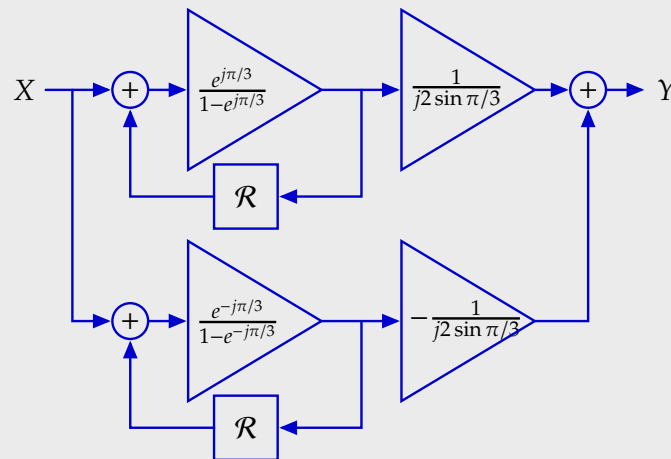
8. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

Homework 4
(not collected)

*Try all of the following warmups and problems. It will **not** be collected. Solutions will be posted in a few days for you to check yourself.*

Warmups

1. **Black's formula**

    Here is a general feedback block diagram where each block can be itself a system with an interesting functional:
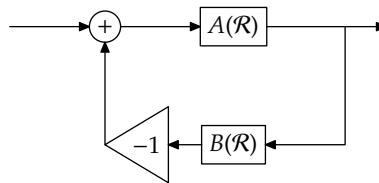
    Show that the functional for the whole system is

    $$F(\mathcal{R}) = \frac{A(\mathcal{R})}{1 + A(\mathcal{R})B(\mathcal{R})}.$$

    This formula is known as *Black's formula* named after Harold Black, one of the inventors of the negative-feedback amplifier.

    > Let X be the input signal and Y the output signal. The output of the adder is $X - B(\mathcal{R})Y$. That output gets fed into the $A(\mathcal{R})$ to produce Y. So
    >
    > $$Y = A(\mathcal{R})(X - B(\mathcal{R})Y).$$
    >
    > Rearranging the terms with Y onto the left side gives
    >
    > $$(1 + A(\mathcal{R})B(\mathcal{R}))Y = A(\mathcal{R})X,$$
    >
    > so
    >
    > $$\frac{Y}{X} = \frac{A(\mathcal{R})}{1 + A(\mathcal{R})B(\mathcal{R})}.$$

2. **System functionals**

    a. Here is a feedback-control system:
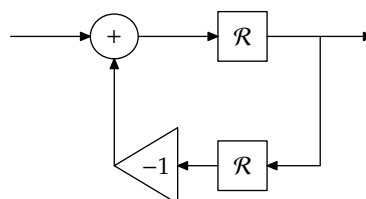
    Find the system functional and the poles, showing them on a pole–zero plot.

Let's use Black's formula for practice. The feedforward functional is $A(\mathcal{R}) = \mathcal{R}$. The feedback functional is $B(\mathcal{R}) = \mathcal{R}$. So the system functional is
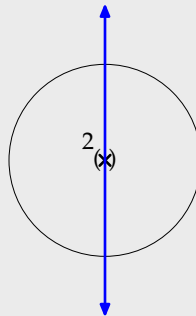
$$\frac{\mathcal{R}}{1 + \mathcal{R}^2}.$$

The denominator factors as $(1 + j\mathcal{R})(1 - j\mathcal{R})$ so the poles are at $\pm j$:
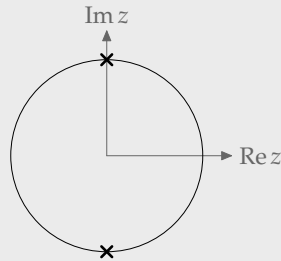
b. In the preceding diagram, let the gain be $-\beta$ instead of -1. On a pole–zero diagram (the z-plane), sketch how the poles move as β varies from 0 to ∞.
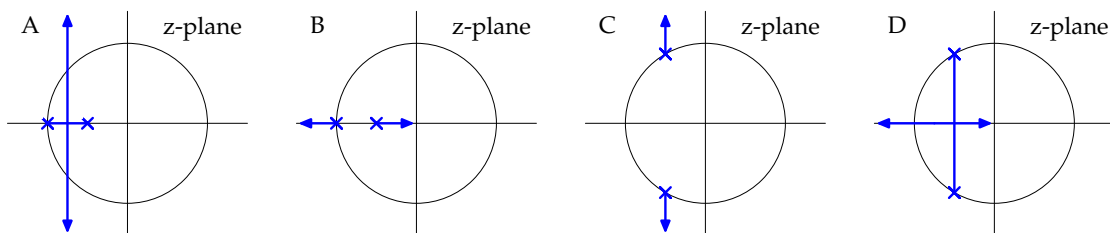
The poles are at $\pm j\sqrt{\beta}$, so the poles start as a double pole at the origin when $\beta = 0$, then move up and down the imaginary axis as β increases to ∞:

Problems

3. Closed-loop poles

The following plots show the positions of closed-loop poles as functions of the gain α (where $\alpha > 0$):

a. Which if any of the plots A, B, C, or D correspond to the following block diagram?
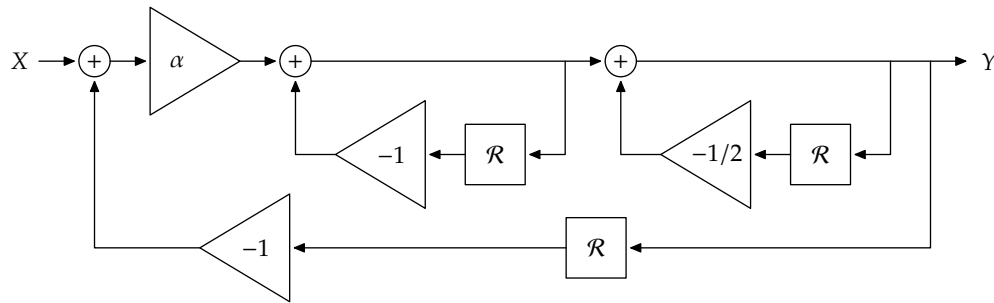
Let's use Black's equation to find the system functional. The feedforward functional is a cascade of three systems:

1. a gain of α,

2. a feedback system with functional $1/(1 + \mathcal{R})$, and

3. a feedback system with functional $1/(1 + \mathcal{R}/2)$.

So the feedforward functional is

$$\frac{\alpha}{(1 + \mathcal{R})(1 + \frac{1}{2}\mathcal{R})}.$$

The feedback functional is $\mathcal{R}$ (not counting the gain of -1, which is already accounted for by the plus sign in Black's formula). Then the overall functional is

$$\frac{\alpha(1 + \mathcal{R})^{-1}(1 + \frac{1}{2}\mathcal{R})^{-1}}{1 + \mathcal{R}\alpha(1 + \mathcal{R})^{-1}(1 + \frac{1}{2}\mathcal{R})^{-1}}.$$

Clear the fractions by multiplying top and bottom by $(1 + \mathcal{R})(1 + \frac{1}{2}\mathcal{R})$ to get

$$\frac{\alpha}{(1 + \mathcal{R})(1 + \frac{1}{2}\mathcal{R}) + \alpha\mathcal{R}}.$$

The denominator expands to

$$1 + \left(\frac{3}{2} + \alpha\right)\mathcal{R} + \frac{1}{2}\mathcal{R}^2,$$

whose poles start at -1 and $-1/2$ when $\alpha = 0$. So only diagrams A and B are possible. The sum of the poles is $-(\alpha + 3/2)$, so as $\alpha \to \infty$, the midpoint moves to the left (while keeping the product at $1/2$). In diagram A, the midpoint stays the same as α changes, whereas in diagram B the midpoint could move left. A quick check with the quadratic formula, if you like it, will show that diagram B is correct.

b. Which if any of the plots A, B, C, or D correspond to the following block diagram?
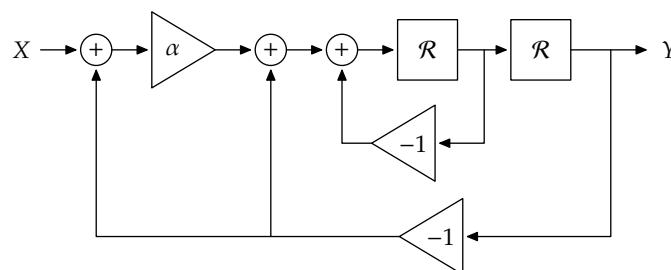
Let's find the system functional directly without Black's formula, since the topology is not the usual one. The output of the first adder is $X - Y$. The output of the second adder is $\alpha(X - Y) - Y$ or $\alpha X - (\alpha + 1)Y$. It is fed through a system with functional $\mathcal{R}^2/(1 + \mathcal{R})$. So

$$Y = (\alpha X - (\alpha + 1)Y) \frac{\mathcal{R}^2}{1 + \mathcal{R}}.$$

Clear the fractions, rearrange the terms, and you find

$$\frac{Y}{X} = \frac{\alpha \mathcal{R}^2}{1 + \mathcal{R} + (1 + \alpha)\mathcal{R}^2}.$$

The sum of the poles is -1, so the midpoint $(p_1 + p_2)/2$ is at $-1/2$. Diagrams A and B are therefore impossible. Since the midpoint is $-1/2$,

$$p_{1,2} = -\frac{1}{2} \pm \beta,$$

where β is unknown for the moment. Then

$$p_1 p_2 = \frac{1}{4} - \beta^2.$$

But this product is $\alpha + 1$ since $\alpha + 1$ is the coefficient of $\mathcal{R}^2$. Therefore β must be imaginary, and grows in magnitude as $p_1 p_2$ grows, which means it grows in magnitude as α grows. Choice D fails the test, whereas choice C does exactly what is needed.

4. **Surviving delay**

   Consider this feedback-control system:
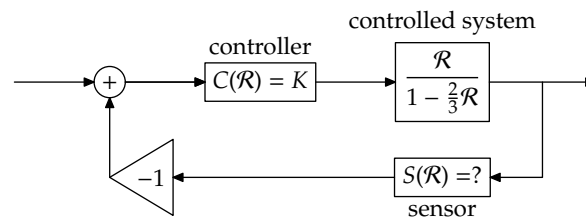
In this problem you simulate to investigate the effect of increasing the delay in the sensor.

Start with the sensor being reasonably fast: $S(\mathcal{R}) = \mathcal{R}$. Let the controller gain K be 1/9. How many poles does the system with feedback have, and where are they? Simulate the system (in Octave/Matlab or Python) to confirm these pole locations.

Now increase the sensor delay by making $S(\mathcal{R}) = \mathcal{R}^2$. Modify your simulation for this case, keeping $K = 1/9$. If your simulation is well written, you should not have much to change to increase the sensor delay. Is the new system stable? If it is, increase the sensor delay until the system becomes unstable, keeping $K = 1/9$. What is the maximum usable sensor delay (i.e. how many powers of $\mathcal{R}$)?

The system functional with $S(\mathcal{R}) = \mathcal{R}$ is (using Black's equation from problem 1):

$$\frac{K\mathcal{R}(1 - \frac{2}{3}\mathcal{R})^{-1}}{1 + K\mathcal{R}^2(1 - \frac{2}{3}\mathcal{R})^{-1}}.$$

Clearing the fractions gives

$$\frac{\text{something}}{1 - \frac{2}{3}\mathcal{R} + K\mathcal{R}^2},$$

where the 'something' might contain zeros, but who cares about it since the poles, and therefore the stability, are determined by the denominator. This second-order denominator produces two poles that have a midpoint at 1/3. With $K = 1/9$, both poles are at 1/3, so the system is stable.

With $S(\mathcal{R}) = \mathcal{R}^2$, the denominator becomes

$$1 - \frac{2}{3}\mathcal{R} + K\mathcal{R}^3.$$

Again $K = 1/9$.

One way to check stability is to write a simulation. An alternative, which we use here for variety, is to ask `maxima` to find the poles. To find the poles, substitute $\mathcal{R} = 1/z$ and solve for z, which gives the equation

$$z^3 - \frac{2}{3}z^2 + K = 0.$$

Ask `maxima` for the magnitudes of the poles with the command:

```
abs(allroots(z^3-(2/3)*z^2+K=0,z)),K=1/9;
```

The poles have magnitude:

0.57735027202848	(two poles),
0.33333333333333	(one pole).

These are inside the unit circle. But the worst pole has moved away from the origin: from 1/3 when $S(\mathcal{R}) = \mathcal{R}$ to $0.577\ldots$ (or $1/\sqrt{3}$). So the extra sensor delay has pushed the system toward instability.

Now add one more sensor delay. The equation to solve is

$$z^4 - \frac{2}{3}z^3 + K = 0.$$

With $K = 1/9$, `maxima` gives these magnitudes:

0.69965250234153	(two poles),
0.47642698656514	(two poles).

The worst pole is now closer the unit circle, so the system is less stable. But as you find out from trying ever more delays, the system never goes unstable (with $K = 1/9$). As the number of delays increases, the poles move ever closer to the unit circle, with no pole ending up outside the unit circle. Maybe you find this result surprising. It surprised us!

Homework 5

Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 10 October 2007**.

Warmups

1. **Impulse responses**

   Find the impulse responses of the following systems:

   1. $\dfrac{1}{1 - \mathcal{L}/2}$

   > The series expansion is
   >
   > $$1 + \frac{1}{2}\mathcal{L} + \frac{1}{4}\mathcal{L}^2 + \frac{1}{8}\mathcal{L}^3 + \cdots.$$
   >
   > So the impulse response is
   >
   > $$y[n] = \begin{cases} 2^n & (n \le 0); \\ 0 & \text{otherwise.} \end{cases}$$

   2. $\dfrac{\mathcal{L}}{1 + 3\mathcal{L}}$

   > The series expansion is
   >
   > $$\mathcal{L} - 3\mathcal{L}^2 + 9\mathcal{L}^3 - 27\mathcal{L}^4 + \cdots.$$
   >
   > So the impulse response is
   >
   > $$y[n] = \begin{cases} \left(-\frac{1}{3}\right)^{n+1} & \text{for } n < 0; \\ 0 & \text{otherwise.} \end{cases}$$

   3. $\dfrac{1 - \mathcal{R}}{1 - \mathcal{L}}$

   > The $1/(1 - \mathcal{L})$ produces a leftward step:
   >
   > $$Y = \cdots 1 1 1 1 \underline{1} 0 0 0 0 \cdots,$$
   >
   > where the rightmost '1' occurs at $n = 0$ (marked with an underline). So the $1 - \mathcal{R}$ takes the leftward step Y and computes these intermediate signals:
   >
   > $$\begin{aligned} Y \quad & \cdots 1 1 1 1 \underline{1} 0 0 0 0 \cdots \\ \mathcal{R}Y \quad & \cdots 1 1 1 1 \underline{1} 1 0 0 0 \cdots \end{aligned}$$
   >
   > and then subtracts them. The two signals match everywhere except at $n = 1$, when their difference is -1. So the response is a right-shifted and negated impulse.

2. **Cascade**

   a. Is the system

   $$\frac{1}{1 - 0.95\mathcal{R}} \frac{1}{1 - 0.95\mathcal{L}}$$

   causal, anticausal, or noncausal?

   It contains $\mathcal{L}$ so it is not causal; it contains $\mathcal{R}$ so it is not anticausal. It is therefore noncausal.

   b. Find the $n = 0$ sample of its impulse response. You can use a computer or, if you are skilled with series, you can do it analytically.

   The causal factor (with the $\mathcal{R}$) produces a response $y[n] = 0.95^n$ for $n \geq 0$ and 0 otherwise. The leftward factor computes a weighted average of those samples. The $y[0]$ sample gets weight 1. The $y[1]$ sample gets weight 0.95. The $y[2]$ sample gets weight 0.95^2. So the $y[n]$ sample gets weight 0.95^n. Since $y[n]$ is itself 0.95^n, the weighted sum is

   $$1 + 0.95^2 + 0.95^4 + \cdots = \frac{1}{1 - 0.95^2} \approx 10.$$

3. **RC filter**

   a. Write the continuous-time system functional (using $\mathcal{A}$) for $V_{\text{out}}/V_{\text{in}}$ in the *RC* circuit:
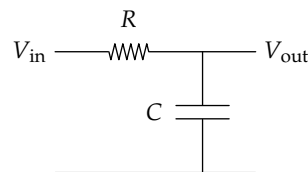
   Expand the functional as a power series in $\mathcal{A}$.

   The capacitor integrates current into stored charge, which generates a voltage across the capacitor. So the capacitor voltage, which is also the output voltage, is

   $$V_{\text{out}} = \frac{Q}{C} = \frac{1}{C} \int_{-\infty}^{t} I \, dt = \mathcal{A}\frac{I}{C}.$$

   The voltage across the resistor is the difference $V_{\text{in}} - V_{\text{out}}$, and it is also IR by Ohm's law. So

   $$I = \frac{V_{\text{in}} - V_{\text{out}}}{R}.$$

   Now replace I in the functional expression for the output voltage:

   $$V_{\text{out}} = \frac{1}{RC} \mathcal{A}(V_{\text{in}} - V_{\text{out}}).$$

   Replacing RC with τ and rearranging gives

   $$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\mathcal{A}/\tau}{1 + \mathcal{A}/\tau}.$$

   The power series expansion is

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\mathcal{A}}{\tau}\right)^n.$$

b. Feed the impulse $\delta(t)$ into it and analytically compute the impulse response of each term in the series. Then add the results to find the impulse response of the system.

We need to apply $(\mathcal{A}/\tau)^n$ to the impulse $\delta(t)$. The first $\mathcal{A}$ produces the step function 1 (for $t \geq 0$), and each $\mathcal{A}$ does one more integration, giving t, then $t^2/2$, then $t^3/6$, ... So

$$\mathcal{A}^n \delta(t) = \frac{t^{n-1}}{(n-1)!} \qquad \text{(for } t \geq 0, n \geq 1\text{)}.$$

Thus the RC functional applied to $\delta(t)$ gives

$$\sum_{n=1}^{\infty} \frac{(-t)^{n-1}/\tau^n}{(n-1)!} = \frac{e^{-t/\tau}}{\tau}.$$

c. Give dimensional and extreme-cases arguments to show why the impulse response that you found is reasonable.

The impulse response really should have a constant in front of it to make it a voltage rather than a pure delta function. But the system is linear, so as long as the input and output signals have the correct ratio of dimensions, then we do not have to worry about that constant (it'll just divide out when taking the ratio).

The dimensions of a delta function are not obvious, but its time integral is 1 by definition. Since integration in time is dimensionally equivalent to multiplication by time (because the dt has dimensions of time), the delta function itself has dimensions of inverse time. Since the system functional is a ratio (loosely speaking) of two voltages, V_{out} and V_{in}, the system functional should be dimensionless. So the output signal should have the same dimensions as the input signal (which are inverse time). The output signal $e^{-t/\tau}/\tau$ indeed has dimensions of inverse time, because the exponential is dimensionless and the τ is a time.

Checking dimensions is useful. The first version of the solutions lacked a factor of τ^{-1} in the impulse response, spotted only thanks to writing the solution to this part!
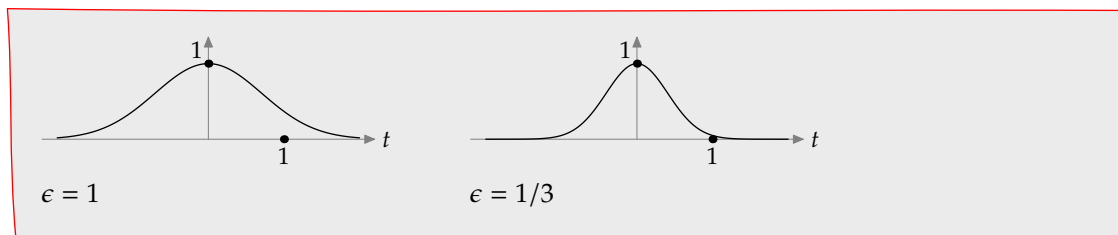
Now investigate extreme cases of τ. First try $\tau \rightarrow \infty$. One way to produce that limit is with a giant capacitor $C = \infty$. The delta function impulse produces a current impulse, which contributes a finite charge. But a finite charge on an infinite capacitor produces zero voltage ($Q = CV$). So the impulse response should go tozero in the limit that $\tau \rightarrow \infty$. And it does, because of the τ in the denominator.

The other extreme $\tau \rightarrow 0$ can be produced by taking $R \rightarrow 0$. Then the circuit becomes a voltage source connected to a capcitor (which leads to classic paradoxes about energy but we will carefully avoid them by not asking dangerous questions). So the entire input voltage should appear across the capacitor, meaning that V_{out} should equal V_{in} in the limit $\tau \rightarrow 0$. With an impulse $\delta(t)$ as the input, the output $e^{-t/\tau}/\tau$ should also be a delta function. In the limit $\tau \rightarrow 0$, it is infinitely high at $t = 0$ and zero elsewhere, and its area is 1. So the output becomes a delta function.
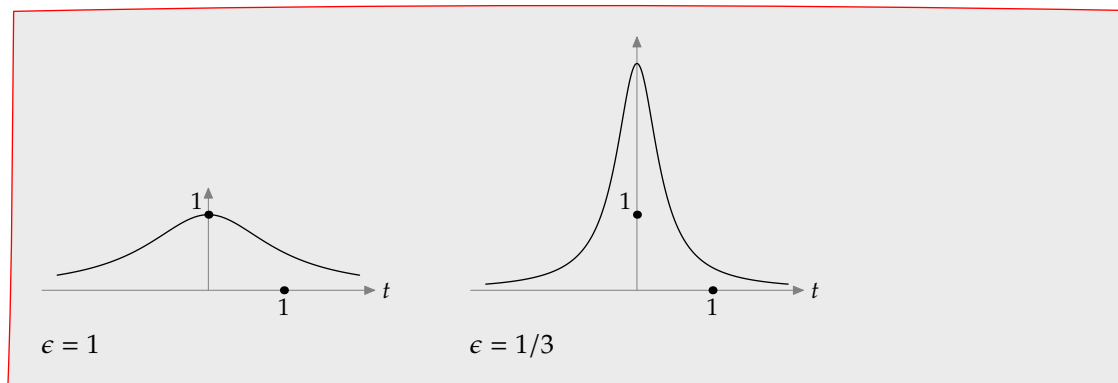
4. Find the delta functions

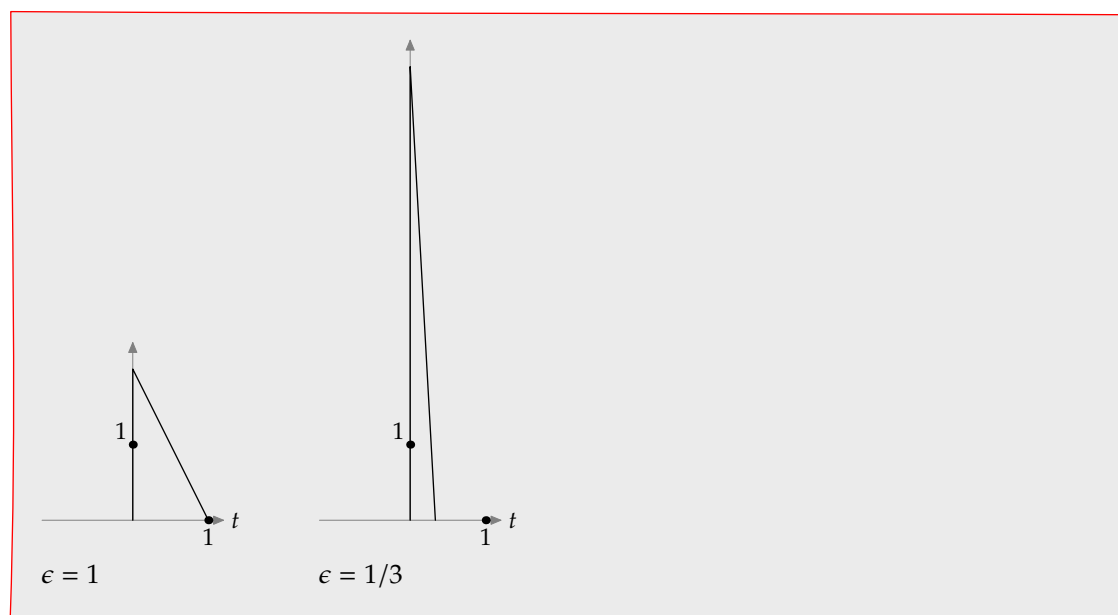**a.** Sketch each function for $\epsilon = 1$ and $\epsilon = 1/3$:

1. $e^{-t^2/\epsilon}$

$\epsilon = 1$ $\qquad\qquad\qquad\qquad$ $\epsilon = 1/3$

2. $\dfrac{\epsilon}{t^2 + \epsilon^2}$

$\epsilon = 1$ $\qquad\qquad\qquad\qquad$ $\epsilon = 1/3$

3. $\begin{cases} \dfrac{2}{\epsilon}\left(1 - \dfrac{t}{\epsilon}\right) & (0 \le t \le \epsilon); \\ 0 & \text{(otherwise)}. \end{cases}$

$\epsilon = 1$ $\qquad\qquad\qquad\qquad$ $\epsilon = 1/3$

b. Which functions, if any, become $\delta(t)$ in the limit $\epsilon \to 0$?

The integral of the bell curve (option 1) goes to zero as $\epsilon \to 0$ because it gets infinitely thin without its height increasing. So it is not a delta function.

Option 2, the Lorentzian shape, gets infinitely high and infinitely thin. Let's check its integral:

$$\int \frac{\epsilon}{t^2 + \epsilon^2}\, dt = \arctan \frac{t}{\epsilon}.$$

Putting in the limits $t = -\infty \ldots \infty$ gives π. At least the integral is not infinite and not zero; but it should be unity. So the proposed function is too large by a factor of π.

Option 3 is a triangle with height $2/\epsilon$ and base ϵ. So it becomes infinitely thin and infinitely high. Its area is

$$\frac{1}{2} \times \frac{2}{\epsilon} \times \epsilon = 1.$$

Great! It is a delta function.

Problems

5. **Image reconstruction**
   Choose one mangled image from each row, sharpen the image, and identify the building. Here are thumbnails of the images:

a1 a2

b1 b2
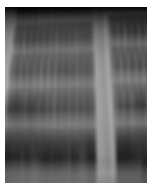
c1 c2

The images are available in machine-readable form (buildings.zip) on the Stellar class site.

The first row of pictures were blurred by passing each row of pixels through a system with the pole at 0.985 processing from left to right. The resulting pictures can be unblurred by passing each row through a system with a zero at 0.985 processing from left to right.

The second row of pictures were blurred by passing each column of pixels through a system with the pole at 0.985 processing from bottom to top. The resulting pictures can be unblurred by passing each column through a system with a zero at 0.985 processing from bottom to top.

The third row of pictures were blurred by passing each column of pixels through a system with the pole at 0.985 processing from top to bottom. The resulting pictures can be unblurred by passing each column through a system with a zero at 0.985 processing from top to bottom.

Results after unblurring are show below.

a1: biology building

a2: Building 39

b1: Medical Center

b2: Building 13

c1: Building 9

c2: Hayden

6. **Leaky tank**

   The leaky-tank equation is

   $$\tau\dot{y} = x - y.$$

   where x is the input flow rate, y is the output flow rate, and τ is the time constant.

   a. Define $p = 1/\tau$. Then find the system functional in terms of the $\mathcal{A}$ operator, and expand the functional in powers of $\mathcal{A}$.

   > The leaky tank has the same equation as the RC circuit (see Recitation 1). So it has the same functional as computed in Problem 3 (with $p = 1/\tau$):
   >
   > $$\frac{p\mathcal{A}}{1 + p\mathcal{A}}.$$
   >
   > Its expansion is also the same:
   >
   > $$p\mathcal{A} - (p\mathcal{A})^2 + (p\mathcal{A})^3 - (p\mathcal{A})^4 + \cdots.$$

   b. Write a program that finds the system's response to any input signal X by using the series expansion in part (a). Try your program with the following input signals (which are all zero for $t < 0$), choosing two or three interesting values for the time constant τ:

   1. $x(t) = 1$.

   2. $x(t) = e^{-t}$.

   For each input signal and value of τ, explain why your program's answer for the output is reasonable.

   > Here is Python code that saves the various plots as `pdf`, trying three time constants for each of the two input signals.
   >
   > ```python
   > from scipy import *
   > from scipy.integrate import *
   > import pylab as p
   >
   > dt = 0.01
   > tmax = 8
   > t = arange(0,tmax+dt/10,dt)
   >
   > # takes a signal X and returns the signal A(X).  Uses
   > # library routine for cumulative integration, with the trapzoidal
   > # approximation, so it's more accurate than the rectangular
   > # approximation R/(1-R).
   > def A(x):
   >     y = 0.0*x
   >     y[1:] = cumtrapz(x,dx=dt)
   >     return y
   > ```

```
# evaluate (A/tau) - (A^2/tau^2) + ... applied to the input signal X
def response(x, tau):
    newx = A(x)/tau
    total = 0*newx
    for i in range(int(tmax/tau*4)):
        total += newx
        newx = -A(newx)/tau
    return total


signals = {'step'  : ones(shape(t)),
           'expdecay' : exp(-t)}


p.hold(False)
for signame in signals:
    for tau in [0.3,1,3]:
        p.plot(t, response(signals[signame],tau))
        p.xlabel('$t$')  # use 't' if the TeX backend isn't working for you
        p.savefig('figs/sol5p6-graphs-%s-%.1f.pdf' % (signame, tau))
```
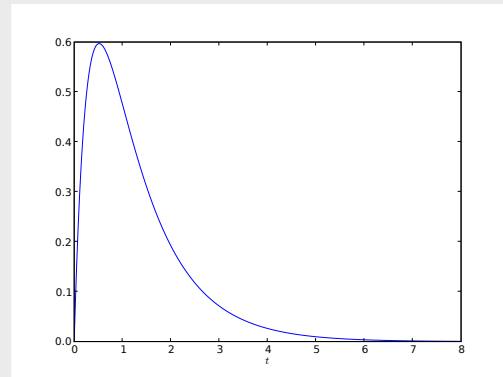
Here are the computed output signals:
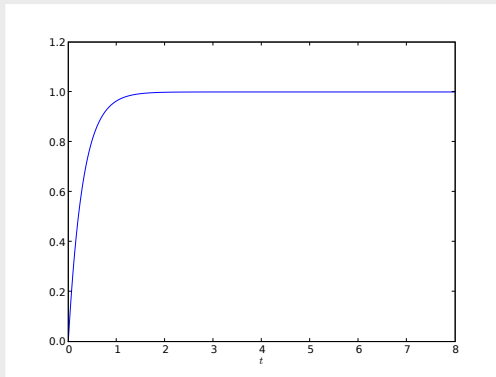
step, $\tau = 0.3$

expdecay, $\tau = 0.3$
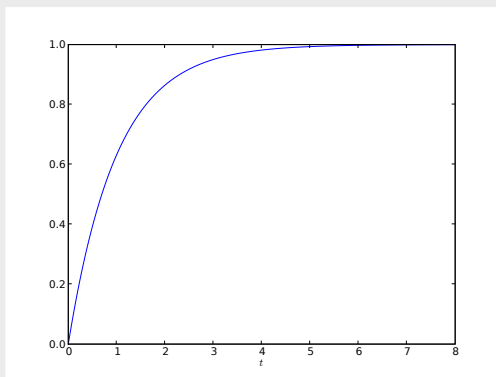
step, $\tau = 1.0$
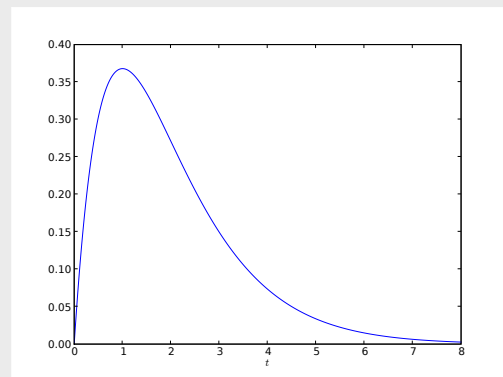
expdecay, $\tau = 1.0$
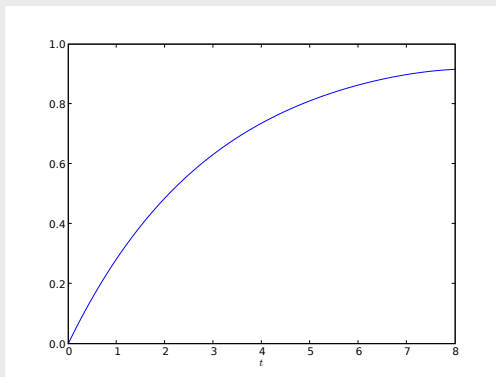
step, $\tau = 3.0$
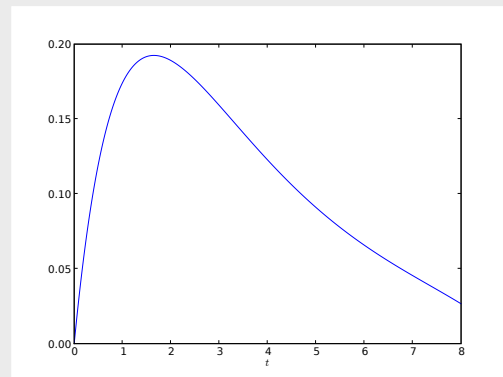
expdecay, $\tau = 3.0$

The step responses are all reasonable. In the steady state, i.e. when $t \to \infty$, the response of the *RC* circuit should be the same as the input. The longer the time constant τ, the longer before the output approximately reaches the steady state. And the three left-hand plots show steadily increasing time to reach the steady state.

The response to the exponential-decay input signal should be zero as $t \to \infty$. The longer the τ, the longer it takes the output to reach the steady state. And the three right-hand plots show steadily increasing time to decay to zero. Furthermore, because the *RC* circuit preserves the *DC* signal (it's a low-pass circuit), the output signal should have the same area as the input signal. So as the output signal gets fatter, it should get lower. And the three right-hand panels show this pattern.

7. **Hours**

   While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.

Homework 6

*Do all of the following warmups and problems, including the question about hours spent on the problem set. Due in recitation on **Wednesday, 17 October 2007**.*

Warmups

1. **Find the nonsense**

   You are handed a giant rat's nest of a linear, passive circuit with L's, R's, and C's and asked to analyze it. Which expression(s) below cannot be its system functional V_{out}/V_{in}? The various τ's are time constants made by combining inductor, resistor, and capacitor values.

   a. $\dfrac{\mathcal{A}/\tau_1}{1 + \mathcal{A}/\tau_3^2}$

   > The $\mathcal{A}$ operator has dimensions of time since it multiplies its input signal by a (tiny) time dt. The denominator adds a 1, which is dimensionless, to a $\mathcal{A}/\tau_3^2$, which has dimensions of frequency. The system functional is dimensional nonsense, and cannot be valid.

   b. $\dfrac{\mathcal{A}^2/\tau_3^2}{1 + \mathcal{A}/\tau_1 + \mathcal{A}^2/\tau_2}$

   > The first two terms in the denominator are dimensionless but the $\mathcal{A}^2/\tau_2$ term is a time. So this functional cannot be right.

   c. $\dfrac{\mathcal{A}^2}{1 + \mathcal{A}/\tau_2 + \mathcal{A}^2/(\tau_1 \tau_2) + \mathcal{A}^3/(\tau_1 \tau_2 \tau_3)}$

   > Each term in the denominator is dimensionless, so the denominator is dimensionally valid. Because of the $\mathcal{A}^2$ in the numerator, the functional itself has dimensions of time squared. But the functional is the ratio of two voltages, V_{in} and V_{out}, so it should be dimensionless. So this functional also cannot be right.

   d. $\dfrac{\tau_1}{\mathcal{A}(1 + \mathcal{A}/\tau_2)}$

   > This functional is dimensionally valid and is dimensionless, so it passes two tests. However, the bare $\mathcal{A}$ in the denominator represents a differentiator. A differentiator requires high gain for rapidly varying signals – and the faster the signal varies, the higher the gain needed. So it is not possible to make a differentiator with passive elements.

2. **Practice with the integration operator**
   a. Sketch the result of applying $\mathcal{A}$ to each of the following signals:

   (i)   (ii)   (iii)

   (i)   (ii)   (iii)
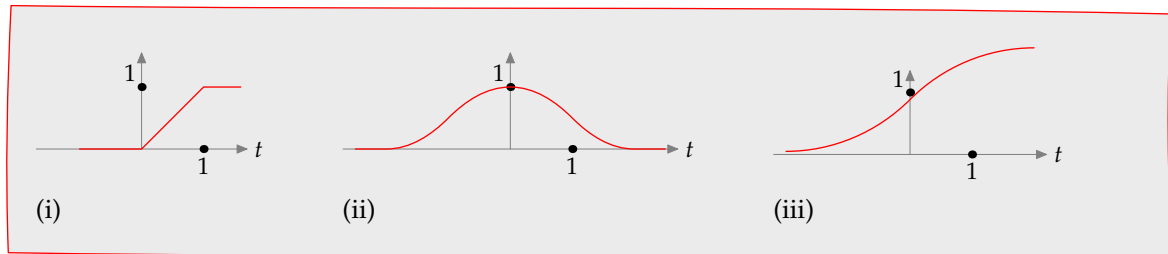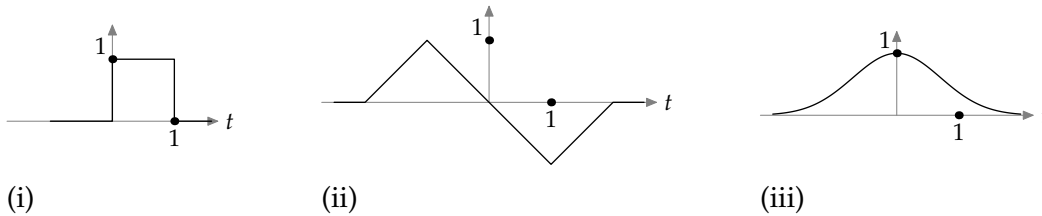
   b. Sketch the result of applying $\mathcal{A} - \mathcal{A}^2 + \mathcal{A}^3 - \mathcal{A}^4 + \ldots$ to the unit pulse (the first signal above).

   > The series $\mathcal{A} - \mathcal{A}^2 + \mathcal{A}^3 - \mathcal{A}^4 + \ldots$ is $\mathcal{A}/(1 + \mathcal{A})$, which is the system functional of an *RC* circuit with $\tau = 1$. When a unit pulse (shown as a dashed line) is sent into an *RC* circuit, the output first approaches 1 exponentially, as if a unit step were sent in. Then, when the falling edge of the pulse arrives, the output decays exponentially towards zero.
   >
   >
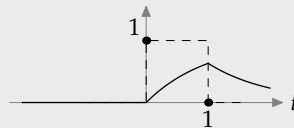
3. **Leaky tanks**
   Here is the impulse response of a cascade of two leaky tanks where the first tank has time constant $\tau_1 = 1$ and the second tank has time constant $\tau_2 = 2$.
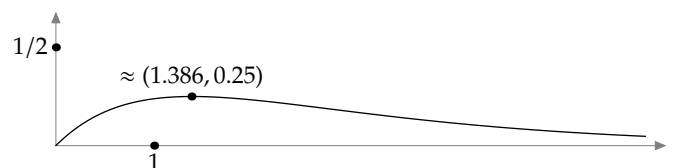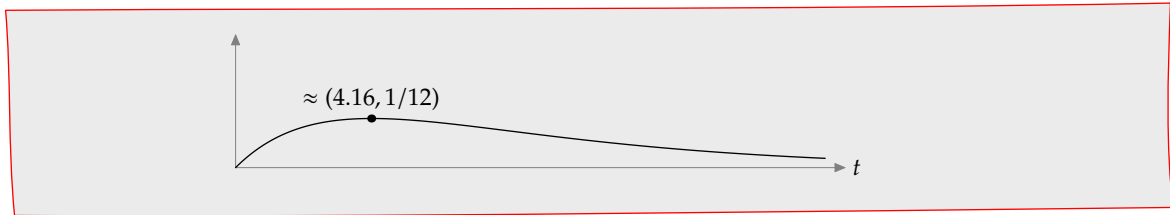

   a. Sketch the impulse response of a cascade with $\tau_1 = 3$ and $\tau_2 = 6$ and label any local maxima or minima.

   > Each new tank is three times slower than its corresponding tank in the original cascade. So the new output can almost be found from stretching the original output horizontally by a factor of three.
   >
   > The other issue is the vertical stretch. All the water that enters the cascade (here, due to the impulse input) eventually leaves the cascade, so the area of the output signal is independent of the time constants. To preserve the area despite stretching the curve horizontally by a factor of three, shrink it vertically by a factor of three.
   >
   > Thus the local maxima goes from $(2 \ln 2, 0.25)$ to $(6 \ln 2, 0.25/3)$, but the curve otherwise looks similar to the original impulse response.

$\approx (4.16, 1/12)$

b. Sketch the impulse response of a cascade with $\tau_1 = 2$ and $\tau_2 = 1$ and label any local maxima or minima.

Switching the order of the tanks switches the order of the factors in the system functional but does not change the product. So the impulse response of this system is identical to the original system.

Problems

4. Double-pole limit

Here is the system functional for a cascade of two identical leaky tanks or *RC* circuits:

$$S_1 = \frac{\mathcal{A}^2}{(1 + \mathcal{A})^2}.$$

Here is the system functional for a cascade of two almost-identical leaky tanks or *RC* circuits:

$$S_2 = \frac{0.9\mathcal{A}}{1 + 0.9\mathcal{A}} \cdot \frac{1.1\mathcal{A}}{1 + 1.1\mathcal{A}}.$$

a. Find the impulse response of the two systems analytically and sketch the results.

Lecture 10 (slide 30 in the handout) analytically computed the impulse response of a double-pole system by expanding the functional in powers of $\mathcal{A}$, then integrating term by term. For the functional

$$\left(\frac{p\mathcal{A}}{1 - p\mathcal{A}} \right)^2,$$

the impulse response is $p^2 t e^{pt}$ (for $t \geq 0$). Here $p = -1$ so the double-pole response is

$$y_1(t) = t e^{-t}.$$

The almost-double-pole system breaks into partial fractions. That calculation was also done in lecture 10 (slide 28 in the handout), but using τ instead of p. So for fun let's redo it with p and find the impulse response of

$$\frac{p_1 \mathcal{A}}{1 - p_1 \mathcal{A}} \frac{p_2 \mathcal{A}}{1 - p_2 \mathcal{A}}.$$

It breaks into partial fractions as

$$\frac{p_1 p_2}{p_1 - p_2} \left(\frac{\mathcal{A}}{1 - p_1 \mathcal{A}} - \frac{\mathcal{A}}{1 - p_2 \mathcal{A}} \right).$$

The first term in the parenthesis produces an impulse response $e^{p_1 t}$ and the second produces $e^{p_2 t}$, so the combined impulse response, including the prefactor, is

$$\frac{p_1 p_2}{p_1 - p_2} \left(e^{p_1 t} - e^{p_2 t} \right).$$

In this problem, $p_1 = -0.9$ and $p_2 = -1.1$, so the impulse response is

$$y_2(t) = 4.95 \left(e^{-0.9t} - e^{-1.1t} \right).$$

We hope that this impulse response looks like the impulse response $y_1(t)$ for the true double pole. One way to check is by making a sketch. To that end it is helpful to know the rough behavior of each curve.

First check the extreme cases of t. For $t \approx 0$, the true double pole response is $y_1(t) = te^{-t} \to 0$. The almost-double-pole response also limits to zero, in that case because the difference of exponentials becomes 0. The slope at $t \approx 0$ is 1 for the double-pole response, and $4.95 \times 0.2 = 0.99$ for the almost-double-pole response. So the slopes at $t = 0$ are almost identical. In the extreme case $t \to \infty$, both $y_1(t)$ and $y_2(t)$ go to zero because of the exponential-decay factors.

Then check the behavior in between the extremes. Each curve must have a local maxima because it starts rising at 0 and somehow has to get back to 0 by $t = \infty$. The double-pole respose has its local maximum at $(1, 1/e)$, as you can check by differentiating it. By the same method, the almost-double-pole response has a local maxima at

$$t = \frac{\ln(p_2/p_1)}{p_1 - p_2} \approx 1.00335,$$

and the signal at this t is approximately 0.3648. So the maximum is $(1.00335, 0.3648)$, which is very close to the maximum at $(1, 1/e)$ for the double pole response.

Here are sketches of the responses:
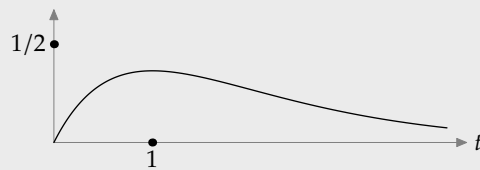
almost-double pole

double pole

b. Find the impulse response of the two systems numerically.

There are many numerical methods. One is to convert each system to a second-order differential equation and solve it numerically. But we haven't discussed how to solve a second-order ODE numerically, so let's not use that method in these solutions (though it's fine if you used it yourself).

Another method is to represent each tank as a first-order ODE, find its impulse response numerically, and then feed that response as the input to the ODE for the second tank. That's also a fine method.

A third method, and the one that we use here because it illustrates the integration operator $\mathcal{A}$, is to represent each tank as a power series in $\mathcal{A}$, and then feed the output of the first tank into the series for the second tank.

The Python code from last week does one tank, so you can reuse most of it to do a cascade of two tanks. Here is the revised code:

```python
# compute two-tank impulse responses for solution set 6, problem 4

from scipy import *
from scipy.integrate import *
import pylab as p

dt = 0.01
tmax = 8
t = arange(0,tmax+dt/10,dt)
# very thin signal of unit area
impulse = 0*t
impulse[0] = 1.0/dt

# takes a signal X and returns the signal A(X).  Uses
# library routine for cumulative integration, with the trapzoidal
# approximation, so it's more accurate than the rectangular
# approximation R/(1-R).
def A(x):
    y = 0.0*x
    y[1:] = cumtrapz(x,dx=dt)
    return y

# evaluate (A/tau) - (A^2/tau^2) + ... applied to the input signal X
def response(x, tau):
    newx = A(x)/tau
    total = 0*newx
    for i in range(int(tmax/tau*4)):
        total += newx
        newx = -A(newx)/tau
    return total

cascades = [(1,1),(1/0.9,1/1.1)]

p.hold(False)
for taus in cascades:
    ir = response(response(impulse,taus[0]), taus[1]) # impulse response
    p.plot(t, ir)
    p.xlabel('$t$')  # use 't' if the TeX backend isn't working for you
    p.savefig('figs/sol6p4-graphs-%.1f-%.1f.pdf' % taus)
```
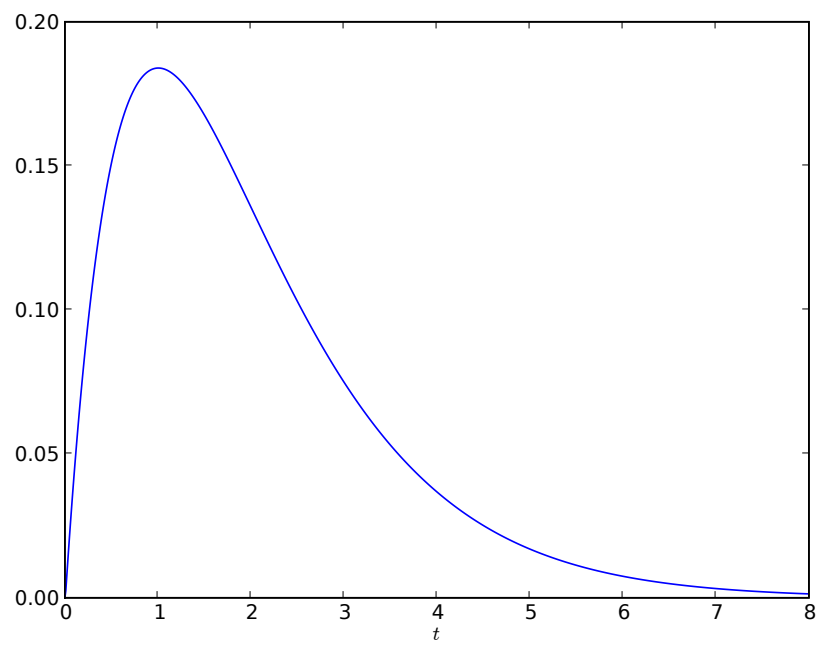
Here are the computed outputs:
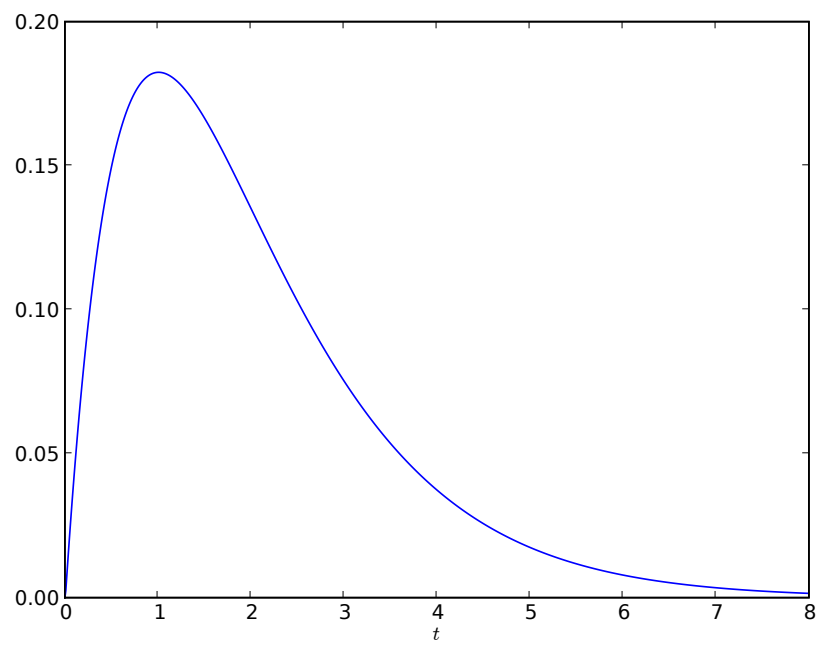
almost-double pole

double pole

5. Differential equation

Imagine feeding the unit step function as the input signal X to the differential equation

$$\ddot{y}(t) + \dot{y}(t) + y(t) = x(t).$$

Find the output signal Y analytically in two ways: (1) by using the system functional Y/X and (2) by using 18.03 methods. Always sketch your results!

First we use the system functional. We have practiced using it to find the impulse response. Here we want the the step response. That problem is equivalent to the impulse response of a cascade of (1) $\mathcal{A}$ and (2) the ODE system. We can swap the two systems, putting the bare $\mathcal{A}$ at the end. This order means integrating the impulse response of the ODE system, and we know how to use the system functional to find the impulse response of the ODE system.

The system functional is

$$\frac{Y}{X} = \frac{\mathcal{A}^2}{1 + \mathcal{A} + \mathcal{A}^2}.$$

Its partial-fractions expansion is

$$\frac{1}{p_1 - p_2}\left(\frac{\mathcal{A}}{1 - p_1\mathcal{A}} - \frac{\mathcal{A}}{1 - p_2\mathcal{A}}\right),$$

where p_1 and p_2 are the complex cube roots of 1, as shown in the pole–zero diagram. With $p_1 = -1/2 + j\sqrt{3}/2$ and $p_2 = -1/2 - j\sqrt{3}/2$, the impulse response from the decomposition is

$$\frac{1}{j\sqrt{3}}e^{-t/2}\left(e^{jt\sqrt{3}/2} - e^{-jt\sqrt{3}/2}\right),$$

You can simplify it to

$$\frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t,$$

but that rewriting makes the next step – integrating the impulse response – more difficult.

The easiest form to integrate is

$$\frac{2}{\sqrt{3}}\operatorname{Im} e^{(-1/2+j\sqrt{3}/2)t}.$$

Don't forget that the signal is zero for $t < 0$, so we integrate this function from 0 to t rather than from $-\infty$ to t. The integral is (before taking the imaginary part):

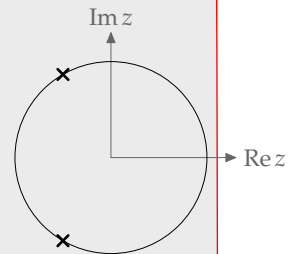$$\int_0^t e^{(-1/2+j\sqrt{3}/2)t}\,dt = \frac{2}{\sqrt{3}}\frac{1}{-1/2 + j\sqrt{3}/2}\left(e^{(-1/2+j\sqrt{3}/2)t} - 1\right).$$

The complex factor before the parentheses is $1/p_1$. Call it $e^{-j\phi}$ with $\phi = 2\pi/3$. Then the integral simplifies to

$$\frac{2}{\sqrt{3}}e^{-j\phi}\left(e^{(-1/2+j\sqrt{3}/2)t} - 1\right) = \frac{2}{\sqrt{3}}\left(e^{-t/2}e^{j(t\sqrt{3}/2-\phi)} - e^{-j\phi}\right).$$

Now take the imaginary part to get the step response:

$$\frac{2}{\sqrt{3}}\left(e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \phi\right) + \sin\phi\right).$$

Since $\sin\phi = \sqrt{3}/2$, this result simplifies to

$$1 + \frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \phi\right).$$

Let's check this horror by looking at the extreme cases. When $t \to \infty$, the exponential decay kills off all but the 1. That result is correct. The system is a damped mass–spring system, so for large t, all the derivatives go away and the system reaches a new equilibrium. The step response means that the driving force moved the driving plate by a unit distance. Eventually the mass catches up to it, the oscillations die, and the spring returns to its natural length. So the large-t signal should be, and is 1.

For $t \approx 0$, the mass has a unit force on it, so it has unit acceleration, which produces a linear velocity $\dot{y}(t) \approx t$ and a quadratic position $y(t) \approx t^2/2$. The Taylor series for the proposed step response does have $t^2/2$ as its lowest-order term, so the response is probably okay.

The 18.03 method involves finding the homogenous and particular solutions and combining them to make sure that the initial conditions are correct. By factoring the system functional we have already done the algebra to find the terms of the homogenous solution. They are the modes e^{pt} for $p = -1/2 \pm j\sqrt{3}/2$. We need the correct linear combination of these two functions. Alternatively, we can find C and θ for the function

$$Ce^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \theta\right).$$

The particular solution is just 1 (for $t \geq 0$). So the general solution is

$$1 + Ce^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \theta\right).$$

The initial conditions are $y(0) = 0$ and $\dot{y}(0) = 0$ because a step-function input, which has a discontinuity at $t = 0$, will produce a discontinuity in the second derivative at $t = 0$, a continuous first derivative at $t = 0$, and a smooth zeroth derivative at $t = 0$. To meet those conditions, it turns out that $C = 2/\sqrt{3}$ and $\theta = 2\pi/3$, as we found before.
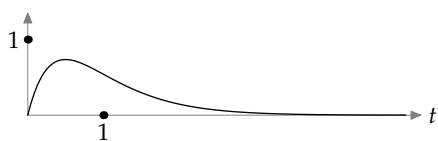
6. **Designing recognizers**

   In this problem you design recognizers for particular sound patterns – such as might be used for speech recognition. For each sound pattern (signal) that follows, give the system functional that turns it into the unit step function. You are therefore designing a system that turns on during that sound pattern.
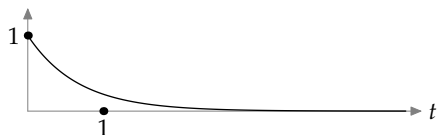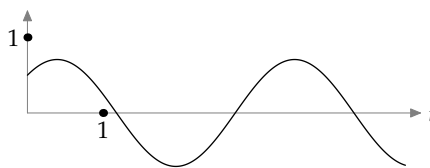
A
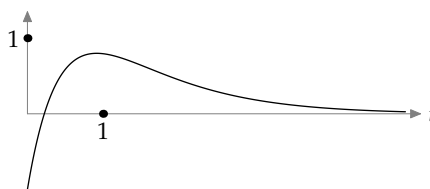
B

C

D

Each signal is available on Stellar as a data file with one $(t, f(t))$ pair on each line.

> For each signal, the first step is *data compression:* Fit it with a simple function. Then design a system with that impulse response. Its reciprocal would turn the mystery signal into an impulse, so multiply by $\mathcal{A}$ to get a system functional that turns the mystery signal into a step.

7. **Hours**

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.