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6.003: Signals and Systems

Lecture 1
Introduction to Signals and Systems

September 6, 2007

6.003: Signals and Systems

Today’s handouts: Single package containing
• Subject Information
• Lecture #1 slides (for today)
• Recitation #2 handout (for tomorrow)

Lecturer Denny Freeman (freeman@mit.edu)
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Secretary Janice Balzer (balzer@mit.edu)
Text Signals and Systems by Oppenheim and Willsky
Web Site mit.edu/6.003

6.003: Signals and Systems

Homework: where subject matter is/isn’t learned.
  equivalent to “practice” in sports or music.
  • Weekly Homework Assignments
    – Conventional Homework Problems plus
    – Engineering Design Problems (often using Matlab, Octave, or Python)
  • Homework Assignments are longer (by about 3 hours) than homework assignments in 12 unit subjects!
    – 15 units + 4 Engineering Design Points
  • Open Office Hours!
    – Stata Basement (32-044)
    – Mondays and Tuesdays, afternoons and evenings
6.003: Signals and Systems

Collaboration Policy
• Discussion of concepts in homework is encouraged
• Sharing of homework or code is not permitted and will be reported to the COD

Firm Deadlines
• Homework must be submitted in recitation on due date
• Late homework will NOT be accepted unless excused by the staff, a Dean, or Physician

Homework Extension Policy
• Every student gets one extension
• Can be used for any weekly homework assignment and for any reason
• Simply ask your TA for an extension before 11:59 pm on the day preceding the due date (cannot be rescinded)

6.003 Calendar
• Basic Representations of Discrete-Time Systems (4 weeks). difference equations, block diagrams, operator expressions, system functions, feedback and control, Z transforms, convolution (O&W Chapters 1, 2, 10, and 11).
• Basic Representations of Continuous-Time Systems (3 weeks). differential equations, block diagrams, operator expressions, system functions, feedback and control, Laplace transforms, convolution (O&W Chapters 1, 2, 9, and 11).
• Signal Processing (2 weeks). Fourier Series, Fourier Transforms, Filtering (O&W Chapters 3, 4, 5, and 6).
• Sampling (2 weeks). Sampling, aliasing, DT processing of CT signals (O&W Chapter 7).
• Communications (2 weeks). modulation, AM, FM (O&W Chapter 8).

6.003: Signals and Systems
Weekly meetings with class representatives
• help staff understand student perspective
• learn about teaching
One representative from each section (6 total)
Tentatively meet on Thursday afternoon
Interested? ...send email to freeman@mit.edu
Lecture 1: The 6.003 Abstraction

**6.003 abstraction:** describe a system (physical, mathematical, or computational) by the way it transforms inputs into outputs.

Example: Mass and Spring

Example: Tanks
Example: Cell Phone System

_signals and systems: uniform representations_

Signals and Systems: Uniformity → Modularity

- focus on the flow of information
- abstract away everything else
Signals and Systems: Broad Applicability

Discrete-Time Systems

Example: Bank account

Transactions (deposits/withdrawals) recorded daily (DT)

*Deposits* are an input (of money) into the system. How are *withdrawals* represented in the framework of signals and systems?

1. as an input signal
2. as an output signal
3. none of the above
Example: Bank Account

Transactions (deposits/withdrawals) recorded daily (DT)

Withdrawals are negative deposits.

Compound interest.

$ deposited today current balance

$ deposited today current balance

$ deposited today current balance
Example: Bank Account

Early Retirement? How soon can you retire if
• living expenses: $25,000 per year
• rate of savings: $10,000 per year
• 5% annual interest
• live off your savings till age 80?

Population Growth

Population Growth
How does the number of pairs of rabbits grow?

1. logarithmic \( f[n] = O(\log n) \)
2. polynomial \( f[n] = O(n^k) \) for some \( k \)
3. exponential \( f[n] = O(z^n) \) for some \( z \)
Multiple Representations of Discrete-Time Systems

Discrete-time systems can be represented in a variety of ways.

Verbal description: ‘To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.’

Difference equation:

\[ y[n] = x[n] - x[n-1] \]

Block diagram:

```
\[ y[n] \quad + \quad x[n-1] \quad + \quad y[n] \quad -1 \quad Delay \]
```

Same input-output behavior, different strengths/weaknesses:

- **word statements** preserve underlying physics.
- **difference equations** are mathematically compact.
- **block diagrams** illustrate signal flow paths.

Step-by-Step solutions

Block diagrams and difference equations are convenient for step-by-step analysis. Let \( x[n] \) equal the “unit sample” \( \delta[n] \).

\[ \delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases} \]

Using the recursion:

\[
\begin{align*}
  y[0] &= x[0] - x[-1] = 1 - 0 = 1 \\
  y[1] &= x[1] - x[0] = 1 - 1 = 0 \\
\end{align*}
\]

\[ x[n] = \delta[n] \quad \text{and} \quad y[n] \]
**Step-by-Step solutions**

Using the block diagram. Start “at rest.”

\[
\begin{align*}
\text{Delay} & \quad \text{Delay}^{-1} \\
0 & \quad +0 0 \\
-1 & \quad 0 1 2 3 4 \\
x[n] = \delta[n] & \quad y[n]
\end{align*}
\]

**From Samples to Signals**

An alternative operator approach focuses on signals rather than samples.

Operator approach: nodes represent whole signals (e.g., \(X\) and \(Y\)) and boxes operate on those signals:
- Delay = shift whole signal to right 1 time step
- Add = sum two signals
- \(-1\): invert whole signal

**Operator Notation**

Symbols can compactly represent diagrams.

Let \(\mathcal{R}\) represent the right-shift operator:

\[
Y = \mathcal{R}\{X\} \equiv \mathcal{R}X
\]

where \(X\) represents the whole input signal (\(x[n]\) for all \(n\)) and \(Y\) represents the whole output signal (\(y[n]\) for all \(n\))

Representing the difference machine

\[
\begin{align*}
\text{Delay} & \quad \text{Delay}^{-1} \\
0 & \quad +0 0 \\
-1 & \quad 0 1 2 3 4 \\
x[n] = \delta[n] & \quad y[n]
\end{align*}
\]

with \(\mathcal{R}\) leads to the equivalent representation

\[
Y = X - \mathcal{R}X = (1 - \mathcal{R})X
\]
Operator Notation: Check Yourself

Let $Y = RX$. Which of the following are true:

1. $y[n] = x[n]$ for all $n$
2. $y[n + 1] = x[n]$ for all $n$
3. $y[n] = x[n + 1]$ for all $n$
4. $y[n - 1] = x[n]$ for all $n$
5. none of the above

Operator Representation of a Cascaded System

System operations have simple operator representations. Cascade systems $\rightarrow$ multiply operator expressions.

Using operator notation:
\[ Y_1 = (1 - R) X \]
\[ Y_2 = (1 - R) Y_1 \]
Substituting for $Y_1$:
\[ Y_2 = (1 - R)(1 - R)X \]

Operator Algebra

Operator expressions expand and reduce like polynomials.

Using difference equations:
\[ y_2[n] = y_1[n] - y_1[n - 1] \]
\[ = (x[n] - x[n - 1]) - (x[n - 1] - x[n - 2]) \]
\[ = x[n] - 2x[n - 1] + x[n - 2] \]
Using operator notation:
\[ Y_2 = (1 - R) Y_1 = (1 - R)(1 - R)X \]
\[ = (1 - R)^2 X \]
\[ = (1 - 2R + R^2) X \]
**Operator Approach**

Applies your existing expertise with polynomials to understand block diagrams, and thereby understand systems.

---

**Operator Algebra**

Operator notation facilitates seeing relations among systems. “Equivalent” block diagrams (assuming both initially at rest):

```
Delay−1 + Delay−1 + X Y1 Y2
```

```
Delay
Delay
−2 + X Y
```

Equivalent operator expression:

```
(1 − R)(1 − R) = 1 − 2R + R^2
```

---

**Operator Algebra**

Operator notation prescribes operations on signals, not samples: e.g., start with \( X \), subtract 2 times a right-shifted version of \( X \), and add a double-right-shifted version of \( X \)!

\[
\begin{align*}
X & : [-1, 0, 1, 2, 3, 4, 5, 6] \\
-2RX & : [-1, 0, 2, 3, 4, 5, 6] \\
+R^2X & : [-1, 0, 1, 2, 3, 4, 5, 6] \\
y & = X - 2RX + R^2X: [-1, 0, 2, 3, 4, 5, 6]
\end{align*}
\]
Operator Algebra

Expressions involving $R$ obey many familiar laws of algebra, e.g., commutativity.

$R(1 − R)X = (1 − R)RX$

This is easily proved by the definition of $R$, and it implies that cascaded systems commute (assuming initial rest)

$\begin{align*}
X & \rightarrow + \rightarrow \text{Delay} \rightarrow Y \\
-1 & \rightarrow \text{Delay}
\end{align*}$

is equivalent to

$\begin{align*}
X & \rightarrow \text{Delay} \rightarrow + \rightarrow Y \\
-1 & \rightarrow \text{Delay}
\end{align*}$

Equivalent operator expression:

$\begin{align*}
R(1 − R) &= R − R^2
\end{align*}$

Operator Algebra

Multiplication distributes over addition.

Equivalent systems

$\begin{align*}
X & \rightarrow \text{Delay} \rightarrow + \rightarrow Y \\
-1 & \rightarrow \text{Delay}
\end{align*}$

$\begin{align*}
X & \rightarrow \text{Delay} \rightarrow + \rightarrow Y \\
-1 & \rightarrow \text{Delay} \rightarrow \text{Delay}
\end{align*}$

Equivalent operator expression:

$\begin{align*}
R(1 − R) &= R − R^2
\end{align*}$

Operator Algebra

The associative property similarly holds for operator expressions.

Equivalent systems

$\begin{align*}
X & \rightarrow \text{Delay} \rightarrow + \rightarrow 2 \rightarrow Y \\
-1 & \rightarrow \text{Delay}
\end{align*}$

$\begin{align*}
X & \rightarrow + \rightarrow 2 \rightarrow \text{Delay} \rightarrow Y \\
-1 & \rightarrow \text{Delay}
\end{align*}$

Equivalent operator expression:

$\begin{align*}
(1 − R)R(2 − R) = (1 − R)(R(2 − R))
\end{align*}$
### Operator Algebra: Explicit and Implicit Rules

Recipes versus constraints.

\[ Y = (1 - R)X \]

**Recipe:** output signal equals difference between input signal and right-shifted input signal.

\[ Y = RY + X \]

\[ (1 - R)Y = X \]

**Constraints:** find the signal \( Y \) such that the difference between \( Y \) and \( RY \) is \( X \). But how?

---

### Example: Accumulator

Try step-by-step analysis: it always works.

\[ x[n] \rightarrow + \rightarrow y[n] \]

\[ \text{Delay} \]

Find \( y[n] \) given \( x[n] = \delta[n] \):

- \( y[0] = x[0] + y[-1] = 1 + 0 = 1 \)
- \( y[1] = x[1] + y[0] = 0 + 1 = 1 \)

\[ x[n] = \delta[n] \quad y[n] \quad \ldots \]

-1 0 1 2 3 4

Persistent response to a transient input!

---

### Example: Accumulator

The response of the accumulator system could also be generated by a system with infinitely many paths from input to output, each with one unit of delay more than the previous.

\[ Y = (1 + R + R^2 + R^3 + \cdots)X \]
Example: Accumulator

These systems are equivalent in the sense that if each is initially at rest, they will produce identical outputs from the same input.

\[(1 - R) Y_1 = X_1 \ \Rightarrow \ Y_2 = (1 + R + R^2 + R^3 + \cdots) X_2\]

Proof: if \(X_2 = X_1\) then \(Y_2 = Y_1\)

\[Y_2 = (1 + R + R^2 + R^3 + \cdots) X_2\]
\[= (1 + R + R^2 + R^3 + \cdots) X_1\]
\[= (1 + R + R^2 + R^3 + \cdots)(1 - R) Y_1\]
\[= ((1 + R + R^2 + R^3 + \cdots) - (R + R^2 + R^3 + \cdots)) Y_1\]
\[= Y_1\]

Example: Accumulator

The system functional for the accumulator is the reciprocal of a polynomial in \(R\).

\[X \rightarrow \text{Delay} \rightarrow Y\]

\[(1 - R) Y = X\]

The product \((1 - R) \times (1 + R + R^2 + R^3 + \cdots)\) equals 1.

Therefore the terms \((1-R)\) and \((1+R+R^2+R^3+\cdots)\) are reciprocals.

Thus we can write

\[\frac{Y}{X} = \frac{1}{1 - R} = 1 + R + R^2 + R^3 + R^4 + \cdots\]

Example: Accumulator

The reciprocal of \(1 - R\) can also be evaluated using synthetic division.

\[
1 - R \overline{\begin{array}{c}
1 + R + R^2 + R^3 + \cdots \\
1 - R \\
R \\
R - R^2 \\
R^2 - R^3 \\
R^3 - R^4 \\
\vdots
\end{array}}
\]

Therefore

\[\frac{1}{1 - R} = 1 + R + R^2 + R^3 + R^4 + \cdots\]
System #1 is represented by the following functional:
\[
\frac{Y}{X} = 1 - \frac{1}{2}R
\]
How many of the following systems are equivalent to System #1 (provided they are all initially at rest)?

\[
\frac{Y}{X} = 1 + \frac{1}{2}R + \frac{1}{2}R^2 + \frac{1}{2}R^3 + \frac{1}{2}R^4 + \cdots
\]

Now you know four representations of discrete-time systems.

**Verbal descriptions:** preserve the underlying physics.

“To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.”

**Difference equations:** mathematically compact.

\[
y[n] = x[n] - x[n-1]
\]

**Block diagrams:** illustrate signal flow paths.

**Operator representations:** analyze systems as polynomials.

\[
Y = (1 - R)X
\]
Multiple Representations of Discrete-Time Systems

Last time: four representations for discrete-time systems

**Verbal descriptions:** preserve the underlying physics.

“To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.”

**Difference equations:** mathematically compact.

\[ y[n] = x[n] - x[n-1] \]

**Block diagrams:** illustrate signal flow paths.

![Delay Block Diagram](image)

**Operator representations:** analyze systems as polynomials.

\[ Y = (1 - R) X \]

Example: Accumulator

Compare the various representations of an accumulator.

- **Verbal description:** “bank account paying no interest.”

- **Difference equation:**

  \[ y[n] = y[n-1] + x[n] \]

- **Block diagram:** Which is the correct block diagram?

  ![Block Diagrams](image)

  (a) (b) (c) neither

- **Operator representation:**

  \[ Y = RY + X \]
Multiple Representations: Check Yourself

How many of the outputs $y_1[2]$, $y_2[2]$, and $y_3[2]$ will equal $\frac{1}{4}$ when the input is a unit sample ($x[n] = \delta[n]$) and the system is initially at rest?

\[
\begin{align*}
    w[n] &= x[n] + \frac{1}{2}x[n-1] \\
    y_1[n] &= w[n] + \frac{1}{2}w[n-1]
\end{align*}
\]

Example: Accumulator

We used the operator representation to gain insight into how an accumulator works.

\[
(1 - R) Y = X
\]

The system functional for the accumulator is the reciprocal of a polynomial in $R$.

\[
\frac{Y}{X} = \frac{1}{1 - R}
\]

Example: Accumulator

The reciprocal of a polynomial in $R$ is equivalent to an infinite-degree polynomial in $R$.

\[
\frac{Y}{X} = \frac{1}{1 - R} = 1 + R + R^2 + R^3 + R^4 + \ldots
\]

The latter polynomial corresponds to an equivalent system with infinitely many paths from input to output, each with one unit of delay more than the previous.
Feedback, Cyclic Signal Paths, and Modes

Today's goals:
- Generalize results for accumulator.
- Show that persistent responses result from feedback.
- Analyze persistent responses to find modes.

Transient inputs can give rise to persistent outputs when a signal depends on a previous value of the same signal.

\[ x[n] = \delta[n] \]
\[ y[n] = x[n] + y[n-1] \]
\[ y[0] = x[0] + y[-1] = 1 + 0 = 1 \]
\[ y[1] = x[1] + y[0] = 0 + 1 = 1 \]

Feedback is pervasive. We've already seen examples.

Example 1: Population growth.
\[ y[n] = y[n-1] + y[n-2] + x[n] \]

Example 2: Bank account with interest \( r \) compounded daily.
\[ y[n] = (1 + r)y[n-1] + x[n] \]
Cyclic Signal Paths, Feedback, and Modes

Block diagrams help visualize feedback. Feedback occurs when there is a cyclic signal flow path.

Acyclic: all paths through system go from input to output with no cycles.
Cyclic: at least one cycle.

Feedback, Cyclic Signal Paths, and Modes

The effect of a cycle can be visualized by unwinding the cycle.

Each cycle creates another sample in the output. The response will persist even though the input is transient. Such persistent responses are called modes.

Feedback, Cyclic Signal Paths, and Modes

The effect of a cycle can be visualized by unwinding the cycle.

Each cycle creates another sample in the output. The response will persist even though the input is transient. Such persistent responses are called modes.
Analysis of Cyclic Systems: Geometric Growth

If traversing the cycle decreases or increases the magnitude of the signal, then the mode will decay or grow, respectively.

These are geometric sequences: \( y[n] = (0.5)^n \) and \((1.2)^n\) for \( n \geq 0 \).

Analysis of Cyclic Systems

The persistent response of more complicated cyclic systems is more complicated.

Not geometric. This response grows then decays.

Analysis of Cyclic Systems: Equivalent forms

Factor the system functional to break the system into two simpler systems (divide and conquer).

\[
Y = X + 1.6RY - 0.63R^2Y \\
(1 - 1.6R + 0.63R^2) Y = X \\
(1 - 0.7R)(1 - 0.9R) Y = X
\]
Analysis of Cyclic Systems: Equivalent forms

Factored form corresponds to a cascade of simpler systems.

\[(1 - 0.7R)(1 - 0.9R) Y = X\]

\[(1 - 0.9R) Y_1 = X \quad (1 - 0.7R) Y = Y_1\]

\[(1 - 0.7R) Y_2 = X \quad (1 - 0.9R) Y = Y_2\]

The order doesn't matter (if systems initially at rest).

The persistent response of the cascaded system can be found by multiplying the polynomial representations of the subsystems.

\[Y = \frac{1}{(1 - 0.7R)(1 - 0.9R)} = \frac{1}{1 - (0.7R) + (0.7)^2R^2 + \cdots} \times \frac{1}{1 - (0.9R) + (0.9)^2R^2 + \cdots}\]

\[= (1 + 0.7R + 0.7^2R^2 + 0.7^3R^3 + \cdots) \times (1 + 0.9R + 0.9^2R^2 + 0.9^3R^3 + \cdots)\]

Polynomial multiplication: collect terms with equal delays.

\[Y = \frac{1}{X} = (1 + (a + b)R + (a^2 + ab + b^2)R^2 + \cdots) \times (1 + bR + b^2R^2 + b^3R^3 + \cdots)\]

\[Y = 1 + (a + b)R + (a^2 + ab + b^2)R^2 + (a^3 + ab^2 + b^3)R^3 + \cdots\]
**Analysis of Cyclic Systems: Equivalent forms**

This construction shows why the persistent response is not geometric.

\[
\frac{Y}{X} = 1 + (a + b)R + (a^2 + ab + b^2)R^2 + (a^3 + a^2b + ab^2 + b^3)R^3 + \ldots
\]

Growth results because the number of terms is increasing linearly.

\[
y[0] = 1
\]
\[
y[1] = 0.7 + 0.9 = 1.6
\]
\[
y[2] = 0.7^2 + 0.7 \times 0.9 + 0.9^2 = 0.49 + 0.63 + 0.81 = 1.93
\]

However, the magnitudes of the terms decrease geometrically with \( n \).

\[
y[8] = \sum_{k=0}^{8} 0.7^{8-k}0.9^k \approx 1.74
\]

More formally, \( y[n] < (n+1)(0.9)^n \), which is the product of \((n+1)\), which increases with time, with \((0.9)^n\), which decreases with time.

---

**Equivalent Systems**

The sum of simpler parts suggests a parallel implementation.

\[
\frac{Y}{X} = \frac{4.5}{1 - 0.9R} - \frac{3.5}{1 - 0.7R}
\]

This representation was derived using partial fraction expansion of the system functional.

It would be difficult (or impossible) to derive this representation directly from the block diagram or difference equation.
**Equivalent Systems**

The persistent response is the sum of geometric sequences.

If \( x[n] = \delta[n] \) then \( y_1[n] = 0.9^n \) and \( y_2[n] = 0.7^n \) for \( n \geq 0 \).

Thus, \( y[n] = 4.5(0.9)^n - 3.5(0.7)^n \) for \( n \geq 0 \).

---

**Persistent Responses in More Complicated Systems**

Systems that can be represented by linear difference equations with constant coefficients have functionals that are the ratio of polynomials in \( R \).

\[
y[n] + a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] + \cdots = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + \cdots
\]

\[
(1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots) Y = (b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots) X
\]

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots}
\]

**Rational Polynomial:** ratio of two polynomials
Persistent Responses in More Complicated Systems

Rational polynomials can be realized with block diagrams of the following form:

\[
\begin{align*}
Y &= b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots \\
\frac{X}{R} &= 1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots
\end{align*}
\]

Modes can be identified by expanding system functional in partial fractions.

\[
\begin{align*}
Y &= \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots} \\
\frac{X}{R} &= \frac{C_0}{1 - p_0 R} + \frac{C_1}{1 - p_1 R} + \frac{C_2}{1 - p_2 R} + \cdots + D_0 + D_1 R + D_2 R^2 + \cdots
\end{align*}
\]

One persistent mode of the form \(p_i^n\) arises from each factor of the denominator.

Modal decomposition provides an alternative block diagram representation.

The upper part is cyclic; the lower part is acyclic.
**Persistent Responses: Check Yourself**

How many of the outputs $y_1[n]$, $y_2[n]$, and $y_3[n]$ will persist indefinitely when the input is a unit sample ($x[n] = \delta[n]$) and the system is initially at rest?

$$w[n] = x[n] + \frac{1}{2}x[n-1]$$

$$y_1[n] = w[n] + \frac{1}{2}w[n-1]$$

**Summary**

Feedback occurs when the value of a signal at one time depends on a previous value of the same signal.

Feedback corresponds to cyclic signal paths in block diagrams.

When a system has feedback, transient inputs can give rise to persistent outputs called modes.

Modes of systems that can be described linear difference equations with constant coefficients are weighted sums of geometric sequences.
Last Time

Systems that can be represented by linear difference equations with constant coefficients have functionals that are the ratio of polynomials in $R$.

\[
y[n] + a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] + \cdots = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + \cdots
\]

\[
(1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots) Y = (b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots) X
\]

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots}
\]

Rational Polynomial: ratio of two polynomials

Block Diagram Representation of Difference Eq.

Block diagram illustrates explicit dependence of $y[n]$ on $x[n]$ and on right-shifted versions of $x[n]$ and $y[n]$. 

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots}
\]
Modal Decomposition

Modes can be identified by expanding system functional in partial fractions.

\[ \frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots} \]

Factor denominator:

\[ \frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R)\cdots} \]

Partial fractions:

\[ \frac{Y}{X} = \frac{C_0}{1 - p_0 R} + \frac{C_1}{1 - p_1 R} + \frac{C_2}{1 - p_2 R} + \cdots + \frac{D_0}{1 - p_0 R} + \frac{D_1 R}{1 - p_1 R} + \frac{D_2 R^2}{1 - p_2 R} + \cdots \]

One persistent mode of the form \( p_k^n \) arises from each of the \( k \) factors of the denominator.

Modal Decomposition

Modal decomposition provides an alternative block diagram representation.

Each mode is explicit in this parallel representation.

Repeated Roots

What if two of the roots are the same?

Example:

\[ \frac{Y}{X} = \frac{1}{(1 - p_0 R)(1 - p_1 R)} = \frac{C_1}{1 - p_0 R} + \frac{C_2}{1 - p_1 R} = \frac{C_1 + C_2}{1 - p_0 R} \]

It doesn’t make sense that two roots, even if equal, create only one partial fraction. What’s going on?
Repeated Roots

To understand what happens when two roots are equal, consider the limiting behavior as two nearly equal roots approach each other.

\[
\frac{Y}{X} = \frac{1}{(1 - pR)(1 - (p + \epsilon)R)} = \frac{1}{\epsilon} \left( \frac{p + \epsilon}{1 - (p + \epsilon)R} - \frac{p}{1 - pR} \right)
\]

There are two modes that, for small \( \epsilon \), are nearly identical:

\[a[n] = \frac{p + \epsilon}{\epsilon} (p + \epsilon)^n \quad \text{and} \quad b[n] = \frac{p}{\epsilon} p^n; \quad n \geq 0.\]

Both are geometric sequences: slightly different amplitudes, slightly different bases.

Repeated Roots

As \( \epsilon \to 0 \), the amplitudes \( \to \infty \), but \( a[n] - b[n] \) is well behaved.

\[a[n] = \frac{p + \epsilon}{\epsilon} (p + \epsilon)^n \quad \text{and} \quad b[n] = \frac{p}{\epsilon} p^n.\]

\( p = 0.7, \epsilon = 0.1 \)

\[
\begin{array}{cccccc}
a[n] & b[n] & a[n] - b[n] \\
8.00 & 6.40 & 5.12 & 4.10 & 3.28 & 2.62 \\
7.00 & 4.90 & 3.43 & 2.40 & 1.68 & 1.18 \\
1.00 & 1.50 & 1.69 & 1.69 & 1.60 & 1.44 \\
\end{array}
\]

\( p = 0.7, \epsilon = 0.01 \)

\[
\begin{array}{cccccc}
a[n] & b[n] & a[n] - b[n] \\
71.00 & 50.41 & 35.79 & 25.41 & 18.04 & 12.81 \\
70.00 & 49.00 & 34.30 & 24.01 & 16.81 & 11.76 \\
1.00 & 1.41 & 1.49 & 1.40 & 1.24 & 1.05 \\
\end{array}
\]

Repeated Roots

As \( \epsilon \to 0 \), the amplitudes \( \to \infty \), but \( a[n] - b[n] \) is well behaved.

\[a[n] = \frac{p + \epsilon}{\epsilon} (p + \epsilon)^n \quad \text{and} \quad b[n] = \frac{p}{\epsilon} p^n.\]

\( p = 0.7, \epsilon = 0.01 \)

\[
\begin{array}{cccccc}
a[n] & b[n] & a[n] - b[n] \\
71.00 & 50.41 & 35.79 & 25.41 & 18.04 & 12.81 \\
70.00 & 49.00 & 34.30 & 24.01 & 16.81 & 11.76 \\
1.00 & 1.41 & 1.49 & 1.40 & 1.24 & 1.05 \\
\end{array}
\]

\( p = 0.7, \epsilon = 0.001 \)

\[
\begin{array}{cccccc}
a[n] & b[n] & a[n] - b[n] \\
701.00 & 491.40 & 344.47 & 241.47 & 169.27 & 118.66 \\
700.00 & 490.00 & 343.00 & 240.10 & 168.07 & 117.65 \\
1.00 & 1.40 & 1.47 & 1.37 & 1.20 & 1.01 \\
\end{array}
\]

The difference \( a[n] - b[n] \) is not geometric: it’s a new mode.
Repeated Roots

Repeated roots introduce a new mode that is not geometric. The new mode is the response of two subsystems in cascade.

\[
\frac{Y}{X} = \frac{1}{(1-pR)(1-pR)} = \left(\frac{1}{1-pR}\right)\left(\frac{1}{1-pR}\right) = (1 + pR + p^2R^2 + \cdots) \times (1 + pR + p^2R^2 + \cdots)
\]

\[
= 1 + 2pR + 3p^2R^2 + 4p^3R^3 + \cdots
\]

Repeated Roots

Use the block diagram representation to visualize signal flow paths.

\[
\frac{Y}{X} = (1 + pR + p^2R^2 + p^3R^3 + \cdots) \times (1 + pR + p^2R^2 + p^3R^3 + \cdots)
\]

\[
\frac{Y}{X} = 1 + 2pR + 3p^2R^2 + 4p^3R^3 + \cdots = \sum_{n=0}^{\infty} (n+1)p^nR^n
\]

Multiplying Polynomials

Tables help visualize the multiplication of high-order polynomials.

\[(1 + pR + p^2R^2 + p^3R^3 + \cdots) \times (1 + pR + p^2R^2 + p^3R^3 + \cdots)\]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>pR</th>
<th>p^2R^2</th>
<th>p^3R^3</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>pR</td>
<td>p^2R^2</td>
<td>p^3R^3</td>
<td>...</td>
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<td>pR</td>
<td>pR</td>
<td>p^2R^2</td>
<td>p^3R^3</td>
<td>p^4R^4</td>
<td>...</td>
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<tr>
<td>p^2R^2</td>
<td>p^3R^3</td>
<td>p^4R^4</td>
<td>p^5R^5</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>p^3R^3</td>
<td>p^4R^4</td>
<td>p^5R^5</td>
<td>p^6R^6</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

Group same powers of \(R\) by following reverse diagonals:

\[
1 + (p + p)R + (p^2 + p^2)R^2 + (p^3 + p^3 + p^3)R^3 + \cdots
\]

\[
= \sum_{n=0}^{\infty} (n+1)p^nR^n
\]
Repeated Roots

When expanding repeated roots with partial fractions, include both the repeated and unrepeated modes.

Example:

\[ Y = \frac{R}{(1-pR)^2} = \frac{C_2}{(1-pR)^2} + \frac{C_2}{1-pR} \]

\[ = \frac{1}{p} \left( \frac{1}{(1-pR)^2} - \frac{1}{1-pR} \right) \]

The unit sample applied to this system (initially at rest) excites two modes:

\[ y[n] = \frac{1}{p} \sum_{n=0}^{\infty} (n+1)p^n - \frac{1}{p} \sum_{n=0}^{\infty} p^n ; n \geq 0 \]

Repeated Roots: Check Yourself

What is the shape of the new mode that results from a triply-repeated root?

1. \( 1 + 2pR + 3p^2R^2 + 4p^3R^3 + \cdots \)
2. \( 1 + 3pR + 5p^2R^2 + 7p^3R^3 + \cdots \)
3. \( 1 + 3pR + 6p^2R^2 + 10p^3R^3 + \cdots \)
4. \( 1 + 4pR + 9p^2R^2 + 16p^3R^3 + \cdots \)
5. none of the above

Multiplying Polynomials

Collecting terms with equal delays is called **convolving**.

\[ (p_0 + p_1R + p_2R^2 + p_3R^3 + \cdots) \times (q_0 + q_1R + q_2R^2 + q_3R^3 + \cdots) \]

Convolution sum:

\[ r_n = \sum_{m=0}^{n} p_m q_{n-m} \]
Modal Decomposition with Repeated Modes

Include all orders of repeated modes in modal decompositions.

Example: \( p_0 \) is repeated twice

\[
\frac{Y}{X} = b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots \frac{1}{(1 - p_0 R)(1 - p_0 R)(1 - p_1 R)} \cdots
\]

Partial fractions:

\[
\frac{Y}{X} = \frac{C_0}{1 - p_0 R} + \frac{C_1}{(1 - p_0 R)^2} + \frac{C_2}{1 - p_1 R} + \cdots + D_0 + D_1 R + D_2 R^2 + \cdots
\]

Two modes for a repeated root.

Each mode is explicit in this parallel representation.

Complex Roots

What if a root has a non-zero imaginary part?

Factor theorem: express a polynomial as a product of factors, with one factor associated with each root of the polynomial.

Fundamental theorem of algebra: a polynomial of order \( n \) has \( n \) roots. The roots can have imaginary parts.

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots}
\]

Factor denominator:

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R)} \cdots
\]

Partial fractions:

\[
\frac{Y}{X} = \frac{C_0}{1 - p_0 R} + \frac{C_1}{1 - p_1 R} + \frac{C_2}{1 - p_2 R} + \cdots + D_0 + D_1 R + D_2 R^2 + \cdots
\]

How does a mode from a complex root behave?
Complex Roots

Complex-valued roots produce complex-valued modes. Because modes are geometric series with ratio \( pR \),
\[
\frac{1}{1 - pR} = 1 + pR + p^2R^2 + \ldots + p^nR^n + \ldots
\]
it is convenient to express the base \( p \) of a complex-valued mode in polar form. Let \( p = re^{j\Omega} \). Then
\[
\frac{1}{1 - re^{j\Omega}R} = 1 + re^{j\Omega}R + r^2e^{2j\Omega}R^2 + \ldots
\]
Magnitude of samples is a geometric sequence with ratio \(|p|\).
Phase angle of the samples grows linearly with time.

Complex Roots

An isolated complex root can result only from a difference equation with complex-valued coefficients.
Example:
\[
Y = \frac{1}{1 - re^{j\Omega}R}X
\]
Corresponding difference equation:
\[
y[n] - re^{j\Omega}y[n - 1] = x[n]
\]

Complex Roots

Difference equations that represent physical systems (e.g., population growth, bank accounts, making sherry) or discrete approximations to physical CT systems have real-valued coefficients.

Bank account with interest
\[
y[n] = (1 + r)y[n - 1] + x[n]
\]
Forward-Euler approximation to leaky tank
\[
r_2[n + 1] = \left(1 - \frac{T}{\tau}\right)r_2[n] + \frac{T}{\tau}r_0[n]
\]
Difference equations with real-valued coefficients generate real-valued outputs from real-valued inputs.
**Complex Roots**

If $p$ is a root of a polynomial with constant real-valued coefficients, then its complex-conjugate $p^*$ is also a root.

**Proof.** Let $D(R)$ represent a polynomial in $R$ with constant real-valued coefficients.

If $p$ is a root of $D(R)$ then $D(p) = 0$.

Since all of the coefficients are real-valued,

$$D(p^*) = (D(p))^* = 0^* = 0.$$  

Thus $p^*$ is also a root.

---

**Complex Roots**

If we pair the factors corresponding to complex-conjugate roots, the resulting polynomial has real-valued coefficients.

$$Y = \frac{1}{(1 - re^{j\Omega R})(1 - re^{-j\Omega R})} = \frac{1}{1 - 2r \cos \Omega R + r^2 \Omega^2 R^2}$$

---

**Complex Roots**

Each of the complex-valued roots corresponds to a complex-valued mode.

$$M_1 = \frac{1}{1 - re^{j\Omega R}} = 1 + r e^{j\Omega R} + r^2 e^{2j\Omega R} + \cdots$$

$$M_2 = \frac{1}{1 - re^{-j\Omega R}} = 1 + r e^{-j\Omega R} + r^2 e^{-2j\Omega R} + \cdots$$

Notice that one mode is the complex conjugate of the other.
Complex Roots

Real-valued inputs always excite combinations of these modes so that the imaginary parts cancel.

Consider the impulse response of the following system.

\[
\frac{Y}{X} = \frac{1}{(1 - re^{j\Omega R})(1 - re^{-j\Omega R})} = \frac{1}{2j\sin\Omega} \left( \frac{e^{j\Omega}}{1 - re^{j\Omega R}} - \frac{e^{-j\Omega}}{1 - re^{-j\Omega R}} \right) = \frac{e^{j\Omega}}{2j\sin\Omega} M_1 + \frac{-e^{-j\Omega}}{2j\sin\Omega} M_2
\]

The modes themselves are complex conjugates, and their coefficients are also complex conjugates. So the sum is a sum of something and its complex conjugate, which is real.

Here is what remains when the imaginary parts cancel.

\[
\frac{Y}{X} = \frac{e^{j\Omega}}{2j\sin\Omega} M_1 + \frac{-e^{-j\Omega}}{2j\sin\Omega} M_2
\]

This signal is a sinusoid, from \(\sin((n+1)\Omega)\), whose amplitude decays or grows due to the factor of \(r^n\).

The frequency is \(\Omega\) (with dimensions of angle/sample).

\[
y[n] = \frac{\sin 0.5(n+1)}{\sin 0.5} \times 0.97^n
\]

\[
y[n] = \frac{\sin 0.2(n+1)}{\sin 0.2} \times 1.02^n
\]
Complex Roots: Check Yourself

How many of the following statements are true?

1. This system has 3 modes.
2. All of the modes can be written as geometrics.
3. Unit-sample response is $y[n] : 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
4. Unit-sample response is $y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
5. One of the modes of this system is the unit step.
Multiple Representations of Discrete-Time Systems

Now you know four representations of discrete-time systems.

Verbal descriptions: preserve the underlying physics.

“To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.”

Difference equations: mathematically compact.

\[ y[n] = x[n] - x[n-1] \]

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.

\[ Y = (1 - \mathcal{R}) X \]

Last Time

The persistent response of a complicated system to a transient input signal can be decomposed into a sum of simpler parts called modes.

\[
\frac{Y}{X} = \frac{b_0 + b_1 \mathcal{R} + b_2 \mathcal{R}^2 + b_3 \mathcal{R}^3 + \cdots}{1 + a_1 \mathcal{R} + a_2 \mathcal{R}^2 + a_3 \mathcal{R}^3 + \cdots}
\]

Factor denominator:

\[
\frac{Y}{X} = \frac{b_0 + b_1 \mathcal{R} + b_2 \mathcal{R}^2 + b_3 \mathcal{R}^3 + \cdots}{(1 - p_0 \mathcal{R})(1 - p_1 \mathcal{R})(1 - p_2 \mathcal{R})(1 - p_3 \mathcal{R}) \cdots}
\]

Partial fractions:

\[
\frac{Y}{X} = \frac{C_0}{1 - p_0 \mathcal{R}} + \frac{C_1}{1 - p_1 \mathcal{R}} + \frac{C_2}{1 - p_2 \mathcal{R}} + \cdots + D_0 + D_1 \mathcal{R} + D_2 \mathcal{R}^2 + \cdots
\]
Modal Decomposition

The sum corresponds to parallel paths through a block diagram representation of the system.

Today: Poles and Zeros

Cascade (series) decomposition: poles and zeros.

Factoring the numerator and denominator of the system functional breaks the system into parts that are multiplied together.

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots}
\]

\[
\frac{Y}{X} = \frac{G_1 (1 - z_0 R)(1 - z_1 R)(1 - z_2 R)(1 - z_3 R) \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots}
\]

Poles and Zeros

We can think of the factors as subsystems that are connected in series.

\[
\frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots}
\]

\[
\frac{Y}{X} = \frac{G_1 (1 - z_0 R)(1 - z_1 R)(1 - z_2 R)(1 - z_3 R) \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots}
\]

cascade subsystems ↔ multiply subsystem functionals
Divide and conquer: figure out how each of the subsystems work, then cascade the subsystems to describe how the system works.

\[ X \xrightarrow{G} \frac{1}{1-p_0R} \xrightarrow{\frac{1}{1-p_1R}} \cdots \xrightarrow{(1-z_0R)} \xrightarrow{(1-z_1R)} \cdots Y \]

The gain \( G \) is easy. It produces an output signal that is \( G \) times the input signal.

\[ Y = GX \]

Each denominator factor contributes a cyclic signal flow path. Therefore, transient inputs can give rise to a persistent response called a mode.

\[ x[n] = \delta[n] \]

The unit sample response is a geometric sequence.
Poles and Zeros: Numerator Sections

Each numerator factor contributes an acyclic signal path. Therefore, the response to a transient signal is transient.

\[ 1 - z_0 R \]

\[ x[n] = \delta[n] \]
\[ y[n] = \delta[n] - \frac{1}{2} \delta[n - 1] \]

Poles and Zeros: Numerator Sections

Numerator and denominator sections have similar functional descriptions, with the roles of input and output exchanged.

\[ (1 - p_0 R) Y = X \]

\[ Y = (1 - z_0 R) X \]

If the unit sample response of a denominator section is a geometric sequence, then the response of a numerator section to a geometric sequence should be a unit sample!

Poles and Zeros: Numerator Sections

Calculate the response of a numerator section to a geometric sequence with base \( z_0 \).

\[ Y = (1 - z_0 R) X \]

\[ X : 1 \ z_0 \ z_0^2 \ z_0^3 \ ... \]
\[ z_0 RX : 0 \ z_0 \ z_0^2 \ z_0^3 \ ... \]
\[ Y = X - z_0 RX : 1 \ 0 \ 0 \ 0 \ ... \]
**Poles and Zeros: Numerator Sections**

The numerator section has a transient response to a persistent input! It "eats" geometric sequences with base $z_0$.

\[ Y = (1 - z_0 R) X \]

- **Gain**: $G$
- **Zeros**: roots of the numerator
- **Poles**: roots of the denominator

---

**Poles and Zeros: Check Yourself**

Design a system that turns the input signal

X : 1, 2, 3, 4, ...

into output signal

Y : 1, 0, 0, 0, ...

---

**Poles and Zeros**

Each of the boxes in the cascaded representation can be represented by a single number.

\[ Y = \frac{(1 - z_0 R)(1 - z_1 R)(1 - z_2 R)(1 - z_3 R) \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots} \]

- **Gain**: $G$
- **Zeros**: roots of the numerator
- **Poles**: roots of the denominator

---
Poles and Zeros

Characterize a system with a handful of numbers.

Example: Fibonacci System

\[ y[n] = y[n-1] + y[n-2] + x[n] \]

Factor the system functional,

\[ \frac{Y}{X} = \frac{1}{1 - R - R^2} = \frac{1}{(1 - \phi R)(1 + \frac{1}{\phi} R)} \]

where \( \phi = (1 + \sqrt{5})/2 \).

\[ G = 1 \]
\[ p_0 = \phi \]
\[ p_1 = -\frac{1}{\phi} \]

Pole-Zero Diagrams

Represent the poles and zeros on the z-plane to graphically depict behaviors of modes.

Example: Fibonacci System

\[ p_0 = \phi \approx 1.618 \]
\[ p_1 = -\frac{1}{\phi} \approx -0.618 \]
Pole-Zero Diagrams

Poles inside/outside the unit circle correspond to modes with amplitudes that decrease/increase with time.

Example: Fibonacci System

\[ p_0 = \phi \approx 1.618 \quad p_1 = -\frac{1}{\phi} \approx -0.618 \]

\[ \frac{-1}{\phi} \]

\[ \phi \]

\[ \text{Re } z \]

\[ \text{Im } z \]

\[ \text{z-plane} \]

Example 2:

\[ + \]

\[ R \quad R \quad R \quad R \quad R \]

\[ X \]

\[ Y \]

\[ \text{Re } z \]

\[ \text{Im } z \]

\[ \text{z-plane} \]

Pole-Zero Diagrams

Poles on/off the positive real axis correspond to modes that are monotonic/oscillatory in time.
### Pole-Zero Diagrams

Poles on/off the positive real axis correspond to modes that are monotonic/oscillatory in time.

**Example: Fibonacci System**

$$z_0 = \phi \approx 1.618$$

$$z_1 = -\frac{1}{\phi} \approx -0.618$$

![Pole-Zero Diagram](image)

---

### Poles and Zeros: Check Yourself

Determine the following numbers:

1. number of complex-valued modes in this system
2. number of oscillatory modes in this system
3. number of growing modes
4. number of decaying modes

---

### Multiple Representations of Discrete-Time Systems

Now you know five representations of discrete-time systems.

**Verbal descriptions:** preserve the underlying physics.

"... record the first number, and then record successive differences."

**Difference equations:** mathematically compact.

$$y[n] = x[n] - x[n-1]$$

**Block diagrams:** illustrate signal flow paths.

**Operator representations:** analyze systems as polynomials.

$$Y = (1 - R) X$$

**Pole-Zero diagrams:** show factors of the system functional.
What's the unit-sample response for this system:

\[
\frac{Y}{X} = \left( \frac{1}{1 - R^3} \right)^2
\]
Feedback and Control

Feedback is pervasive in natural and artificial systems.

Turn steering wheel to stay centered in the lane.

Feedback and Control

Feedback is useful for regulating a system’s behavior, as when a thermostat regulates the temperature of a house.
Feedback and Control

Concentration of glucose in blood is highly regulated and remains nearly constant despite episodic ingestion and use.

Motor control relies on feedback from pressure sensors in the skin as well as proprioceptors in muscles, tendons, and joints.

Try building a robotic hand to unscrew a lightbulb!

Shadow Dexterous Robot Hand (Wikipedia)

Today's goal: use systems theory to gain insight into how to control a system.
**Feedback and Control**

Steering a car: controlling its lateral position.

Algorithm: steer left when right of center and vice versa.

- straight ahead?
- steer right
- steer right
- steer right
- straight ahead?
- steer left
- steer left

**Feedback and Control**

Bad algorithm → poor performance.

Here we get persistent oscillations!

- straight ahead?
- steer right
- steer right
- steer right
- straight ahead?
- steer left
- steer left

**Feedback and Control**

The persistent oscillation is a mode of the system.

Control almost always involves feedback.

Feedback → cyclic paths → persistent outputs (modes).

Studying the behaviors of modes helps you design control algorithms.
Example

Guide a person to walk down the center of a lane by telling how far to move left/right on next step.

Example

First consider how the person processes motion commands.

- 1. command rightward +1
- 2. command rightward +1
- 3. command rightward 0
- 4. command rightward 0

Motion commands “accumulate” to give person's position.

\[ P = \frac{R}{1 - R} \]

Example

Now add feedback:

- Observe current position \( p[n] \).
- Tell person to move rightward by \(-p[n]\).

- 1. see −1: command rightward +1
- 2. see 0: command rightward 0
- 3. see 0: command rightward 0

Easy.
Example

What if you cannot see the current position, but only the prior position.

1. see previous (−1) → command +1
2. see previous (0) → command 0
3. see previous (+1) → command −1
4. see previous (+1) → command −1
5. see previous (0) → command 0

Introducing delay can destabilize a control system.
Key issue in biological and artificial control systems.

Feedback and Control

Make model to understand destabilizing effect of delay.

Check Yourself: Which is True?

1. \( \frac{Y}{X} = \frac{\alpha R}{1 - R} \)
2. \( \frac{Y}{X} = \frac{\alpha R}{1 - R + \alpha R^2} \)
3. \( \frac{Y}{X} = \frac{\alpha R}{1 + R - \alpha R^2} \)
4. \( \frac{Y}{X} = \frac{\alpha R}{1 - R - \alpha R} \)
Feedback and Control

To find the modes of this system, put denominator of system functional in the form \((1 - p_0 R)(1 - p_1 R)\), then the modes are at \(z = p_0\) and \(z = p_1\).

\[
\begin{align*}
\frac{Y}{X} &= \frac{aR}{1 - R + aR^2} = \frac{aR}{(1 - p_0 R)(1 - p_1 R)} \\
1 - R + aR^2 &= 1 - (p_0 + p_1)R + p_0 p_1 R^2 \\
p_0, p_1 &= \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \alpha}
\end{align*}
\]

Feedback and Control: Persistent Responses

If \(\alpha\) is small, the modes occur at \(z \approx \alpha\) and \(z \approx 1 - \alpha\).

\[
p_0, p_1 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \alpha} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4\alpha}\right) \approx \frac{1}{2} \left(1 \pm (1 - 2\alpha)\right) = 1 - \alpha, \alpha
\]

As \(\alpha\) increases, the modes move toward each other and collide at \(z = 1/2\) when \(\alpha = 1/4\).

\[
p_0, p_1 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \alpha} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \frac{1}{4}} = \frac{1}{2} \pm \frac{1}{2}
\]

Persistent responses decay. Position of person \(\rightarrow 0\).
If $\alpha > \frac{1}{4}$, the modes become complex.

$$p_0, p_1 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \alpha} = \frac{1}{2} \pm j\sqrt{\alpha - \frac{1}{4}}$$

Complex modes $\rightarrow$ oscillations.

We saw this behavior when we were guiding the person.

1. see previous $(-1)$ $\rightarrow$ command $+1$
2. see previous $(0)$ $\rightarrow$ command $0$
3. see previous $(+1)$ $\rightarrow$ command $-1$
4. see previous $(+1)$ $\rightarrow$ command $-1$
5. see previous $(0)$ $\rightarrow$ command $0$

Try using less feedback.

The pole-zero diagram shows why we got oscillations.

We tried to correct the error all at once: $\alpha = 1$.

Too much feedback!

Try using less feedback.
Change the control algorithm, command smaller steps: 
\( \alpha = 0.5 \).

1. see previous (−1) → command +1/2
2. see previous (0) → command 0
3. see previous (+1/2) → command −1/4
4. see previous (+1/4) → command −1/8
5. see previous (+1/8) → command −1/16

Original “guide the person problem” had no sensor delay.

Illustrate how the pole position(s) depend on \( \alpha \).

Feedback is useful for controlling systems. Feedback → cyclic signal paths → persistent outputs (modes). Pole-zero plots help visualize how modes depend on amount of feedback.
Feedback and Control: Robotic Driver

Design a system to automatically steer a car.

A moving car, velocity $V$

$p$

Assume a sensor reports position $p$ within the lane:

- $p = 0$: in center
- $p > 0$: right of center
- $p < 0$: left of center
Feedback and Control: Robotic Driver

Use feedback to position the car.
Let $X$ represent the desired position in the lane (normally 0). Turn the steering wheel ($\phi$) in proportion to the difference between the desired and current positions.

$$X \xrightarrow{\alpha} \phi \xrightarrow{\text{steering system}} P$$

What’s the continuous-time relation between $\phi$ and $P$?
What’s the corresponding difference equation?

Robotic Driver: Check Yourself

Find the continuous-time relation between $\phi$ and $\theta$.

1. $\theta \propto \phi$
2. $\dot{\theta} \propto \phi$
3. $\theta \propto \dot{\phi}$
4. $\phi \propto \sin \theta$

Feedback and Control: Robotic Driver

Let $\phi$ represent the angle of the steering wheel. If you hold $\phi$ constant, you will drive in circles. The angle of the car, $\theta$, increases with time at a constant rate, proportional to $\phi$.

$$\theta[n] = \theta[n-1] + \beta \phi[n-1]$$
$$\Theta = R\Theta + \beta R\Phi$$
$$\Theta = \beta R$$
$$\Phi = \frac{1}{1 - R}$$
Feedback and Control: Robotic Driver

If the angle of the car ($\theta$) is not zero, then the position of the car within the lane ($p$) increases linearly with time.

\[
p[n] = p[n-1] + \gamma' V \sin \theta[n-1] \\
\approx p[n-1] + \gamma' V \theta[n-1]
\]

\[
P = \gamma R p + \gamma R \Theta \\
P = \gamma R \\
\Theta = \gamma R \frac{1}{1 - R}
\]

\[
Y = P = \gamma R \Theta = \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} \Theta = a \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} E = a \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} (X - Y)
\]

Solving:

\[
Y = \frac{a \beta \gamma R^2}{1 - 2R + (1 + \alpha \beta \gamma) R^2} = \frac{KR^2}{1 - 2R + (1 + K) R^2}
\]

where $K = a \beta \gamma$.

Feedback and Control: Robotic Driver

To find the modes, put denominator of system functional in the form $(1 - p_0 R)(1 - p_1 R)$; then the modes are at $z = p_0$ and $z = p_1$.

\[
Y = \frac{KR^2}{1 - 2R + (1 + K) R^2} = \frac{KR^2}{(1 - p_0 R)(1 - p_1 R)}
\]

$p_0, p_1 = 1 \pm j \sqrt{K}$
**Feedback and Control: Robotic Driver**

If \( K = 0 \), there is a double pole at \( z = 1 \).

\[
\frac{Y}{X} = \frac{KR^2}{1 - 2R + (1 + K)R^2} = \frac{KR^2}{(1 - p_0 R)(1 - p_1 R)}
\]

\( p_0, p_1 = 1 \pm j\sqrt{K} \)

The output diverges – the system is “unstable.”

---

**Feedback and Control: Robotic Driver**

If \( K = 1 \), there are complex poles at \( z = 1 \pm j \).

\[
\frac{Y}{X} = \frac{KR^2}{1 - 2R + (1 + K)R^2} = \frac{KR^2}{(1 - p_0 R)(1 - p_1 R)}
\]

\( p_0, p_1 = 1 \pm j\sqrt{K} \)

The output oscillates and diverges – the system is “unstable.”

---

**Feedback and Control: Robotic Driver**

No values of \( K = \alpha \beta \gamma \) result in acceptable performance.

Need a better controller.

The system is the cascade of two accumulators.

Proportional control was fine for a system with one accumulator ("guiding person" system in last lecture).
Feedback and Control: Robotic Driver

Try a difference controller ("derivative control").

\[
Y = P = \frac{\gamma R}{1 - R} \Theta = \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} \Phi = a(1 - R) \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} E = \frac{a \beta \gamma R^2}{1 - R} (X - Y)
\]

Solving:
\[
\frac{Y}{X} = \frac{KR^2}{1 - R + KR^2} \quad \text{where} \quad K = a \beta \gamma
\]

Feedback and Control: Robotic Driver

This is the same functional that we found for the “guiding person” problem (last lecture). The modes of the closed-loop system are stable for \(0 < K < 1\).

Feedback and Control: Robotic Driver

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Feedback and Control: Robotic Driver

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Feedback and Control: Robotic Driver

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Feedback and Control: Robotic Driver

But can we really cancel the pole at $z = 1$ with a zero at $z = 1$?

Try step-by-step analysis of the block diagram.

Feedback and Control: Robotic Driver

Try step-by-step analysis of the block diagram.

<table>
<thead>
<tr>
<th>$\alpha$ = 0.5</th>
<th>$\beta$ = $\gamma$ = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{\theta}[0]$ = 0.1</td>
<td>$p[0]$ = 0.5</td>
</tr>
</tbody>
</table>

With time, the value of $\theta$ goes to zero, but the value of $p$ does not.
Feedback and Control: Robotic Driver

The angle $\theta$ goes to zero, but the position $p$ does not.

This suggests that the closed loop response has a pole at $z = 1$ and a corresponding mode $1^n$, $n \geq 0$.

Feedback and Control: Robotic Driver

But this is not consistent with our analysis.

Feedback and Control: Robotic Driver

If $K = 0.5$, the poles are at $z = \frac{1}{2} \pm \frac{1}{2}j$.

These poles correspond to decaying modes.
**Feedback and Control: Robotic Driver**

Is the persistent value of $p$ caused by pole-zero cancellation?

\[
\begin{align*}
\alpha(1-R) & \beta R \\
1-R & 1-R \\
-1 & X \ Y E \ \Phi \ \Theta \ P
\end{align*}
\]

controller

steering system

sensor

\[
\begin{align*}
+K R & -1 \\
+ & + R + R \\
-1 & X \ Y E \ \Theta \ P
\end{align*}
\]

**Feedback and Control: Robotic Driver**

Analyse of the block diagram that results when we remove the parts that correspond to the pole and cancelling zero (red).

\[
\begin{align*}
p[0] = 0.5 & \quad a = 0.5 \quad \beta = \gamma = 1 \\
\theta[0] = 0.1 & \quad n
\end{align*}
\]

Now the value of $p$ also goes to zero with time.

**Cancelling a Pole with a Zero**

We did a step-by-step analysis of block diagrams for systems with the following open-loop poles and zeros:

- $z_0 = 1$ and $p_0, p_1 = 1$
- $p_0 = 1$

and we got different answers. This suggests that you cannot cancel a pole at 1 with a zero at 1.

What’s going on?

Consider some simpler systems.
Cancelling a Pole with a Zero

Accumulate and difference.

\[ W = X + RW \]
\[ (1 - R)W = X \]
\[ Y = (1 - R)W \]
\[ Y = X \]

The pole and zero cancelled.

---

Cancelling a Pole with a Zero

Difference then accumulate.

\[ W = (1 - R)X \]
\[ Y = W + RY \]
\[ (1 - R)Y = W \]
\[ (1 - R)Y = (1 - R)X \]

The pole and zero almost cancelled.

---

Cancelling a Pole with a Zero

If the system is initially at rest, then \( Y = X \).

\[ (1 - R)X = (1 - R)Y \]
\[ y[0] = x[0] = 0 \]
\[ \cdots \]
Cancelling a Pole with a Zero

If the system is not initially at rest, then \( Y \) and \( X \) are not always equal.

\[
(1 - R)X = (1 - R)Y
\]

\[
y[0] = 5 \quad x[0] = 0
\]

\[
\]

\[
\]

...
Check yourself: Will feedback make $\theta \to 0$?

Feedback and Control: Robotic Driver

Derivative feedback never eliminates the constant outputs. Therefore, derivative controllers are not good for feedback systems that are intended to control position.

Need an even better controller.

Feedback and Control: Robotic Driver

Use combination of proportional and derivative control to eliminate unwanted persistent behavior.

Solving:

\[
Y = \frac{(a_0 + a_1 - a_1 R) \beta \gamma R^2}{(1 - R)(1 - R)} (X - Y)
\]

Solving:

\[
\frac{Y}{X} = \frac{(a_0 + a_1 - a_1 R) \beta \gamma R^2}{1 - 2R + (1 + a_0 + a_1) R^2 - a_1 \beta \gamma R^3}
\]
Feedback and Control: Robotic Driver

Try step-by-step analysis of the block diagram.

\[ E = \alpha_0 + \alpha_1 (1 - R) \]
\[ \beta R \]
\[ 1 - R \]
\[ \gamma R \]
\[ 1 - R \]

controller

steering system

sensor

Feedback and Control: Summary

Now you know about two kinds of controllers:

- proportional
- proportional plus derivative

Adding delays to a loop tends to destabilize the loop.

Adding accumulators to a loop tends to destabilize the loop.

Derivative feedback can help to stabilize such loops.
Causal Systems

If the output of a system at time $n_0$ can be determined from prior values of the output and from values of the input for $n \leq n_0$, then the system is causal.

Example: Fibonacci machine

$$y[n] = y[n - 1] + y[n - 2] + x[n]$$

For $n_0 = 6$,


Causal Systems

Delay elements and the $\mathcal{R}$ operator are useful abstractions for representing causal systems.

Example: Fibonacci machine

$$y[n] = y[n - 1] + y[n - 2] + x[n]$$

$$Y = \mathcal{R}Y + \mathcal{R}^2Y + X$$
Noncausal Systems

If the output of a system at time \( n_0 \) cannot be determined from prior values of the output and from values of the input for \( n \leq n_0 \), then the system is not causal.

Example:

\[
y[n] = y[n-1] + y[n-2] + x[n+1]
\]

For \( n_0 = 6 \),

\[
\]

Anticausal Systems

This system is anticausal: its output at time \( n_0 \) can be determined from future values of the output and from values of the input for \( n \geq n_0 \).

Example:

\[
y[n] = y[n-1] + y[n-2] + x[n+1]
\]

Solve for \( y[n-2] \):

\[
y[n-2] = y[n] - y[n-1] - x[n+1]
\]

Then, for \( n_0 = 6 \),

\[
\]

Anticausal systems:

One can represent an anticausal system with a block diagram using Advance elements that are analogous to Delay elements.

Anticausal system:

\[
y[n] = y[n+1] + x[n]
\]
Step-by-Step solutions

Step-by-step analysis of Advance elements is analogous to that of Delay elements, with time proceeding backwards.

"Initial" rest: \( y[n] = 0, \ n > 0 \)

Then use the recursion:

\[
y[n] = y[n + 1] + x[n] \\
y[0] = y[1] + x[0] = 0 + 1 = 1 \\
y[-1] = y[0] + x[-1] = 1 + 0 = 1 \\
y[-2] = y[-1] + x[-2] = 1 + 0 = 1 \\
y[-3] = y[-2] + x[-3] = 1 + 0 = 1 \\
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**Anticausal Systems**

Expressions in $L$ can be expanded in series.

Example:

\[ y[n] = y[n+1] + x[n] \]

\[ Y = LY + X \]

\[ \frac{Y}{X} = \frac{1}{1-L} = 1 + L + L^2 + L^3 + \cdots \]

If traversing the cycle decreases the magnitude of the signal, then the mode will decay as time decreases!

\[ y[n] = \left(\frac{3}{4}\right)^n = \left(\frac{4}{3}\right)^n \quad n \leq 0 \]
Anticausal Systems

If traversing the cycle decreases the magnitude of the signal, then the mode will decay as time decreases!

\[ y[n] = \left(\frac{3}{4}\right)^n, \quad n \leq 0 \]
\[ y[n] = \left(\frac{4}{3}\right)^n, \quad n \geq 0 \]

Anticausal Systems

System functionals for anticausal systems can be written as a sum of simpler parts using partial fractions.

\[
Y = X + 1.6LY - 0.63L^2Y \\
(1 - 1.6L + 0.63L^2) Y = X
\]

\[
\frac{Y}{X} = \frac{1}{1 - 1.6L + 0.63L^2} = \frac{1}{(1 - 0.9L)(1 - 0.7L)} = \frac{4.5}{1 - 0.9L} - \frac{3.5}{1 - 0.7L}
\]

Anticausal Systems

The sum of simper parts suggests a parallel implementation.

\[
\frac{Y}{X} = \frac{4.5}{1 - 0.9L} - \frac{3.5}{1 - 0.7L}
\]

Each parallel path represents a mode.

Impulse response:

\[ y[n] = 4.5 \left(\frac{1}{0.9}\right)^n - 3.5 \left(\frac{1}{0.7}\right)^n, \quad n \leq 0 \]
Anticausal Systems: Summary

Analysis of anticausal systems (with advance elements and $L$) is remarkably similar to analysis of causal systems (with delays and $R$).

- Operator expressions in $L$ can be manipulated as though they are polynomials in $L$.
- Anticausal systems have persistent responses (modes) that extend infinitely into the past.
- The modes are sums of (left-sided) geometric sequences that can be determined using partial fractions.

Virtually all behaviors of $L$ operators are analogous to $R$ operators, except that time runs backwards.

But is this useful for anything in the real world?

Noncausal Systems

Example: image processing.

Think of a digital image as a stack of rows of pixels. Each row can be represented as a 1D signal:
- independent variable is space (not time)
- dependent variable is brightness

There is nothing causal or anticausal about space!

Noncausal Systems: Check Yourself

Model a blurred image as the output signal from a system whose input is an unblurred image.

What should the impulse response of such a system look like?
Noncausal Systems

Cascading a causal system with an anticausal system produces a system that is neither causal nor anticausal.

\[
\frac{Y}{X} = \frac{1}{1 - 0.8R} \times \frac{1}{1 - 0.8L}
\]

Such a system can be used to model the blurring of images.

\[
\frac{Y}{X} = \frac{1}{1 - 0.95R} \times \frac{1}{1 - 0.95L}
\]

You can get more blur by changing the positions of the poles.
Noncausal Systems: Check Yourself

Is it possible to build a system to deblur an image?
If yes, describe the system.
If no, explain why not.

Noncausal Systems

Use zeros to cancel the previous poles.

Blurring system:

\[
\begin{align*}
X & \xrightarrow{1 \ 1 - 0.8R} Y_R \\
Y_R & \xrightarrow{1 \ 1 - 0.8L} Y
\end{align*}
\]

Deblurring system:

\[
\begin{align*}
X & \xrightarrow{1 - 0.8R} Y_R \\
Y_R & \xrightarrow{1 - 0.8L} Y
\end{align*}
\]
Noncausal Systems

Such a system deblurs quite nicely.

\[
\frac{Y}{X} = (1 - 0.8R) \times (1 - 0.8L)
\]

Noncausal Systems

Even for radical blurring.

\[
\frac{Y}{X} = (1 - 0.95R) \times (1 - 0.95L)
\]

Noncausal Systems

It’s difficult to distinguish the original image from the blurred then deblurred image.

\[
\frac{Y}{X} = \frac{1}{1 - 0.95R} \times \frac{1}{1 - 0.95L} \times (1 - 0.95R) \times (1 - 0.95L)
\]
System Functions

System functions unify the rules for finding modes in causal and anticausal systems.

\[ y[n] = \left(\frac{3}{4}\right)^{-n} = \left(\frac{4}{3}\right)^n \quad n \leq 0 \]

\[ y[n] = \left(\frac{4}{3}\right)^n \quad n \geq 0 \]

---

**Example: blurring system**

\[
\frac{Y}{X} = \frac{1}{1 - 0.8R \times \frac{1}{1 - 0.8L}} = \frac{1}{1 - 0.8/z \times \frac{1}{1 - 0.8z}} = \frac{z}{(z - 0.8)(1 - 0.8z)}
\]

Poles are at the roots of the denominator: \( z = 0.8 \) and \( z = \frac{1}{0.8} \).
Summary

The $R$ operator is useful for representing causal systems.

The $L$ operator is useful for representing anticausal systems.

Some systems are neither causal nor anticausal. Such systems require both $R$ and $L$ in their functional descriptions.

Causal, anticausal, and noncausal systems can be represented by their system functions $H(z)$

$$H(z) = \frac{Y}{X} | R \leftarrow z^{-1}, R \leftarrow z$$

Poles are the roots of the denominator polynomial of $H(z)$.

Zeros are the roots of the numerator polynomial of $H(z)$.

Multiple Representations of Discrete-Time Systems

Verbal descriptions: preserve the underlying physics.

“... record the first number, and then record successive differences.”

Difference equations: mathematically compact.

$$y[n] = x[n] - x[n-1]$$

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.

$$Y = (1 - R)X$$

Pole-Zero diagrams: represent system functional in factored form.

System functions: represent systems as polynomials in $z$. 

Multiple Representations of Discrete-Time Systems

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System functions: represent systems as polynomials in \( z \).

Multiple Representations of CT Systems

Verbal descriptions of continuous time systems typically specify relations among signals (e.g., inputs and outputs) and their rates of change.

“Water flows into a tank at rate \( r_0(t) \) and flows out at a rate that is proportional to the depth of water in the tank. Determine ...”

Verbal descriptions preserve the underlying physics of both DT and CT systems.
Differential equations are mathematically compact.
\[ \dot{y}(t) - py(t) = x(t) \]
Notice that derivatives play a key role for CT systems, while time shifts played a key role for DT systems.

Block diagrams illustrate signal flow paths.
DT: block diagrams contained adders, scalers, delays, and advancers – sufficient for any system described by a linear difference equation with constant coefficients.
CT: block diagrams will contain adders, scalers, and integrators – sufficient for any system described by a linear differential equation with constant coefficients.

Why integrators and not differentiators?
One reason is –
In DT, delays (and advance elements) retain all of the “state” of a system (the minimum history (or future) that is required to iteratively solve for the next (or previous) time).
In CT, integrators serve the same function. The outputs of integrators hold the “state” in an analogous fashion.
Multiple Representations of CT Systems

We will define the \( A \) operator for functional analysis of CT systems.

Applying \( A \) to a CT signal generates a new signal that is equal to the integral of the first signal at all points in time.

\[ Y = A X \]

is equivalent to

\[ y(t) = \int_{-\infty}^{t} x(t) \, dt \]

for all time.

Check Yourself

Which of the following system functionals represents this system?

\[ x(t) \xrightarrow{\int_{-\infty}^{t}} y(t) \]

1. \( \frac{Y}{X} = \frac{A}{1 + pA} \)
2. \( \frac{Y}{X} = \frac{1}{p + A} \)
3. \( \frac{Y}{X} = \frac{A}{1 - pA} \)
4. \( \frac{Y}{X} = \frac{1}{p - A} \)

Multiple Representations of CT Systems

As with the \( R \) and \( L \) operators, \( A \) expressions can be manipulated as polynomials.

- Commutativity: \( A(1 - A)X = (1 - A)AX \)
- Distributivity: \( A(1 - A)X = (A - A^2)X \)
- Associativity: \( (1 - A)A(2 - A)X = (1 - A)(A - A)X \)
Does feedback produce persistent responses to transient inputs?

\[ y(t) = x(t) + py(t) \]

Find the impulse response.
First we have to define a CT impulse function.

The CT impulse function can be thought of as a pulse of unit area whose width is reduced to zero.

\[ \delta(t) = \lim_{\varepsilon \to 0} p\varepsilon(t) \]

Its integral is one.

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \]

It follows that the integral of the unit impulse is the unit step function.

\[ u(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
Check Yourself

Which of the following expressions gives the impulse response of this system?
\[ \dot{y}(t) = x(t) + py(t) \]

1. \( e^{pt} \), \( t \geq 0 \)
2. \( e^{-pt} \), \( t \geq 0 \)
3. \( pe^{pt} \), \( t \geq 0 \)
4. \( pe^{-pt} \), \( t \geq 0 \)

Multiple Representations of CT Systems

Feedback introduces persistent response to transient inputs.

\[ X(t) = \int_{-\infty}^{t} \cdot \, dt \]

Transient input: \( x(t) = \delta(t) \).
Persistent output: \( y(t) = e^{pt} \), \( t \geq 0 \).

\[ Y \]

\[ \frac{X}{Y} = \frac{A}{1 - pA} = A (1 + pA + p^2A^2 + p^3A^3 + \ldots) \]

If \( x(t) = \delta(t) \), then \( y(t) \) for \( t > 0 \) is a sum of
- \( A\delta(t) = 1 \) (step function)
- \( pA\delta(t) = pt \) (ramp function)
- \( p^2A\delta(t) = (pt)^2/2 \)
- \( \ldots \)
- \( p^kA^{k+1}\delta(t) = (pt)^k/k! \)
Multiple Representations of CT Systems

Determine persistent response directly from system functional.

\[
\frac{Y}{X} = \frac{\mathcal{A}}{1-p\mathcal{A}} = \mathcal{A} \sum_{k=0}^{\infty} p^k \mathcal{A}^k
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \sum_{k=0}^{\infty} p^k \mathcal{A}^{k+1} \delta(t)
\]

\[= 1 + \cdots\]
Check Yourself

Which of the following differential equations applies to the block diagram?

1. \( \dot{y}(t) - 2y(t) = x(t) \)
2. \( \dot{y}(t) + 2y(t) = x(t) \)
3. \( \dot{y}(t) - 2y(t) = -x(t) \)
4. \( \dot{y}(t) - 2y(t) = -x(t) \)

Check Yourself

Which of the following plots shows the impulse response of this differential equation?

\( \dot{y}(t) + 2y(t) = x(t) \)

Modes of Discrete-Time Systems
Multiple Representations of CT Systems

Consider the impulse responses that result for different values of $p$.

\[
\begin{align*}
Y &= \frac{A}{1 - pA} \\
\end{align*}
\]

Check Yourself

Indicate regions of the p-plane with convergent, divergent, monotonic, and non-monotonic growth.

Multiple Representations of CT Systems

**Verbal descriptions**: preserve the underlying physics.

“... the rate at which water flows from a leaky tank ...”

**Differential equations**: mathematically compact.

\[
y'(t) = x(t) + py(t)
\]

**Block diagrams**: illustrate signal flow paths.

\[
x(t) \quad \int_{-\infty}^{t} (\cdot) dt \quad y(t)
\]

**Operator representations**: analyze systems as polynomials.

\[
(1 - pA)Y = AX
\]
Multiple Representations of CT Systems

Remember from last lecture:

Verbal descriptions: preserve the underlying physics.

"... the rate at which water flows from a leaky tank ..."

Differential equations: mathematically compact.

\[ \dot{y}(t) = x(t) + py(t) \]

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.

\[ (1 - pA)Y = AX \]

Today we will look at two applications: a spring-mass system and two leaky tanks.

Example: Mass and Spring
Multiple Representations of CT Systems

\[ F = K(x(t) - y(t)) = M\ddot{y}(t) \]

\[ \frac{Y}{X} = \frac{\frac{K}{M}A^2}{1 + \frac{K}{M}A^2} \]

These complex-valued poles are complex conjugates of each other.

\[ p_0 = j\sqrt{\frac{K}{M}} \quad p_1 = -j\sqrt{\frac{K}{M}} \]
Multiple Representations of CT Systems

Each of these purely imaginary poles corresponds to a complex-valued mode.

\[ \begin{align*}
\text{p-plane} & \quad \text{Im } p \\
\sqrt{\frac{K}{M}} & \equiv \omega_0 \quad \text{Re } p \\
-\sqrt{\frac{K}{M}} & \equiv -\omega_0
\end{align*} \]

\[ e^{p_0 t} = e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \]
\[ e^{p_1 t} = e^{-j\omega_0 t} = \cos \omega_0 t - j \sin \omega_0 t \]

Multiple Representations of CT Systems

Real-valued inputs always excite combinations of these modes so that the imaginary parts cancel.

Multiple Representations of CT Systems

Find the modes to find the impulse response of the previous system.

\[ Y \Bigg| X = \frac{K}{M} \frac{A^2}{1 + \frac{K}{M} A^2} = \frac{K}{M} \frac{A}{p_0 - p_1} \left( \frac{A}{1 - p_0 A} - \frac{A}{1 - p_1 A} \right) \]
\[ = \frac{\omega_0^2}{2j\omega_0} \left( \frac{A}{1 - j\omega_0 A} - \frac{A}{1 + j\omega_0 A} \right) \]
\[ = \frac{\omega_0}{2j} \left( \frac{A}{1 - j\omega_0 A} - \frac{\omega_0}{2j} \left( \frac{A}{1 + j\omega_0 A} \right) \right) \]

makes mode 1 \quad makes mode 2

The modes themselves are complex conjugates, and their coefficients are also complex conjugates. So the sum is a sum of something and its complex conjugate, which is real.
Multiple Representations of CT Systems

The impulse response is therefore real.

\[
\frac{Y}{X} = \frac{\omega_0}{2j} \left( \frac{A}{1 - j\omega_0 A} \right) - \frac{\omega_0}{2j} \left( \frac{A}{1 + j\omega_0 A} \right)
\]

\[
= \omega_0 \text{Im} \frac{A}{1 - j\omega_0 A}
\]

When \( x(t) = \delta(t) \) the output is real:

\[
y(t) = \omega_0 \sin \omega_0 t, \quad t \geq 0
\]

Multiple Representations of CT Systems

Find impulse response by expanding the system functional.

\[
Y = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} = \omega_0^2 A^2 - \omega_0^4 A^4 + \omega_0^6 A^6 - \cdots
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - \cdots , \quad t \geq 0
\]

Multiple Representations of CT Systems

Look at successive approximations to this infinite series.

\[
Y = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} = \omega_0^2 A^2 \sum_{l=0}^{\infty} (-\omega_0^2 A^2)^l
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \sum_{l=0}^{\infty} \omega_0^2 (-\omega_0^2)^l A^{2l+2} \delta(t)
\]

\[
= \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - \omega_0^8 \frac{t^7}{7!} + \omega_0^{10} \frac{t^9}{9!} - \cdots = \omega_0 \sin \omega_0 t
\]
Multiple Representations of CT Systems

Now you know how to find the position directly from the system functional without having to guess solutions to differential equations.

\[ x(t) \xrightarrow{+} \omega_0^2 y(t) \xrightarrow{-1} y(t) \]

Now use velocity as the output signal.

\[ x(t) \xrightarrow{+} \omega_0^2 \dot{y}(t) \xrightarrow{-1} \ddot{y}(t) \]

Q for you: Find the system functional.

\[ \dot{Y} = \frac{\omega_0^2 A}{1 + \omega_0^2 A^2} = \omega_0^2 A - \omega_0^4 A^3 + \omega_0^6 A^5 - \cdots \]

When \( x(t) = \delta(t) \) the output is

\[ \dot{y}(t) = \omega_0^2 1 - \omega_0^4 \frac{t^2}{2!} + \omega_0^6 \frac{t^4}{4!} - \cdots, \quad t \geq 0 \]
Look at successive approximations to this infinite series.

\[ \dot{y}(t) = \sum_{l=0}^{\infty} \omega_0^2 (-\omega_0^2)^l A^{2l+1} \delta(t) \]

\[ = \omega_0^2 \frac{1}{2!} - \omega_0^4 \frac{1}{4!} + \omega_0^6 \frac{1}{6!} - \omega_0^8 \frac{1}{8!} + \cdots = \omega_0^2 \cos \omega_0 t \]

Now you know how to find the velocity directly from the system functional without having to guess solutions to differential equations.

Now use acceleration as the output signal.

Q for you: Find the system functional.
Multiple Representations of CT Systems

Compare the acceleration functional
\[ \ddot{Y} = \frac{\omega_0^2}{1 + \omega_0^2 A^2} X \]

to the position functional
\[ Y = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} X \]

Q for you: How can you find the impulse response of the acceleration from the impulse response of the position?

\[ \ddot{y}(t) = \omega_0^2 \delta(t) - \omega_0^3 \sin \omega_0 t ; \quad t \geq 0 \]

Since the impulse response of the displacement system was \( y(t) = \omega_0 \sin \omega_0 t \), it follows that the impulse response of the acceleration system is

Two integrators in forward path → continuous impulse response at \( t = 0 \).
**Multiple Representations of CT Systems**

One integrator in forward path → step discontinuity in the impulse response at \( t = 0 \).

\[
x(t) \rightarrow \frac{x^2}{A} \rightarrow \dot{y}(t)
\]

\[
\dot{y}(t)
\]

**Multiple Representations of CT Systems**

No integrators in forward path → impulse in the impulse response at \( t = 0 \).

\[
x(t) \rightarrow \frac{x^2}{A} \rightarrow \ddot{y}(t)
\]

\[
\ddot{y}(t)
\]

**Example: Tanks**

- \( r_0(t) \)
- \( r_1(t) \)
- \( r_2(t) \)
- \( h_1(t) \)
- \( h_2(t) \)

- \( t \)
Multiple Representations of CT Systems

\[ \tau_1 r_1(t) = r_0(t) - r_1(t) \]
\[ \tau_2 r_2(t) = r_1(t) - r_2(t) \]

\[ \frac{R_2}{R_0} = \frac{A/\tau_1}{1 + A/\tau_1} \times \frac{A/\tau_2}{1 + A/\tau_2} \]

Modal decomposition of second-order CT system.

\[ \frac{R_2}{R_0} = \frac{A/\tau_1}{1 + A/\tau_1} \times \frac{A/\tau_2}{1 + A/\tau_2} = \frac{C_1 A}{1 - p_1 A} + \frac{C_2 A}{1 - p_2 A} \]

\[ p_1 = \frac{1}{\tau_1}, \quad p_2 = \frac{1}{\tau_2} \]
Multiple Representations of CT Systems

Modal decomposition of second-order CT system.

\[
\frac{R_2}{R_0} = \frac{A}{\tau_1} + A/\tau_1 \times A/\tau_2
\]

\[
p_1 = \frac{1}{\tau_1} \quad p_2 = \frac{1}{\tau_2}
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \frac{1}{\tau_1 - \tau_2} \left( e^{-t/\tau_1} - e^{-t/\tau_2} \right) \quad ; \; t \geq 0
\]

Impulse response. If \( x(t) = \delta(t) \) then

\[
y(t) = \frac{1}{\tau_1 - \tau_2} \left( e^{-t/\tau_1} - e^{-t/\tau_2} \right) \quad ; \; t \geq 0
\]

As \( \tau_2 \to \tau_1 \), the amplitudes of the modes \( \to \infty \) but the difference approaches a limit.

\[
y(t) = \frac{1}{\tau_1 - \tau_2} \left( e^{-t/\tau_1} - e^{-t/\tau_2} \right) \quad ; \; t \geq 0
\]

Let \( \frac{1}{\tau_1} = -p - \Delta \) and \( \frac{1}{\tau_2} = -p \). Then

\[
y(t) = \frac{1}{\tau_1 - \tau_2} \left( e^{-t/\tau_1} - e^{-t/\tau_2} \right)
\]

\[
= \frac{p(p + \Delta)}{\Delta} \left( e^{(p+\Delta)t} - e^{pt} \right)
\]

\[
\lim_{\Delta \to 0} \frac{p(p + \Delta)}{\Delta} \left( \frac{e^{(p+\Delta)t} - e^{pt}}{\Delta} \right) = p^2 \frac{d}{dp} e^{pt} = p^2 t e^{pt}
\]
Multiple Representations of CT Systems

Analyze double pole system with $A$.

$$\frac{R_2}{R_0} = \frac{pA}{1-pA} \times \frac{pA}{1-pA}$$

$$= pA \left(1 + pA + p^2A^2 + p^3A^3 + \cdots\right) pA \left(1 + pA + p^2A^2 + p^3A^3 + \cdots\right)$$

\[
\begin{array}{cccccc}
1 & pA & p^2A^2 & p^3A^3 & \cdots \\
1 & 1 & pA & p^2A^2 & p^3A^3 & \cdots \\
pA & pA & p^2A^2 & p^3A^3 & p^4A^4 & \cdots \\
p^2A^2 & p^2A^2 & p^3A^3 & p^4A^4 & p^5A^5 & \cdots \\
p^3A^3 & p^3A^3 & p^4A^4 & p^5A^5 & p^6A^6 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

$$\frac{R_2}{R_0} = p^2A^2 \left(1 + 2pA + 3p^2A^2 + 4p^3A^3 + 5p^4A^4 + \cdots\right)$$

Multiple Representations of CT Systems

Find the impulse response.

$$\frac{R_2}{R_0} = p^2A^2 \left(1 + 2pA + 3p^2A^2 + 4p^3A^3 + 5p^4A^4 + \cdots\right)$$

When $r_0(t) = \delta(t)$ the output is

$$y(t) = p^2\left(t + 2p\frac{t^2}{2!} + 3p^2\frac{t^3}{3!} + 4p^3\frac{t^4}{4!} + \cdots\right)$$

$$= p^2\left(t + pt^2 + p^2\frac{t^3}{2!} + p^3\frac{t^4}{3!} + \cdots\right)$$

$$= p^2\left(t + pt + p^2\frac{t^2}{2!} + p^3\frac{t^3}{3!} + \cdots\right)$$

$$y(t) = p^2te^pt; \ t \geq 0$$

Check Yourself

What’s the impulse response for the following system functional, which has a triply-repeated pole?

$$\left(\frac{pA}{1-pA}\right)^3$$
Last Time

We analyzed mass and spring system.

\[ F = K(x(t) - y(t)) = M\ddot{y}(t) \]

\[ F = K\left(x(t) - y(t)\right) = M\ddot{y}(t) \]

Last Time

We developed a block diagram representation.

\[ F = K\left(x(t) - y(t)\right) = M\ddot{y}(t) \]

\[ F = K\left(x(t) - y(t)\right) = M\ddot{y}(t) \]
Last Time

We developed a functional representation.

\[
\begin{align*}
Y &= \frac{K}{M}\mathcal{A}^2 \\
\frac{X}{Y} &= \frac{1 + \frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2}
\end{align*}
\]

Last Time

We factored system functional to find the poles.

\[
\begin{align*}
Y &= \frac{K}{M}\mathcal{A}^2 \\
\frac{X}{Y} &= \frac{1 + \frac{K}{M}\mathcal{A}^2}{1 + \frac{K}{M}\mathcal{A}^2} = \frac{1}{(1 - p_0\mathcal{A})(1 - p_1\mathcal{A})} \\
1 + \frac{K}{M}\mathcal{A}^2 &= 1 - (p_0 + p_1)\mathcal{A} + p_0p_1\mathcal{A}^2 \\
p_0 &= j\sqrt{\frac{K}{M}} \\
p_1 &= -j\sqrt{\frac{K}{M}}
\end{align*}
\]

Last Time

We found complex-valued poles → complex-valued modes.

\[
\begin{align*}
e^{p_0t} &= e^{j\omega_0t} = \cos \omega_0t + j\sin \omega_0t \\
e^{p_1t} &= e^{-j\omega_0t} = \cos \omega_0t - j\sin \omega_0t
\end{align*}
\]
Last Time

The impulse response is a sinusoidal function of time.
\[
\frac{Y}{X} = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} = \frac{\omega_0}{2j} \left( \frac{A}{1 - j\omega_0 A} \right) - \frac{\omega_0}{2j} \left( \frac{A}{1 + j\omega_0 A} \right)
\]
\[= \omega_0 \text{Im} \frac{A}{1 - j\omega_0 A}\]

When \( x(t) = \delta(t) \) the output is real:
\[ y(t) = \omega_0 \sin \omega_0 t, \quad t \geq 0 \]

Transient input → persistent output
and the amplitude does not decay: persists to infinity!

Second-Order Systems

Today: Look more carefully at growth and decay of oscillatory responses by studying an analogous electrical circuit.

Second-Order Systems

Solve with state variable approach.
State variables represent the minimum knowledge of the past \( t < t_0 \) needed to propagate the output into the future \( t > t_0 \).
Check Yourself

Which of the following can be state variables?

1. $v_C$ and $v_L$
2. $i_C$ and $v_L$
3. $i_C$ and $i_L$
4. $v_C$ and $i_L$
5. $i_C$ and $v_C$ and $i_L$ and $v_L$
6. none of above

Second-Order Systems

State variable approach: determine expressions for derivatives of state variables in terms of (undifferentiated) state variables.

$$\frac{dv_C}{dt} = \frac{1}{C}i_C = \frac{1}{C}i_L \quad \text{(KCL)}$$

$$\frac{di_L}{dt} = \frac{1}{L}v_L = \frac{1}{L}(v_i - Ri_L - v_C) \quad \text{(KVL)}$$

Check Yourself

Determine the system functional.

$$\frac{dv_C}{dt} = \frac{1}{C}i_C$$
$$\frac{di_L}{dt} = \frac{1}{L}(v_i - Ri_L - v_C)$$
Second-Order Systems

The functional for the RLC circuit is the same as that of the mass and spring system (last lecture) if \( R = 0 \) and \( \frac{1}{LC} = \frac{1}{KM} \).

RLC circuit:

\[
\frac{V_o}{V_i} = \frac{1}{\bar{A}} \bar{A}^2 \\
1 + \frac{R}{L} \bar{A} + \frac{1}{LC} \bar{A}^2
\]

Mass and Spring system (last lecture):

\[
\frac{Y}{X} = \frac{\bar{A}}{1 + \frac{K}{M} \bar{A}^2}
\]

Such systems are said to be “analogous.”

By analogy with the mass and spring problem, if \( R = 0 \), then this electrical system will have poles at \( \pm j\omega_0 \), where \( \omega_0 = \sqrt{\frac{1}{LC}} \).

\[
\frac{V_o}{V_i} = \frac{\omega_0^2 \bar{A}^2}{1 + \frac{R}{L} \bar{A} + \omega_0^2 \bar{A}^2}
\]

Focus on effect of \( R \) by scaling time. Let \( \bar{A} = \omega_0 \bar{A} \). Since \( \bar{A} = \int_0^t \bar{A} \, dt \), \( \bar{A} = \int_{\omega_0 t}^{\omega_0 t} \bar{A} \, d(\omega_0 t) \). Thus \( \bar{A} \) is dimensionless.

\[
\frac{V_o}{V_i} = \frac{\bar{A}^2}{1 + \frac{R}{\omega_0 L} \bar{A} + \bar{A}^2}
\]

Reduce the number of parameters by replacing \( \frac{R}{\omega_0 L} \) with \( \frac{1}{Q} \).

\[
\frac{V_o}{V_i} = \frac{\bar{A}^2}{1 + \frac{1}{Q} \bar{A} + \bar{A}^2}
\]

Now \( R = 0 \) corresponds to \( Q \to \infty \).

Find the poles by factoring the denominator of the system functional.

\[
\frac{V_o}{V_i} = \frac{\bar{A}^2}{1 + \frac{1}{Q} \bar{A} + \bar{A}^2} = \frac{\bar{A}^2}{(1 - \bar{p}_0 \bar{A})(1 - \bar{p}_1 \bar{A})}
\]

\[
\bar{p}_0 + \bar{p}_1 = -\frac{1}{Q}
\]

\[
\bar{p}_0 \bar{p}_1 = 1
\]

Substituting,

\[
\bar{p}_0 (-\frac{1}{Q} - \bar{p}_0) = 1
\]

\[
\bar{p}_0^2 + \frac{1}{Q} \bar{p}_0 + 1 = 0
\]

Solving,

\[
\bar{p}_0, \bar{p}_1 = \frac{1}{2Q} \pm \sqrt{\left(\frac{1}{2Q}\right)^2 - 1}
\]
Second-Order Systems

Map pole locations as a function of $Q$.
If $Q \approx 0$, then $\frac{1}{Q} \gg 1$.

$$e^{p_0} + \frac{1}{Q} e^{p_0} + 1 = 0$$

$\gg 1$

Now if $|p_0| \ll 1$, the $p_0^2$ term can be neglected and $p_0 \approx -Q$.
If $|p_0| \gg 1$, the $+1$ term can be neglected and $p_0 \approx -\frac{1}{Q}$.

---

Second-Order Systems

Map pole locations as a function of $Q$.

$$p_0, p_1 = -\frac{1}{2Q} \pm \sqrt{\left(\frac{1}{2Q}\right)^2 - 1}$$

As $Q \to \frac{1}{2}$ the two poles converge to a double pole at $p = -1$. 

---

Second-Order Systems

Map pole locations as a function of $Q$.

$-\frac{1}{Q}$

$-Q$
Second-Order Systems

Map pole locations as a function of $Q$.

\[ \vec{p}_0 + \vec{p}_1 = -\frac{1}{Q} \]

\[ \vec{p}_0 \vec{p}_1 = 1 \]

For $Q > \frac{1}{2}$, the poles have imaginary parts. Furthermore, they are complex conjugates, and their product is 1. Therefore, they lie on the unit circle.

For $Q > 3$, frequency of oscillation is nearly 1 ($0.986 < \omega_d < 1$).

$Q > 3$ is often called the “high-$Q$” region.
Second-Order Systems

We have now found the locus of pole locations for the dimensionless $\bar{A}$ version of the system functional.

$$\frac{1}{R} = \frac{1}{1 + \frac{1}{Q} \bar{A} + \bar{A}^2} = \frac{1}{(1 - \bar{p}_0 \bar{A})(1 - \bar{p}_1 \bar{A})}.$$ 

Check Yourself

Find the locus of pole locations for the original system functional.

$$\frac{\omega_0^2 \bar{A}^2}{1 + \omega_0^2 \bar{A} + \omega_0^2 \bar{A}^2} = \frac{\omega_0^2 \bar{A}^2}{(1 - p_0 \bar{A})(1 - p_1 \bar{A})}.$$ 

Second-Order Systems

The impulse response of the RLC circuit is then a weighted sum of the modes.

$$V_0 \bar{V}_i = \frac{\alpha_0^2 \bar{A}^2}{1 + \frac{1}{Q} \alpha_0 \bar{A} + \alpha_0^2 \bar{A}^2} = \frac{\alpha_0^2 \bar{A}^2}{p_0 - p_1} \left( \frac{\bar{A}}{1 - p_0 \bar{A}} - \frac{\bar{A}}{1 - p_1 \bar{A}} \right).$$

For $0 < Q < \frac{1}{2}$ the modes are real-valued exponentials.

When $x(t) = \delta(t),\ y(t) = \frac{\alpha_0^2}{p_0 - p_1} \left( e^{p_0 t} - e^{p_1 t} \right) ; \ t \geq 0$

For $Q > \frac{1}{2}$, the modes are complex-valued: $-\frac{\omega_0}{2Q} \pm j\omega_d$

When $x(t) = \delta(t),\ y(t) = \frac{\alpha_0^2}{\omega_d} e^{-\omega_d t} \sin \omega_d t$
**Second-Order Systems**

Sketch the result.

\[ Q = 3 \]

\[ Q = 10 \]

\[ Q = 30 \]

**Second-Order Systems: Resonance**

In the high-\( Q \) region, the impulse response is oscillatory.

Capacitor energy \( \frac{1}{2}Cv_C^2 \) is large when \( |v_C(t)| \) is large.

Inductor energy \( \frac{1}{2}L^2i_L^2 \) is large when \( |i_L(t)| = \left| \frac{C}{m} \frac{dv_C}{dt} \right| \) is large.

Thus energy flows back and forth between the capacitor and inductor, a phenomenon we call **resonance**.

**Second-Order Systems**

If \( R > 0 \) then the impulse response decays with time.

Such a system is **lossy**.

As the energy flows back and forth between the capacitor and inductor, current flows through \( R \) on each cycle, and energy is dissipated.
Second-Order Systems

If $R > 0$ then the impulse response decays with time. Such a system is **lossy**.

As the energy flows back and forth between the capacitor and inductor, current flows through $R$ on each cycle, and energy is dissipated.

$$y(t) = \frac{a_0^2}{\omega_d} e^{-\omega_0 t} \sin \omega_d t \; ; \; t \geq 0$$

The exponential term $e^{-\omega_0 t}$ represents this dissipation.

In the high-$Q$ region, the number of cycles before the amplitude diminishes by a factor of $e$ is approximately $\frac{Q}{\pi}$.

$$y(t) = \frac{a_0^2}{\omega_d} e^{-\omega_0 t} \sin \omega_d t$$

Amplitude decays by factor of $e$ in one time constant $= \frac{2Q}{\omega_0}$.

Cycle time $= \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_0}$.

Number of cycles till amplitude down by $e$: $\frac{2Q/\omega_0}{2\pi/\omega_0} = \frac{Q}{\pi}$.

If $R = 0$ then $Q \to \infty$ and the oscillations in the impulse response do not decay with time.

$$y(t) = \frac{a_0^2}{\omega_d} e^{-\omega_0 t} \sin \omega_d t \; ; \; t \geq 0$$

Such a system is **lossless**.
### Second-Order Systems

If $R < 0$ then $Q = \frac{R}{\omega_0 L} < 0$ and the impulse response grows with time.

$$y(t) = \omega_0^2 \omega_d e^{-\omega_0 t} \sin \omega_0 t ; \quad t \geq 0$$

Negative resistors are **active** circuit elements: they add energy to the system.

### Summary

State variable method is a convenient way to translate a circuit description of a system into a functional description.

To analyze a second order system, put the denominator of the system functional in the form $1 + \frac{\omega_0^2}{\omega_d^2} A + \omega_0^2 A^2$.

The poles of a second order system can have real and imaginary parts, depending on the value of $Q$.

High $Q$ systems are **resonant**.

If $Q > 3$, the impulse response of a second order system decays exponentially with time constant $\frac{2Q}{\omega_0}$.

If $Q > 3$, the number of cycles before the amplitude diminishes by a factor of $e$ is $\approx \frac{Q}{\pi}$. 
Multiple Representations of CT Systems

**Verbal descriptions**: preserve the underlying physics.

"... the rate at which water flows from a leaky tank ..."

**Differential equations**: mathematically compact.

\[ \dot{y}(t) = x(t) + py(t) \]

**Block diagrams**: illustrate signal flow paths.

![](image)

**Operator representations**: analyze systems as polynomials.

\[(1 - pA)Y = AX\]

**Pole-Zero diagrams**: represent system functional in factored form.

**System functions**: represent systems as polynomials in \( s \).

Continuous-Time System Functions

Substituting \( \frac{1}{s} \) for \( A \) converts a system functional into its corresponding **system function** \( H(s) \).

\[ H(s) = \frac{Y}{X} |_{A=\frac{1}{s}} \]
Continuous-Time System Functions

Poles and zeros are easily identified using the system function. Poles are the roots of the denominator function of \( s \). Zeros are the roots of the numerator function of \( s \).

\[
H(s) = \frac{Y}{X} = \frac{1}{1 + 2A + 100A^2}
\]

Check Yourself

System functional:

\[
\frac{Y}{X} = \frac{1}{100A} \frac{1}{1 + 2A + 100A^2}
\]

Determine the poles and zeros of the corresponding system function.

1. it has 2 poles \((s \approx -0.01 \pm 0.1j)\) and no zeros
2. it has 2 poles \((s \approx -0.01 \pm 0.1j)\) and 1 zero \((s = 0)\)
3. it has 2 poles \((s \approx -1 \pm 10j)\) and no zeros
4. it has 2 poles \((s \approx -1 \pm 10j)\) and 1 zero \((s = 0)\)
5. none of above

Continuous-Time System Functions

Let \( M \) represent an operator that is the inverse of \( A \). Replacing \( A \) with \( \frac{1}{s} \) is then the same as replacing \( M \) with \( s \).
Check Yourself

What operation does $M = \frac{1}{A}$ perform?

1. $MX = \int_{-\infty}^{\infty} x(t')dt'$ (anti-causal integrator)
2. $MX = \frac{dx(t)}{dt}$ (differentiator)
3. $MX = -\frac{dx(t)}{dt}$ (anti-causal differentiator)
4. $MX = \frac{1}{x(t)}$ (reciprocal)

Continuous-Time System Functions

The $A$ operator is useful in the construction of block diagrams.

Continuous-Time System Functions

The $D$ operator is useful when your starting point is a differential equation.

Example: Find the system function $H(s)$ for the system described by the following differential equation.

$\dot{y}(t) + cy(t) = ax(t) + bx(t)$

Rewrite the differential equation with the $D$ operator.

$DY + cY = aDX + bX$

Solve.

$Y = \frac{aD + b}{D + c}$

Substitute $s$ for $D$.

$H(s) = \frac{as + b}{s + c}$
Continuous-Time System Functions

You can also operate directly on signals and their derivatives.

Example: Find the system function $H(s)$ for the system described by the following differential equation.

$$
\dot{y}(t) + cy(t) = ax(t) + bx(t)
$$

Let $X$ and $Y$ represent $x(t)$ and $y(t)$ (for all time). Then $DX = sX$ and $DY = sY$ represent their derivatives.

$$
sY + cY = asX + bX
$$

$$
H(s) = \frac{Y}{X} = \frac{as + b}{s + c}
$$

Continuous-Time System Functions

Summary.

The $\mathcal{A}$ operator is useful for constructing block diagrams.

The $\mathcal{D}$ operator is useful when your starting point is a differential equation.

A system functional in $\mathcal{A}$ or $\mathcal{D}$ can be represented by its corresponding system function

$$
H(s) = \left. \frac{Y}{X} \right|_{\mathcal{A} \leftarrow \frac{1}{s}, \mathcal{D} \leftarrow s}
$$

Eigenfunctions

The system function can be used to determine response of a system to exponential inputs.

Example:

$$
\dot{y}(t) + cy(t) = ax(t) + bx(t)
$$

Let $x(t) = e^{pt}$ (for all time).

Try $y(t) = \lambda e^{pt}$. Then $\dot{y}(t) = p\lambda e^{pt}$.

Substitute and solve for $\lambda$.

$$
p\lambda e^{pt} + c\lambda e^{pt} = ape^{pt} + be^{pt}
$$

$$
\lambda = \frac{ap + b}{p + c} = H(p)
$$

where

$$
H(s) = \frac{as + b}{s + c}
$$
**Eigenfunctions**

If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the scale multiplier as the eigenvalue.

**Check Yourself: Eigenfunctions**

Consider the system described by
\[ \dot{y}(t) + 2y(t) = x(t). \]
Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue. If it is not, show why not.

1. \( e^{-t} \) for all time
2. \( e^{t} \) for all time
3. \( e^{jt} \) for all time
4. \( \cos(t) \) for all time
5. \( u(t) \) for all time

**Vector Diagrams**

The value of \( H(s) \) at a point \( s = s_0 \) can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.
\[
H(s_0) = \frac{k(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots}
\]

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here \( z_0 \)) to \( s_0 \), the point of interest in the \( s \)-plane.
**Vector Diagrams**

Example: Find the response of the system described by
\[ H(s) = \frac{1}{s + 2} \]
to the input \( x(t) = e^{2jt} \) (for all time).

The denominator of \( H(s) \) is \( s + 2 \), a vector with length \( 2\sqrt{2} \) and angle \( \pi/4 \). Therefore, the response of the system is
\[ y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}}e^{-j\pi/4}e^{2jt}. \]

**Vector Diagrams**

The value of \( H(s) \) at a point \( s = s_0 \) can be determined by combining the contributions of the vectors associated with each of the poles and zeros.
\[ H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots} \]

The magnitude is determined by the product of the magnitudes.
\[ |H(s_0)| = |K| \frac{|s_0 - z_0||s_0 - z_1||s_0 - z_2| \cdots}{|s_0 - p_0||s_0 - p_1||s_0 - p_2| \cdots} \]

The angle is determined by the sum of the angles.
\[ \angle H(s_0) = \angle K + \angle(s_0 - z_0) + \angle(s_0 - z_1) + \cdots - \angle(s_0 - p_0) - \angle(s_0 - p_1) - \cdots \]

**Frequency Response**

The system function can be used to determine the responses of a system to sinusoidal inputs.

Let \( x(t) = \cos \omega t \) (for all time), which can be written as
\[ x(t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t}) \]

The response to a sum is the sum of the responses.
\[ y(t) = \frac{1}{2} \left( H(j\omega_0)e^{j\omega t} + H(-j\omega_0)e^{-j\omega t} \right) \]
Conjugate Symmetry

The complex conjugate of $H(j\omega)$ is $H(-j\omega)$.

Let

$$H(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in $s$.

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

If the coefficients of the polynomials are real-valued (as they are for physical systems) then

$$(H(j\omega))^* = \frac{N(-j\omega)}{D(-j\omega)} = H(-j\omega)$$

Frequency Response

The system function can be used to determine the responses of a system to sinusoidal inputs.

Let $x(t) = \cos(\omega_0 t)$ (for all time), which can be written as

$$x(t) = \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t})$$

The response to a sum is the sum of the responses,

$$y(t) = \frac{1}{2} \left( H(j\omega_0) e^{j\omega_0 t} + H(-j\omega_0) e^{-j\omega_0 t} \right)$$

$$= \text{Re} \left[ H(j\omega_0) e^{j\omega_0 t} \right]$$

$$= \text{Re} \left[ |H(j\omega_0)| e^{j\theta} e^{j\omega_0 t} \right]$$

$$= |H(j\omega_0)| \text{Re} \left[ e^{j\omega_0 t} \right]$$

$$y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle(H(j\omega_0))).$$
Vector Diagrams

$H(s) = s - z_1$

$|H(j\omega)|$

$\angle H(j\omega)$

$22$

Vector Diagrams

$H(s) = \frac{g}{s - p_1}$

$|H(j\omega)|$

$\angle H(j\omega)$

$23$

Vector Diagrams

$H(s) = \frac{s - z_1}{s - p_1}$

$|H(j\omega)|$

$\angle H(j\omega)$

$24$
Check Yourself: Frequency Response
Consider the system represented by the following poles.

Let \( x(t) = \cos \omega_d t \) (for all time). Is the phase of the response

\[ = -\frac{\pi}{2} \text{ or } > -\frac{\pi}{2} \text{ or } < -\frac{\pi}{2} \]

If not \( -\frac{\pi}{2} \), should \( \omega > \omega_d \) or \( \omega < \omega_d \) to get the phase to be \( -\frac{\pi}{2} \)?

Check Yourself: Frequency Response
For the geometrically inclined:

Check Yourself: Frequency Response
Consider the system represented by the following poles.

Let \( x(t) = \cos \omega t \) (for all time). Find the frequency \( \omega \) at which the magnitude of the response \( y(t) \) is greatest.

Check Yourself: Frequency Response
Consider the system represented by the following poles.
Multiple Representations of CT Systems

Verbal descriptions: preserve the underlying physics.

“... the rate at which water flows from a leaky tank ...”

Differential equations: mathematically compact.

\[ y(t) = x(t) + py(t) \]

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.

\[(1 - pA)Y = AX\]

Pole-Zero diagrams: represent system functional in factored form.

System functions: represent systems as polynomials in \( s \).
Last Time: System Functions

Substituting $\frac{1}{s}$ for $A$ and $s$ for $D$ converts a CT system functional into its corresponding system function $H(s)$.

$$H(s) = \frac{Y}{X} = \left. \frac{1}{s} \right|_{A \rightarrow \frac{1}{s}, D \leftarrow s}$$

Poles (or zeros) are roots of the denominator (or numerator) polynomial of $H(s)$.

$$H(s) = \frac{Y}{X} = \frac{100s}{s^2 + 2s + 100}$$

Last Time: Eigenfunctions and Eigenvalues

The system function can be used to determine response of a system to exponential inputs.

If $x(t) = e^{s_0 t}$ (for all time) then $y(t) = H(s_0)e^{s_0 t}$ (for all time).

Exponentials are eigenfunctions of systems that can be represented by linear differential equations with constant coefficients.

The eigenvalue associated with the eigenfunction $e^{s_0 t}$ is the value of the system function $H(s)$ at $s = s_0$. 
The value of $H(s_0)$ at a point $s = s_0$ can be determined graphically using vector analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots}$$

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here $z_0$) to $s_0$, the point of interest in the $s$-plane.

The magnitude of $H(s_0)$ is the product of the magnitudes of the vectors associated with the zeros divided by the product of the magnitudes of the vectors associated with the poles.

$$|H(s_0)| = \left| K \prod_{q=1}^{Q} \frac{|s_0 - z_q|}{|s_0 - p_q|} \right| = |K| \frac{\prod_{q=1}^{Q} |s_0 - z_q|}{\prod_{p=1}^{P} |s_0 - p_p|}$$
**Last Time: Vector Diagrams**

The angle of $H(s_0)$ is the sum of the angles of the vectors associated with the zeros minus the sum of the angles of the vectors associated with the poles.

$$\angle H(s_0) = \angle K + \sum_{q=1}^{Q} \angle (s_0 - z_q) - \sum_{p=1}^{P} \angle (s_0 - p_p)$$

The angle of $K$ can be 0 or $\pi$ for systems described by linear differential equations with constant, real-valued coefficients.

**Last Time: Frequency Response**

The frequency response of a system is given by $H(j\omega)$.

If $x(t) = \cos \omega_0 t$ then

$$y(t) = |H(j\omega)| \cos(\omega_0 t + \angle H(j\omega))$$

The frequency response is a complex-valued function of $\omega$. The magnitude gives the gain of the system for each frequency. The angle gives the phase.

**Frequency Response: $H(s)|s\leftarrow j\omega$**

$H(s) = s - z_1$

$|H(j\omega)|$

$\angle H(j\omega)$
**Frequency Response: \( H(s) \) → \( j\omega \)**

\[
H(s) = \frac{9}{s - p_1}
\]

\[
|H(j\omega)|
\]

\[
\angle H(j\omega)
\]

**Check Yourself**

Could the phase plots of any of the systems represented by the following pole-zero plots be equal to each other? [caution: this could be a trick question]
From Frequency Response to Bode Plot

The magnitude of \( H(j\omega) \) is a product of magnitudes.

\[
|H(j\omega)| = |K| \prod_{q=1}^{Q} \frac{|j\omega - z_q|}{|j\omega - p_p|}
\]

The angle of \( H(j\omega) \) is a sum of angles.

\[
\angle H(j\omega) = \angle K + \sum_{q=1}^{Q} \angle (j\omega - z_q) - \sum_{p=1}^{P} \angle (j\omega - p_p)
\]

The angle of \( K \) can be 0 or \( \pi \) for systems described by linear differential equations with constant, real-valued coefficients.

Bode Plot: Isolated Zero

The two asymptotes are a good approximation to \( \log|H(j\omega)| \).

\[
H(s) = s - z_1, \quad z_1 < 0 \quad \frac{|H(j\omega)|}{|z_1|}
\]

\[
\begin{align*}
\lim_{\omega \to 0} |H(j\omega)| &= |z_1| \\
\lim_{\omega \to \infty} |H(j\omega)| &= \omega
\end{align*}
\]
**Bode Plot: Isolated Zero**

Straight-line approximation to $\angle H(j\omega)$.

$$H(s) = s - z_1, \quad z_1 < 0$$

![Bode Plot: Isolated Zero](image)

$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = \frac{\pi}{2}$$

---

**Bode Plot: Isolated Pole**

The two asymptotes are a good approximation to $\log |H(j\omega)|$.

$$H(s) = \frac{1}{s - p_1}, \quad p_1 < 0$$

![Bode Plot: Isolated Pole](image)

$$\lim_{\omega \to 0} |H(j\omega)| = \frac{1}{|p_1|}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \frac{1}{\omega}$$

---

**Bode Plot: Isolated Pole**

Straight-line approximation to $\angle H(j\omega)$.

$$H(s) = \frac{1}{s - p_1}, \quad p_1 < 0$$

![Bode Plot: Isolated Pole](image)

$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\frac{\pi}{2}$$
Check Yourself

\[ H_1(s) = \frac{1}{s + 1} \quad \text{and} \quad H_2(s) = \frac{10}{s + 10} \]

The Bode magnitude plot for \( H_2(s) \) can be obtained from that for \( H_1(s) \) by

1. shifting it horizontally
2. scaling it horizontally
3. shifting and scaling it horizontally
4. shifting and scaling both horizontally and vertically
5. none of the above

Bode Plot: More Complicated
Bode Plot: More Complicated

\[ H(s) = \frac{s}{(s + 1)(s + 10)} \]

\[ \angle s \]

\[ \angle \frac{1}{s + 1} \]

\[ \angle \frac{1}{s + 10} \]

Bode Plot: dB

\[ H(s) = \frac{10s}{(s + 1)(s + 10)} \]

\[ \log |H(j\omega)| \]

\[ \angle H(j\omega) \]
**Bode Plot: dB**

\[ H(s) = \frac{10s}{(s + 1)(s + 10)} \]

The straight-line approximations are surprisingly accurate.

\[ H(j\omega) = \frac{1}{j\omega + 1} \]

Without using a calculator (other than the one in your head) determine

1. the frequency that is 6dB below 1kHz
2. the amplitude that is 10 dB above 1
3. the value of 500 in dB
4. how many dB/octave correspond to a slope of 20dB/decade
Frequency Response of a High-\(Q\) System

The magnitude of the frequency response of a high-\(Q\) system is peaked.

\[
H(s) = \frac{1}{1 + \frac{1}{Q} s + \left(\frac{s}{\omega_0}\right)^2}
\]

Check Yourself

Estimate the peak value of the magnitude function as a function of \(Q\) assuming \(Q\) is large (e.g., \(Q > 3\)).
As $Q$ increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Check Yourself

Estimate the “3dB bandwidth” of the peak.

Let $\omega_l$ (or $\omega_h$) represent the lowest (or highest) frequency for which the magnitude is greater than the peak value divided by $\sqrt{2}$. The 3dB bandwidth is then $\omega_h - \omega_l$. 

As $Q$ increases, the phase changes more abruptly with $\omega$. 

$$H(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$
**Frequency Response of a High-Q System**

As Q increases, the phase changes more abruptly with ω.

\[
H(s) = \frac{1}{1 + \frac{1}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}
\]

**Check Yourself**

Estimate the change in phase that occurs over the 3dB bandwidth.

\[
H(s) = \frac{1}{1 + \frac{1}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}
\]

**Multiple Representations of CT Systems**

**Verbal descriptions:** preserve the underlying physics.

**Differential equations:** mathematically compact.

\[
\dot{y}(t) = x(t) + py(t)
\]

**Block diagrams:** illustrate signal flow paths.

**Operator representations:** analyze systems as polynomials.

\[(1 - pA)Y = AX\]

**Pole-Zero diagrams:** represent factors of system functional.

**System functions:** represent systems as polynomials in s.

**Bode plots:** quickly sketch the frequency response.
Feedback and Control

Feedback is pervasive in natural and artificial systems. Turn steering wheel to stay centered in the lane.

Feedback and Control

Feedback is useful for regulating a system’s behavior, as when a thermostat regulates the temperature of a house.
Concentration of glucose in blood is highly regulated and remains nearly constant despite episodic ingestion and use.

Motor control relies on feedback from pressure sensors in the skin as well as proprioceptors in muscles, tendons, and joints.

Try building a robotic hand to unscrew a lightbulb!

We've previously used systems theory to gain insight into how to control a discrete-time system.

Today's goal: use systems theory to gain insight into how to control a continuous-time system.
Example: Op Amp

Feedback is often used to improve the performance of a system as characterized by some figure of merit.
Example: Design an amplifier for processing audio signals using an op amp.

An op amp is a device with 2 inputs ($V_+$ and $V_-$) and one output ($V_o$) which is approximately equal to a gain ($F(s)$) times the difference between the inputs ($V_+ - V_-$).

Op Amp:

\[ V_+ \quad F(s) \quad V_o = F(s)(V_+ - V_-) \]

Example: Op Amp

Op amps have high gain at low frequencies, but the gain decreases with frequency.

We can represent the frequency dependence of a typical op amp (LM741) with a system function of the form

\[ F(s) = \frac{F_0}{1 + \tau s} \]

where $F_0 \approx 2 \times 10^5$ and $\tau \approx 1/40$ seconds.

Example: Op Amp

Without feedback, the LM741 is a poor audio amplifier.

\[ F(s) = \frac{F_0}{1 + \tau s} ; \; F_0 \approx 2 \times 10^5 ; \; \tau \approx 1/40 \text{ seconds} \]
Example: Op Amp

Without feedback, the LM741 is slow.
Impulse response
\[ y(t) = \frac{F_0}{\tau} e^{-t/\tau}, \quad t > 0 \]
Step response
\[ s(t) = F_0 (1 - e^{-t/\tau}), \quad t > 0 \]

Fast for humans ... slow for electronics!

Check Yourself: Op Amp

What is the most negative value of the closed-loop pole that can be achieved with feedback?

\[ s \text{-plane} \]

\[ \begin{array}{c}
V_i \\
V_o
\end{array} \quad \begin{array}{c}
F(s) \\
\frac{F(s)}{1 + \beta F(s)}
\end{array} \quad \begin{array}{c}
\beta V_o \\
V_o
\end{array} \quad \begin{array}{c}
\frac{F_0}{1 + \frac{1}{1 + \frac{1}{1 + \beta F_0}}}
\end{array} \]

\[ \frac{V_o}{V_i} = \frac{F(s)}{1 + \beta F(s)} \]

\[ V_- = \beta V_o = \left( \frac{R_2}{R_1 + R_2} \right) V_o \]
Example: Op Amp

Feedback extends the frequency response by as much as a factor of $F_0 = 2 \times 10^5$.

Example: Op Amp

Feedback produces higher bandwidths by reducing the gain at low frequencies. It trades gain for bandwidth.

Example: Op Amp

Feedback makes the time response faster by as much as a factor of $F_0 = 2 \times 10^5$.

Step response

$$s(t) = \frac{F_0}{1 + \beta F_0} \left(1 - e^{-t/(\tau/(1+\beta F_0))}\right), \quad t > 0$$
Example: Op Amp

Feedback produces faster responses by reducing the final value of the step response. It trades gain for speed.

Step response

\[ s(t) = \frac{F_0}{1 + \beta F_0} \left(1 - e^{-t/(\tau/(1+\beta F_0))}\right), \quad t > 0 \]

Summary

- feedback can extend frequency response
- feedback can increase speed

These performance enhancements are achieved through a reduction of gain.

Feedback and Control

Feedback can also be used to control a system.

Black’s equation for the closed-loop gain \( H(s) \):

\[
\frac{Y}{X} = H(s) = \frac{K(s)F(s)}{1 + K(s)F(s)G(s)}
\]

forward gain

loop gain
Example: Motor Controller

We wish to build a robot arm (actually its elbow). The input should be voltage \( v(t) \), and the output should be the elbow angle \( \theta(t) \).

\[
\begin{align*}
  v(t) & \rightarrow \text{robotic arm} \rightarrow \theta(t) \propto v(t) \\
\end{align*}
\]

We wish to build the robot arm with a DC motor.

\[
\begin{align*}
  v(t) & \rightarrow \text{DC motor} \rightarrow \theta(t) \\
\end{align*}
\]

Example: Motor Controller

What is the relation between \( v(t) \) and \( \theta(t) \) for a DC motor?

\[
\begin{align*}
  v(t) & \rightarrow \text{DC motor} \rightarrow \theta(t) \\
\end{align*}
\]

Example: Motor Controller

Use proportional feedback to control the angle of the motor’s output.

\[
\begin{align*}
  \Theta = \frac{a \gamma A_s}{1 + a \beta A_s} = \frac{a \gamma}{1 + a \beta} = \frac{a \gamma}{s + a \beta} = \frac{1/\beta}{1 + s/(a \beta)} \\
\end{align*}
\]
Example: Motor Controller

The closed loop system has a single pole at $s = -\alpha \beta \gamma$.

$$\frac{\Theta}{V} = \frac{\alpha \gamma}{s + \alpha \beta \gamma} = \frac{1/\beta}{1 + s/(\alpha \beta \gamma)}$$

If the gain $\alpha$ is zero, then the closed loop pole is at $s = 0$ (i.e., same as open loop).

If the gain increases, the closed loop pole moves to the left.

Example: Motor Controller

Find the step response by integrating the impulse response.

$$\Theta = \frac{\alpha \gamma}{1 + \alpha \beta \gamma \alpha}$$

If $v(t) = \delta(t)$ then $\theta(t) = a e^{-\alpha \beta \gamma t}$; $t \geq 0$.

If $v(t) = u(t) = \int_{-\infty}^{t} \delta(t') dt'$ then

$$\theta(t) = \int_{-\infty}^{t} a e^{-\alpha \beta \gamma t'} u(t') dt' = \int_{0}^{t} a e^{-\alpha \beta \gamma t'} u(t') dt' = \frac{1}{\beta} \left(1 - e^{-\alpha \beta \gamma t}\right); t \geq 0.$$

The response is faster for larger values of $\alpha$.

Try it: Demo.

Example: Motor Controller

The speed of a DC motor does not change instantly if the voltage is stepped. There is lag due to rotational inertia.

Second-order model: integrator with lag

$$V \rightarrow \gamma A \left(\frac{p A}{1 + p A}\right) \rightarrow \Theta$$

Step response:

$$v(t) \quad \theta(t) = v_{\text{step}} \left(\gamma t - \frac{v_{\text{step}}}{p}(1 - e^{-pt})\right) \quad \text{for } t \geq 0$$
Example: Motor Controller

Analyze second-order model.

\[
\Theta \frac{\alpha \gamma p A^2}{1 + p A} = \frac{\alpha \gamma p A^2}{1 + p A + \alpha \beta \gamma p A^2} = \frac{\alpha \gamma p}{s^2 + ps + \alpha \beta \gamma p}.
\]

\[
s = \frac{-p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - \alpha \beta \gamma p}.
\]

Example: Motor Controller

For second-order model, the closed-loop poles are at 0 and \(-p\) when \(\alpha = 0\). Increasing \(\alpha\) causes the poles to approach each other, collide at \(s = -p\) then split into two poles with imaginary parts.

Feedback and Control: Summary

CT feedback is useful for many reasons. Today we saw two:

1. to improve the performance of a system (op amp) as characterized by a figure of merit (e.g. bandwidth, speed), and
2. to control a system (DC motor) so that the output is position rather than speed.
Feedback and Control

CT feedback is useful for many reasons. Last time we saw two:

1. to improve the performance of a system (op amp) as characterized by a figure of merit (e.g. bandwidth, speed), and
2. to control a system (DC motor) so that the output is position rather than speed.

Today we will look at a few others.

Feedback and Control

Feedback can be used to compensate for parameter variation.

Changes in $F_0$ (due to changes in temperature, for example) lead to undesired changes in sound level.
Feedback and Control

Feedback can be used to compensate for parameter variation.

\[ H(s) = \frac{KF_0}{1 + \beta K F_0} \]

If \( K \) is made large, so that \( \beta K F_0 \gg 1 \) then
\[ H(s) \approx \frac{1}{\beta} \] independent of \( K \)!

Feedback reduces the change in gain due to change in \( F_0 \).

If \( K = 100 \) and \( \beta = 1/10 \) then
\[ H(s) = \frac{100F_0}{1 + \frac{100F_0}{10}} \]

[Check Yourself: don’t we need to worry about \( \beta \) varying?]
Crossover Distortion

Feedback can compensate for parameter variation even when the variation occurs rapidly. This circuit illustrates a method by which transistors can be used to amplify power.

Crossover Distortion

This circuit introduces “crossover distortion.” For the upper transistor to conduct, $V_i - V_o > V_T$. For the lower transistor to conduct, $V_i - V_o < -V_T$.

Crossover Distortion

Crossover distortion can have dramatic effects. This figure shows effects of crossover distortion when the input is $V_i(t) = B \sin(\omega_0 t)$.
Crossover Distortion

Feedback can reduce the effects of crossover distortion.

As $K$ increases, the effect of crossover distortion is reduced by feedback.

Feedback and Control: Summary

CT feedback is useful for many reasons.

1. to improve the performance of a system (op amp) as characterized by a figure of merit (e.g. bandwidth, speed).
2. to control a system (DC motor) so that the output is position rather than speed.
3. to reduce sensitivity to parameter variation (unwanted gain variation).
4. to reduce distortions (crossover distortion).
Feedback is also useful for controlling \textit{unstable} systems. Example: Magnetic levitation.

\begin{align*}
i(t) & = i_0 \\
y(t) & = \text{equilibrium position of the body}
\end{align*}

Magnetic levitation is unstable. Let $y = 0$ denote the “equilibrium” position of the body. Then the magnetic force $f(t)$ is equal to the weight $Mg$.

If the body moves a bit closer to (or further from) the magnet, the force is greater (or smaller) than the weight and the body is accelerated upwards (or downwards).

The instability of magnetic levitation is similar to that of a ball poised at the apex of perfectly smooth hill.
“Levitation” with a Spring

By contrast, the mass and spring system is not unstable.

\[ F = K(x(t) - y(t)) = M\ddot{y}(t) \]

The spring and mass system is a stable system, analogous to a ball in a valley.

The spring force decreases (or increases) and the body tends to fall back (or rise up).

\[ F = K(x(t) - y(t)) = M\ddot{y}(t) \]
“Levitation” with a Spring

The block diagram for the spring and mass system has negative feedback, because the slope of the force curve is negative.

\[ F = K(x(t) - y(t)) = M\ddot{y}(t) \]

This system is marginally stable (poles on imaginary axis), or stable (poles in left half plane) if effects of friction are included.

\[ \omega_0 \equiv \sqrt{\frac{K}{M}} \]
\[ \omega_0 \equiv -\sqrt{\frac{K}{M}} \]

Magnetic Levitation

By contrast, the system representing magnetic levitation has positive feedback.

\[ f(t) \]
\[ i(t) = i_0 \]
\[ Mg \]
\[ y(t) \]

\[ x(t) \]
\[ \frac{p}{i_0} \]
\[ -1 \]
Magnetic Levitation

The poles are at $s = \pm p$.

We can stabilize this system by adding an additional feedback loop to control $i(t)$.

Try it. Demo. [designed by Prof. James Roberge]

Inverted Pendulum

As a final example of stabilizing an unstable system, consider an inverted pendulum.

$$\int \frac{d^2 \theta(t)}{dt^2} = ml^2 \frac{d^2 \theta(t)}{dt^2} = mgl \sin \theta(t) - ml \cos \theta(t) \frac{d^2 x(t)}{dt^2}$$
Check Yourself: Inverted Pendulum

Where are the poles of this system?

\[ \frac{d\theta(t)}{dt} = \frac{mg}{l} \sin \theta(t) - \frac{ml\cos \theta(t)}{d\theta(t)/dt} \]

Inverted Pendulum

This unstable system can be stabilized with feedback.

Try it. Demo. [originally designed by Marcel Gaudreau]

Feedback and Control: Summary

CT feedback is useful for many reasons.

1. to improve the performance of a system (op amp) as characterized by a figure of merit (e.g. bandwidth, speed).
2. to control a system (DC motor) so that the output is position rather than speed.
3. to reduce sensitivity to parameter variation (unwanted gain variation).
4. to reduce distortions (crossover distortion).
5. to stabilize unstable systems (magnetic levitation and inverted pendulum).
Multiple Representations of CT and DT Systems

Throughout the semester, we have seen a variety of representations of systems, and how they can each be useful.
Today we add yet another, this one based on the impulse response.

Cascading Systems

Cascade two systems → multiply their system functions and functionals.
Cascade:

\[ H_1(s) \times H_2(s) \]

Equivalent representation:

\[ H_1(s) \times H_2(s) \]

Cascading Systems

Example:

\[ X \rightarrow 1 + R + R^2 \rightarrow Y_1 \rightarrow 1 + R + R^2 \rightarrow Y_2 \]

\[ \frac{Y_2}{X} = (1 + R + R^2) \times (1 + R + R^2) \]

\[
\begin{array}{ccc}
  & 1 & +R & +R^2 \\
1 & 1 & R & R^2 \\
+R & R & R^2 & R^3 \\
+R^2 & R^2 & R^3 & R^4
\end{array}
\]

\[ \frac{Y_2}{X} = 1 + 2R^2 + 3R^3 + 2R^4 + R^5 \]

Cascading Systems

Example:

\[ (1 + R + R^2) \times (1 + R + R^2) = 1 + 2R^2 + 3R^3 + 2R^4 + R^5 \]

Cascading Systems

\[ h_1[n] \]

\[ -1 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \]

\[ h_2[n] \]

\[ -1 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \]

\[ h[n] \]

\[ -1 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \]

Cascading Systems

Generalize to arbitrary causal DT systems.

\[ \frac{Y_1}{X} = a_0 + a_1R + a_2R^2 + \cdots \]

\[ \frac{Y_2}{Y_1} = b_0 + b_1R + b_2R^2 + \cdots \]

\[ \frac{Y_1}{X} = (a_0 + a_1R + a_2R^2 + \cdots) \times (b_0 + b_1R + b_2R^2 + \cdots) \]

\[
\begin{array}{cccc}
  & a_0 & +a_1R & +a_2R^2 & \cdots \\
b_0 & a_0b_0 & a_1b_0R & a_2b_0R^2 & \cdots \\
+R & a_0b_1R & a_1b_1R^2 & a_2b_1R^3 & \cdots \\
+R^2 & a_0b_2R^2 & a_1b_2R^3 & a_2b_2R^4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

\[ \frac{Y_2}{X} = a_0b_0 + (a_0b_1 + a_1b_0)R + (a_0b_2 + a_1b_1 + a_2b_0)R^2 + \cdots \]
Cascading Systems

Polynomial multiplication: collect terms with equal delays.

\[
\frac{Y}{X} = \left(1 + aR + a^2R^2 + a^3R^3 + \cdots \right) \times \left(1 + bR + b^2R^2 + b^3R^3 + \cdots \right)
\]

Cascading Systems

Generalize to arbitrary causal DT systems.

\[
\frac{Y_2}{X} = a_0b_0 + (a_0b_1 + a_1b_0)R + (a_0b_2 + a_1b_1 + a_2b_0)R^2 + \cdots
\]

Let

\[
\frac{Y_2}{X} = c_0 + c_1R + c_2R^2 + \cdots
\]

Then

\[
c_n = \sum_{k=0}^{n} a_kb_{n-k}
\]

Cascading Systems

Generalize to non-causal DT systems.

\[
\frac{Y_2}{X} = (L+1+R) \times (L+1+R)
\]

\[
\begin{array}{c|cc}
   & L & +1 & +R \\
\hline
   L & L^2 & L & 1 \\
   +1 & L & 1 & R \\
   +R & 1 & R & R^2 \\
\end{array}
\]

\[
\frac{Y_2}{X} = L^2 + 2L + 3 + 2R + R^2
\]
Cascading Systems

Example:

\[(L + 1 + R) \times (L + 1 + R) = L^2 + 2L + 3 + 2R + R^2\]

Generalize to non-causal DT systems.

\[Y_2 = ... + a_{-2}L^2 + a_{-1}L + a_0 + a_1R + a_2R^2 + ...\]

\[Y_1 = ... + b_{-2}L^2 + b_{-1}L + b_0 + b_1R + b_2R^2 + ...\]

\[
\begin{array}{cccc}
... & a_{-1}L & +a_0 & +a_1R & ...
\end{array}
\]

\[
\begin{array}{cccc}
b_{-1}L & a_{-1}b_{-1}L & a_0b_{-1}L & a_1b_{-1} & ...
\end{array}
\]

\[
\begin{array}{cccc}
b_0 & a_{-1}b_0L & a_0b_0 & a_1b_0R & ...
\end{array}
\]

\[
\begin{array}{cccc}
b_1R & a_{-1}b_1R & a_0b_1R & a_1b_1R^2 & ...
\end{array}
\]

\[
\begin{array}{cccc}
... & a_{-1}b_1 & a_0b_1 & a_1b_1 & ...
\end{array}
\]

Cascading Systems

Generalize to non-causal DT systems.

\[Y_2 = ... + c_{-2}L^2 + c_{-1}L + c_0 + b_1R + c_2R^2 + ...\]

\[
\begin{array}{cccc}
... & a_{-1}L & +a_0 & +a_1R & ...
\end{array}
\]

\[
\begin{array}{cccc}
b_{-1}L & a_{-1}b_{-1}L & a_0b_{-1}L & a_1b_{-1} & ...
\end{array}
\]

\[
\begin{array}{cccc}
b_0 & a_{-1}b_0L & a_0b_0 & a_1b_0R & ...
\end{array}
\]

\[
\begin{array}{cccc}
b_1R & a_{-1}b_1R & a_0b_1R & a_1b_1R^2 & ...
\end{array}
\]

\[
\begin{array}{cccc}
... & a_{-1}b_1 & a_0b_1 & a_1b_1 & ...
\end{array}
\]

\[c_3 = ... + a_{-2}b_5 + a_{-1}b_4 + a_0b_3 + a_1b_2 + a_2b_1 + ... = \sum_{k=-\infty}^{\infty} a_k b_{3-k}\]

\[c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k} \quad \text{Convolution Sum!}\]
Cascading Systems

Cascade two systems → convolve their impulse responses.
Let $h_1[n]$ and $h_2[n]$ represent the impulse responses of two systems in cascade:

$$x[n] \xrightarrow{h_1[n]} y_1[n] \xrightarrow{h_2[n]} y_2[n]$$

The impulse response of the cascade is then the convolution of $h_1[n]$ and $h_2[n]$:

$$h[n] = (h_1 * h_2)[n] = \sum_{\infty} h_1[k]h_2[n-k]$$

Equivalent representation:

$$x[n] \xrightarrow{(h_1 * h_2)[n]} y_2[n]$$

Signals as Systems

The response of a system with impulse response $h[n]$ is the convolution of its input signal $x[n]$ with $h[n]$.
Consider the cascade of a system with impulse responses $x[n]$ and $h[n]$.

$$\delta[n] \xrightarrow{x[n]} x[n] \xrightarrow{h[n]} y[n] = (x * h)[n]$$

The impulse response of the cascade is the convolution of the impulse responses of the individual systems.
It follows that the response of the second system to $x[n]$ is the convolution of $x[n]$ with the impulse response of the second system.

Impulse Response

The impulse response is a new representation for systems.
This representation is sometimes called the “time-domain” representation, since the signals and systems are both represented as functions of time.
This is in contrast to representing systems by system functions (e.g., functions of $z$) or functionals (e.g., $\mathcal{L}$ and $\mathcal{R}$).
Superposition
Convolution is closely associated with superposition. All systems composed of delays, anticipators, scalers, and adders are “linear” and “time-invariant.”

Linearity
A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs. Given:

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]

The system is linear if the following input-output relation is true for all \( \alpha \) and \( \beta \).

\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]

Time-Invariance
A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time. Given:

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

The system is time invariant if the following input-output relation is true for all \( n_0 \).

\[ x[n - n_0] \rightarrow \text{system} \rightarrow y[n - n_0] \]
LTI Systems

If a system is linear and time-invariant (LTI) then its output can be determined using superposition.

\[ x[n] \]
\[ + \]
\[ + \]
\[ = \]
\[ y[n] \]

Structure of Convolution

\[ y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] \]

Structure of Convolution

\[ y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] \]
**Structure of Convolution**

\[ y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] \]

**Convolution of CT signals is completely analogous to convolution of DT signals.**

**DT:** \[ y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \]

**CT:** \[ y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]
Impulse Response: Summary

The impulse response is a complete description of a linear, time-invariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems.

Microscope

Images from even the best microscopes are blurred.

A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.

Blurring is inversely related to the diameter of the lens.
**Microscope**

We can characterize the blurring of a microscope by measuring its “impulse response.”

Image diameter ≈ 6 times target diameter: target → impulse.

![Image of a 0.09 μm bead](image)

**Microscope**

Images at different focal planes can be assembled to form a three-dimensional impulse response (point-spread function).

![Images by Anthony Patire](image)

**Microscope**

Blurring along the optical axis is better visualized by resampling the three-dimensional impulse response.

![Images by Anthony Patire](image)
Microscope

Blurring is much greater along the optical axis than it is across the optical axis.

The point-spread function (3D impulse response) is a useful way to characterize a microscope. It provides a direct measure of blurring, which is an important figure of merit for optics.

Hubble Space Telescope

Hubble Space Telescope (1990-)

http://hubblesite.org
Hubble Space Telescope

Why build a space telescope?
Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.

\[ h_t(x, y) = (h_a \ast h_d)(x, y) \]

X \hspace{1cm} h_a(x, y) \hspace{1cm} h_d(x, y) \hspace{1cm} Y

atmospheric blurring \hspace{1cm} blur due to mirror size

X \hspace{1cm} h_t(x, y) \hspace{1cm} Y

ground-based telescope

Hubble Space Telescope

Telescope blur can be represented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.

\[ h_a(\theta) \ast h_d(\theta) = h_t(\theta) \]

Hubble Space Telescope

The main optical components of the Hubble Space Telescope are two mirrors.

http://hubblesite.org

http://hubblesite.org
**Hubble Space Telescope**

The diameter of the primary mirror is 2.4 meters.

http://hubblesite.org

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**Hubble Space Telescope**

Hubble's first pictures of distant stars (May 20, 1990) were more blurred than expected.

- expected point-spread function
- early Hubble image of distant star

http://hubblesite.org

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**Hubble Space Telescope**

The parabolic mirror was ground 4 µm too flat!

http://hubblesite.org
Hubble Space Telescope

Corrective Optics Space Telescope Axial Replacement (COSTAR): eyeglasses for Hubble!

Hubble images before and after COSTAR.

http://hubblesite.org

Hubble images before and after COSTAR.

http://hubblesite.org
**Hubble Space Telescope**

Images from ground-based telescope and Hubble.

http://hubblesite.org

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**Impulse Response: Summary**

The impulse response is a complete description of a linear, time-invariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.
Multiple Representations of DT Systems

A major focus of this subject has been the development of multiple representations of systems.

Today we will spend some time thinking about relations among these representations.

\[ f(R) = \frac{1}{1 - R - R^2} \]

\[ H(z) = \frac{z^2}{z^2 - z - 1} \]

\[ y[n] = x[n] + y[n-1] + y[n-2] \]

\[ \sum_{n=0}^{\infty} h[n] R^n \]

Series

Partial fractions

\[ \mathbf{R} \rightarrow \frac{1}{z} \]

Impulse response

1, 1, 2, 3, 5, 8, ...

Block diagram

System functional

Difference Equation

System function

Z transform
**Multiple Representations: Check Yourself**

Determine an expression for $H(z)$ in terms of $h[n]$.

**Multiple Representations of DT Systems**

In general, we should also include noncausal terms.

\[
\mathcal{L} \text{ terms simply extend the sum to include negative values of } n.
\]

\[
f(R, L) = \sum_{n=-\infty}^{\infty} h[n]L^{-n} + h[0] + \sum_{n=1}^{\infty} h[n]R^n
\]

Replace $R$ with $\frac{1}{z}$ and $L$ with $z$ to get

\[
H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} + h[0] + \sum_{n=1}^{\infty} h[n]z^{-n}
\]

\[
H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}
\]

This relation is called the **Z Transform**.
**Z Transforms**

Example: Find the Z transform of $h_1[n]$: $h_1[n] = \begin{cases} (\frac{7}{8})^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

We can represent $h_1[n]$ by the system functional $1 + \frac{7}{8}R + \left(\frac{7}{8}\right)^2 R^2 + \cdots = \frac{1}{1 - \frac{7}{8}R}$

Substitute $R = z^{-1}$ to get $H_1(z) = \frac{1}{1 - \frac{7}{8}z^{-1}}$

**Z Transforms: Check Yourself**

Find the Z transform of $h_2[n]$: $h_2[n] = \begin{cases} -\left(\frac{1}{2}\right)^n & \text{if } n < 0 \\ 0 & \text{otherwise} \end{cases}$

What is the relation between $H_2(z)$ and $H_1(z)$?

1. $H_2(z) = -H_1(z)$
2. $H_2(z) = H_1(z)$
3. $H_2(z) = \frac{1}{H_1(z)}$

**Z Transforms**

The Z transform $H(z)$ is completely specified by $h[n]$. $H(z) = \sum_{n=-\infty}^{\infty} h[n]$

However, $h[n]$ is not completely specified by $H(z)$. In the previous example, $H_1(z) = H_2(z)$ even though $h_1[n] \neq h_2[n]$. 
When we represent a system with its system function, we cannot tell whether the system is causal, anticausal, or non-causal.

Similarly, when we represent a signal by its Z transform, we cannot tell whether the signal is left-sided, right-sided, or both-sided.

Thus the Z transform is not a complete representation of a signal. We need additional information: the region of convergence (ROC).

Consider the Z transform of
\[ h[n] = \begin{cases} p^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \frac{1}{1 - p/z} \]

But the sum only converges if \(|p/z| < 1\), i.e., \(|z| > |p|\).

Similarly, consider the Z transform of
\[ h[n] = \begin{cases} p^n, & \text{if } n \leq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \sum_{l=-\infty}^{\infty} p^{-l}z^{-l} = \frac{1}{1 - z/p} \]

And the sum only converges if \(|z/p| < 1\), i.e., \(|z| < |p|\).
Z Transforms

Knowing the ROC of \( \frac{1}{1-p^{-1}} \) tells you whether it came from \( \frac{1}{1-pR} \) or \( \frac{L}{L-p} \).

Z Transforms: Check Yourself

What is the ROC of the Z transform of the following function.

\[ h[n] = p|n| \]

-4 -3 -2 -1 0 1 2 3 4

1. \(|z| < |p|\)
2. \(|z| < |1/p|\)
3. \(|p| < |z| < |1/p|\)
4. \(|1/p| < |z| < |p|\)
5. none of the above

Summary

Z transform + ROC: complete description.

\( f(R) = \frac{1}{1-R-R^2} \)

\( H(z) = \frac{\sum_{n=0}^{\infty} h[n]z^{-n}}{1-Rz^{-1}} + \frac{\sum_{n=0}^{\infty} h[n]z^{-n}}{1-Rz^{-1}} \)

\( |z| > \phi \)
Multiple Representations of CT Systems

We can develop similar relations among the variety of representations that we have developed for CT systems.

Block diagram

System functional

Impulse response

Differential Equation

System function

Laplace Transforms

We can find the relation between $H(s)$ and $h(t)$ using the eigenfunction property and convolution.

Eigenfunction property:

If $x(t) = e^{st}$ (for all time $t$), then $y(t) = H(s)e^{st}$.

Calculate $y(t)$ using convolution.

$$y(t) = H(s)e^{st} = (x * h)(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st} \int_{-\infty}^{\infty} e^{-st}h(t)dt$$

Therefore

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$
Z transforms

You could have similarly derived the Z transform from the eigenfunction property and convolution. We didn’t because $h[n] \rightarrow f(R, L)$ is so easy.

Eigenfunction property:

If $x[n] = z^n$ (for all time $n$), then $y[n] = H(z)z^n$.

Calculate $y[n]$ using convolution.

$$y[n] = H(z)z^n = (x * h)[n] = \sum_{k=\infty}^{\infty} z^{n-k} h[k] = z^n \sum_{k=\infty}^{\infty} z^{-k} h[k]$$

Therefore

$$H(z) = \sum_{n=\infty}^{\infty} h[n]z^{-n}$$

Laplace Transforms

Example: Find the Laplace transform of $h_1(t)$:

$$h_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_1(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} e^{-t}e^{-st}dt = \left. \frac{e^{-(1+s)t}}{1+s} \right|_{0}^{\infty} = \frac{1}{1+s}$$

provided Re[1+s] > 0 which implies that Re{s} > -1.

Laplace Transforms: Check Yourself

Find the function $h_2(t)$ whose Laplace transform is $H_2(s) = \frac{1}{1+s}$ with ROC Re{s} < -1.
**Laplace Transforms: Check Yourself**

Let $X(s)$ represent the Laplace transform of $x(t)$.

Determine the Laplace transform $Y(s)$ of $y(t) = x(t-T)$ where $T$ is a real-valued constant. Express $Y(s)$ in terms of $X(s)$.

---

**Z Transforms: Check Yourself**

Let $X(z)$ represent the Z transform of $x[n]$.

Determine the Z transform $Y(z)$ of $y[n] = x[n-n_0]$ where $n_0$ is an integer. Express $Y(z)$ in terms of $X(z)$.

---

**Summary**

Today: new relations among system representations.

- Block diagram
- System functional
- Impulse response
- Differential Equation
- Series
- Partial fractions
- Laplace transform

$f(A) = \frac{A}{1-pA}$

$H(s) = \frac{1}{s-p}; \text{ Re}[s] > p$

$e^{pt}u(t)$
Multiple Representations of Signals

Today, we will develop a new representation for signals: **Fourier Series**, in which signals are constructed from sums of sinusoids.

We will also see how this new representation for signals leads to a new representation for systems as **filters**.

---

Fourier Series

In Fourier series, signals are described by the amplitudes and phases of harmonic components.
Fourier Series

Describing signals by their harmonic content is very natural for some kinds of signals.
Example: musical instruments
Musical Instruments

The frequency representation highlights qualities of the sounds.

Fourier Series

Harmonic structure provides a way to think about consonance and dissonance.

Fourier Series

Signals that are constructed from harmonic components are periodic in time because all of the harmonics of $\omega_0$ are periodic in $T = \frac{2\pi}{\omega_0}$. 
Fourier Series

What sort of periodic signals can be represented by Fourier series?

Harmonics are all continuous functions. Is it possible to represent discontinuous functions?

Fourier claimed that ANY periodic signal could be represented by a weighted sum of harmonics.

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

Fourier Series

There are many ways to approximate a periodic signal by a weighted sum of harmonics. One simple way takes advantage of orthogonality of harmonics.

We wish to write \( x(t) \) as a weighted sum of harmonics.

These harmonics are orthogonal: the integral of the product of one harmonic times the complex conjugate of a different harmonic is zero.

\[
\int_{T} e^{j2\pi k t} e^{-j2\pi l t} dt = \begin{cases} 
0 & ; k \neq l \\
T \delta[k-l] & ; k = l 
\end{cases}
\]

We can use orthogonality to “sift” out the weights in the weighted sum.

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k t} \\
\int_{0}^{T} x(t)e^{-j2\pi lt} dt = \sum_{k=-\infty}^{\infty} a_k \int_{0}^{T} e^{j2\pi k t} e^{-j2\pi lt} dt \\
= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = Ta_l \\
a_k = \frac{1}{T} \int_{0}^{T} x(t)e^{-j2\pi lt} dt
\]
Fourier Series

Since $x(t)$ and the harmonics are periodic in $T$, the integral can be evaluated over any time interval of length $T$.

Notation. If $f(t)$ is periodic in $T$ then

$$\int_{0}^{T} f(t)\,dt = \int_{t_0}^{t_0+T} f(t)\,dt = \int_{T} f(t)\,dt$$

Fourier Series

The continuous-time Fourier series representation is then defined as follows:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}kt}\,dt \quad \text{("analysis" equation)}$$

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \text{("synthesis" equation)}$$

Fourier Series

Determine the Fourier Series coefficients for the following square wave.
Fourier Series

If a signal is differentiated in time, its Fourier coefficients are multiplied by \( j \frac{2\pi}{T} k \).

If

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt} \]

then

\[ \dot{x}(t) = \dot{x}(t + T) = \sum_{k=-\infty}^{\infty} \left( j \frac{2\pi}{T} k a_k \right) e^{j2\pi kt} \]

Fourier Series

Determine the Fourier Series coefficients for the following triangle waveform.

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[ \sum_{k=-5}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt} \]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
X_k = \sum_{k=-39}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}
\]

Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[
X_k = \sum_{k=-5}^{5} \frac{1}{jk\pi} e^{j2\pi kt}
\]

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[
X_k = \sum_{k=-39}^{39} \frac{1}{jk\pi} e^{j2\pi kt}
\]
Fourier Series

Partial sums of Fourier series of discontinuous functions “ring” near discontinuities: Gibb’s phenomenon.

This ringing results because the magnitude of the Fourier coefficients is only decreasing as \( \frac{1}{k} \) (while they decreased as \( \frac{1}{k^2} \) for the triangle).

You can decrease (and even eliminate the ringing) by decreasing the magnitudes of the Fourier coefficients at higher frequencies (see homework).

Filtering

The Fourier series representation of a periodic signal provides a new way to think about systems: systems “filter” signals by their frequency content.

A Fourier series represents an arbitrary periodic signal as a sum of complex exponentials.

\[
x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi}{T} kt}
\]

Complex exponentials are eigenfunctions of LTI systems.

\[
e^{j \frac{2\pi}{T} kt} \rightarrow H(j \frac{2\pi}{T} k) e^{j \frac{2\pi}{T} kt}
\]

The system “filters” the input by adjusting the amplitude and phase of each harmonic component.

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi}{T} kt} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(j \frac{2\pi}{T} k) e^{j \frac{2\pi}{T} kt}
\]

Filtering Example: Speech

Speech is generated by the passage of air from the lungs, through the vocal cords, mouth, and nasal cavity.
Filtering Example: Speech

Controlled by complicated muscles, the vocal cords are set into vibrational motion by the passage of air from the lungs.

Looking down the throat:

Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.

Source-filter model of speech production.
Filtering Example: Speech

We detect changes in the filter function to recognize vowels ... at least sometimes. Demonstration.

Continuous-Time Fourier Series: Summary

Fourier series represent signals by their frequency content. Representing a signal by its frequency content is useful for many signals, e.g., music. Fourier series motivate a new representation of a system as a filter. Representing a system as a filter is useful for many systems, e.g., speech synthesis.
Multiple Representations of Signals

Last time, we developed a new representation for signals: Fourier Series, in which signals are constructed from sums of sinusoids.

That new representation for signals led to a new representation for systems as filters.

Today, we will generalize those results for periodic signals to aperiodic signals.

Fourier Transforms

A signal that is not periodic can be thought of as being periodic with an infinite period.

Let $x(t)$ represent an aperiodic signal.

\[
\text{"Periodic extension": } x_T(t) = \sum_{k=-\infty}^{\infty} x(t + kT)
\]

Then $x(t) = \lim_{T \to \infty} x_T(t)$. 
Fourier Transforms

Represent \( x_T(t) \) by its Fourier series.

\[
a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-j2\pi nk/T} dt = \frac{1}{T} \int_{-S}^{S} e^{-j2\pi nk} dt = \sin \frac{2\pi S}{\pi k} = \frac{2\sin \omega S}{\omega}
\]

Doubling the period doubles the number of harmonics in a given frequency interval.

As the period goes to infinity, the amplitudes of the harmonics converge to a function of \( \omega \).

\[
\lim_{T \to \infty} a_k = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \frac{2}{\omega} \sin \omega S = E(\omega)
\]
**Fourier Transforms**

In the limit $T \to \infty$, the Fourier series representation of $x(t)$ as a sum of harmonics passes to an integral of the function $E(\omega)$.

\[ x_T(t) \]

\[ \lim_{T \to \infty} T a_k = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

\[ x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} E(\omega) e^{j\omega k t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} E(\omega) e^{j\omega t} \to \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega) e^{j\omega t} d\omega \]

\[ \omega_0 \approx 2\pi/T \]

\[ \omega = k\omega_0 = k\frac{2\pi}{T} \]

---

**Fourier Transforms**

The function $E(\omega)$ is the Laplace transform of $x(t)$ evaluated at $s = j\omega$. Replacing $E(\omega)$ by $X(j\omega)$ yields the Fourier transform relations.

Fourier transform

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]  ("analysis" equation)

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]  ("synthesis" equation)

---

**Relation between Fourier and Laplace Transforms**

If the Laplace transform of a signal exists and if the ROC includes the $j\omega$ axis, then the Fourier transform is equal to the Laplace transform evaluated on the $j\omega$ axis.

Laplace transform:

\[ X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \]

Fourier transform:

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = H(s) \big|_{s=j\omega} \]
Relation between Fourier and Laplace Transforms

Compare Fourier and Laplace transforms of $x(t) = e^{-t}u(t)$.

Laplace transform

$$X(s) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+1)t}dt = \frac{1}{1+s}; \quad \text{Re}\{s\} > -1$$

Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(j\omega+1)t}dt = \frac{1}{1+j\omega}$$

Laplace Transforms

The Laplace transform maps a function of time $t$ to a complex-valued function of complex-valued domain $s$.

$$|X(s)| = \left|\frac{1}{1+s}\right|$$

Fourier Transforms

The Fourier transform maps a function of time $t$ to a complex-valued function of real-valued domain $\omega$.

$$|X(j\omega)| = \left|\frac{1}{1+j\omega}\right|$$
**Fourier Transforms**

A function of real domain $\omega$, the Fourier transform is often easier to visualize than the equivalent Laplace transform.

Example: square pulse

Laplace transform:

$$X(s) = \int_{-1}^{1} e^{-st} dt = \left[ \frac{e^{-st}}{-s} \right]_{-1}^{1} = \frac{1}{s} (e^s - e^{-s}) \quad [\text{function of } s = \sigma + j\omega]$$

Fourier transform

$$X(j\omega) = \int_{-1}^{1} e^{-j\omega t} dt = \left[ \frac{e^{-j\omega t}}{-j\omega} \right]_{-1}^{1} = \frac{2\sin \omega}{\omega} \quad [\text{function of } \omega]$$

**Laplace Transform**

The magnitude of this Laplace transform grows exponentially as $\text{Re}(s)$ increases or decreases.

$$|X_1(s)| = \left| \frac{1}{s} (e^s - e^{-s}) \right|$$

**Fourier Transform**

The Fourier transform is easier to visualize. It is a function of a single variable: frequency $\omega$.

Time representation:

Frequency representation:

$$X_1(j\omega) = \frac{2\sin \omega}{\omega}$$
Fourier Transforms: Check Yourself

The Fourier transform of $x_2(t)$ is $X_2(j\omega)$ shown below.

Which of the following is true?

1. $b = 1$ and $\omega_0 = \pi/2$
2. $b = 1$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

Fourier Transforms

One of the most useful features of the Fourier transform (and Fourier series) is the simple “inverse” Fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \text{(Fourier transform)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \text{("inverse" Fourier transform)}$$

The Fourier transform and its inverse have very similar forms.

Convert one to the other by

- $t \rightarrow \omega$
- $\omega \rightarrow -t$
- scale by $2\pi$

Fourier Transforms: Check Yourself

Find the impulse response of an “ideal” low pass filter.

$$H(j\omega)$$
Filtering

Representing signals by their Fourier transforms allows us to think about systems by the way they “filter” signals based on their frequency content.

The Fourier transform represents a signal as a sum of complex exponentials.
\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \]

Complex exponentials are eigenfunctions of LTI systems.
\[ e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t} \]

The system “filters” the input by adjusting the amplitude and phase of each frequency component.
\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t} d\omega \]

Filtering

Systems can be designed to selectively pass certain frequency bands. Examples: low-pass filter (LPF) and high-pass filter (HPF).
Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.

ECG and analysis by T. F. Weiss
Filtering Example: Electrocardiogram

Filter design: low-pass filter + high-pass filter + notch.

\[
f = \frac{\omega}{2\pi} \text{ [Hz]} \]

Electrocardiogram: Check Yourself

Which poles and zeros are associated with
- the high-pass filter?
- the low-pass filter?
- the notch filter?

Filtering Example: Electrocardiogram

By placing the poles of the notch filter very close to the zeros, the width of the notch can be made quite small.

\[
f = \frac{\omega}{2\pi} \text{ [Hz]} \]
Filtering Example: Electrocardiogram

Comparision of filtered and unfiltered electrocardiograms.

Unfiltered ECG

Filtered ECG

Continuous-Time Fourier Transforms: Summary

Fourier transforms represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., electrocardiogram.

Representing a signal by its frequency content motivates representing a system as a filter.

Representing a system as a filter is useful for many systems, e.g., electrocardiogram.
Applications of CT Fourier Series and Transforms

CT Fourier series and transforms allow us to represent signals by their frequency components. Representing signals by their frequency components is particularly useful for some types of signals.

Example: musical instruments

Musical Instruments

The frequency representation highlights qualities of the sounds.
Applications of CT Fourier Series and Transforms

Representing signals by their frequency components also motivates thinking about systems as “filters.”

Example: speech

Filtering Example: Speech

Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.

Electrocardiogram

The Fourier transform of the electrocardiogram revealed how to design a filter to eliminate unwanted “noise.”
DT Fourier series

Today, we will start to develop similar representations for DT signals and systems.

Check Yourself

Why bother with DT Fourier Series and Transforms?
Check Yourself

Why bother with DT Fourier Series and Transforms?

\[ x(t) \xrightarrow{A/D} x[n] \xrightarrow{\text{DT filter}} y[n] \xrightarrow{D/A} y(t) \]

Texas Instruments TAS3004
- 2 channels
- 24 bit ADC, 24 bit DAC
- 48 kHz sampling rate
- 100 MIPS
- $7.70 ($4.69 in bulk)

DT Fourier Series

Today we will develop Fourier Series representations for DT signals.

We will start by developing the notion of DT frequencies and frequency responses of DT systems.
**DT Eigenfunctions**

Geometric sequences are eigenfunctions of systems described by linear difference equations with constant coefficients.

Example:

\[ y[n] = x[n] + y[n-1] + y[n-2] \]

Let \( x[n] = p^n \) (for all time).

Try \( y[n] = \lambda p^n \).

Substitute and solve for \( \lambda \).

\[
\lambda p^n = p^n + \lambda p^{n-1} + \lambda p^{n-2} = p^n + \lambda p^{-1}p^n + \lambda p^{-2}p^n
\]

\[
\lambda = \frac{1}{1 - p^{-1} - p^{-2}} = H(z)|_{z=p}
\]

where

\[
H(z) = \frac{\sum \lambda_n z^{-n}}{\sum p_n z^{-n}} \rightarrow \frac{1}{z}
\]

**DT Vector Diagrams**

The value of \( H(z) \) at a point \( z = z_0 \) can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

\[
H(z_0) = \frac{\prod (z_0 - q_0)(z_0 - q_1) \cdots}{\prod (z_0 - p_0)(z_0 - p_1) \cdots}
\]

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here \( q_0 \)) to \( z_0 \), the point of interest in the \( z \)-plane.

**DT Frequency Response**

The system function can be used to determine the responses of a system to sinusoidal inputs.

Let \( x[n] = \cos \Omega_0 n \) (for all time), which can be written as a sum of eigenfunctions,

\[
x[n] = \frac{1}{2} (e^{j\Omega_0 n} + e^{-j\Omega_0 n}) = \frac{1}{2} (z_0^n + z_1^n)
\]

where \( z_0 = e^{j\Omega_0} \) and \( z_1 = e^{-j\Omega_0} \). The response to a sum is the sum of the responses.

\[
y[n] = \frac{1}{2} \left( H(z_0)z_0^n + H(z_1)z_1^n \right)
\]

\[
= \frac{1}{2} \left( H(e^{j\Omega_0})e^{j\Omega_0 n} + H(e^{-j\Omega_0})e^{-j\Omega_0 n} \right)
\]

This can be simplified because \( H(e^{j\Omega_0}) \) is conjugate symmetric.
**Conjugate Symmetry**

The complex conjugate of \( H(e^{j\Omega}) \) is \( H(e^{-j\Omega}) \).

Let

\[
H(z) = \frac{N(z)}{D(z)}
\]

where \( N(z) \) and \( D(z) \) are polynomials in \( z \).

\[
H(e^{j\Omega}) = \frac{N(e^{j\Omega})}{D(e^{j\Omega})}
\]

If the coefficients of the polynomials are real-valued (as they are for physical systems) then

\[
(H(e^{j\Omega}))^* = \frac{(N(e^{j\Omega}))^*}{(D(e^{j\Omega}))^*} = \frac{N(e^{-j\Omega})}{D(e^{-j\Omega})} = H(e^{-j\Omega})
\]

**DT Frequency Response (continued)**

The response of a system to a sinusoidal input

\[
x[n] = \cos \Omega_0 n = \frac{1}{2} (e^{j\Omega_0 n} + e^{-j\Omega_0 n})
\]

is

\[
y[n] = \frac{1}{2} \left( H(e^{j\Omega_0}) e^{j\Omega_0 n} + H(e^{-j\Omega_0}) e^{-j\Omega_0 n} \right)
\]

\[
= \text{Re} \left\{ H(e^{j\Omega_0}) e^{j\Omega_0 n} \right\}
\]

\[
= \text{Re} \left\{ H(e^{j\Omega_0}) |e^{j\Omega_0 n}| e^{j\Omega_0 n} \right\}
\]

\[
= |H(e^{j\Omega_0})| \text{Re} \left\{ e^{j\Omega_0 n+jH(e^{j\Omega_0})} \right\}
\]

\[
y[n] = |H(e^{j\Omega_0})| \cos \left( \Omega_0 n + \angle H(e^{j\Omega_0}) \right).
\]

**DT Frequency Response: Summary**

The frequency response of a DT system is determined by the values of the system function \( H(z) \) on the unit circle \( e^{j\Omega} \).

\[
H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}}
\]
## Finding Frequency Response with Vector Diagrams

The transfer function in the $z$-plane is given by:

$$ H(z) = \frac{z - q_1}{z - p_1} $$

The magnitude and phase of $H(e^{j\omega})$ can be visualized using vector diagrams.

### Comparision of CT and DT Frequency Responses

- **CT frequency response:** $H(s)$ on the imaginary axis, i.e., $s = j\omega$.
- **DT frequency response:** $H(z)$ on the unit circle, i.e., $z = e^{j\Omega}$.

### Periodicity of DT Frequency Responses

DT frequency responses are periodic functions of $\Omega$, with period $2\pi$.

If $\Omega_2 = \Omega_1 + 2\pi k$ where $k$ is an integer then

$$ H(e^{j\Omega_2}) = H(e^{j\Omega_1 + 2\pi k}) = H(e^{j\Omega_1}e^{2\pi jk}) = H(e^{j\Omega_1}) $$

The periodicity of $H(e^{j\Omega})$ results because $H(e^{j\Omega_1})$ is a function of $e^{j\Omega}$, which is itself periodic in $\Omega$. Thus DT complex exponentials have many “aliases.”

$$ e^{j\Omega_2} = e^{j\Omega_1}e^{2\pi jk} = e^{j(\Omega_1 + 2\pi k)n} $$

Because of this aliasing, there is a “highest” DT frequency: $\Omega = \pi$. 

**Check Yourself**

What filtering is done by a DT system with the following pole-zero diagram?

1. high pass  
2. low pass  
3. band pass  
4. band stop (notch)  
5. none of above

---

**DT Fourier Series**

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

\[ x[n] = \sum a_k e^{j\Omega_0 n} \]

The period N of all harmonic components is the same.

---

**Check Yourself**

What is the fundamental (shortest) period of each of the following signals?

1. \( x_1[n] = \cos \frac{\pi n}{12} \)
2. \( x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} \)
3. \( x_3[n] = \cos \pi + \cos 2\pi + \cos 3\pi \)
DT Fourier Series

There are N complex exponential functions of time with period N.

If \( e^{j\Omega n} \) is periodic in N then
\[ e^{j\Omega n} = e^{j\Omega (n+N)} = e^{j\Omega n} e^{j\Omega N} \]
and \( e^{j\Omega N} \) must be 1, and \( \Omega \) must be one of the Nth roots of 1.
Example: \( N = 8 \)

![z-plane](image)

DT Fourier Series

There are N distinct complex exponentials periodic with period N. These can be combined via Fourier series to produce periodic time signals with N independent samples.

Example: periodic in N=3
3 repeated samples in time

Example: periodic in N=4
4 repeated samples in time

DT Fourier Series

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

\[ x[n] = x[n+N] = \sum_{k=0}^{N-1} a_k e^{j\Omega_k n} \quad ; \quad \Omega_0 = \frac{2\pi}{N} \]

There are N equations (one for each point in time n) in n unknowns (\( a_k \)).

Example: \( N = 4 \)

\[
\begin{bmatrix}
    x[0] \\
    x[1] \\
    x[2] \\
    x[3]
\end{bmatrix}
= \begin{bmatrix}
    e^{j\frac{2\pi}{4}0} & e^{j\frac{2\pi}{4}1} & e^{j\frac{2\pi}{4}2} & e^{j\frac{2\pi}{4}3} \\
    e^{j\frac{2\pi}{4}1} & e^{j\frac{2\pi}{4}2} & e^{j\frac{2\pi}{4}3} & e^{j\frac{2\pi}{4}0} \\
    e^{j\frac{2\pi}{4}2} & e^{j\frac{2\pi}{4}3} & e^{j\frac{2\pi}{4}0} & e^{j\frac{2\pi}{4}1} \\
    e^{j\frac{2\pi}{4}3} & e^{j\frac{2\pi}{4}0} & e^{j\frac{2\pi}{4}1} & e^{j\frac{2\pi}{4}2}
\end{bmatrix}
\begin{bmatrix}
    a_0 \\
    a_1 \\
    a_2 \\
    a_3
\end{bmatrix}
\]
DT Fourier Series

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

\[ x[n] = x[n+N] = \sum_{k=0}^{N-1} a_k e^{j\Omega_0 n}; \quad \Omega_0 = \frac{2\pi}{N} \]

There are \( N \) equations (one for each point in time \( n \)) in \( N \) unknowns \( (a_k) \).

Example: \( N = 4 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & j & -1 & -j \\
1 & -1 & 1 & -1 \\
1 & -j & -1 & j
\end{bmatrix} = 
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]

Solving these equations is simple because these complex exponentials are orthogonal to each other.

\[
\sum_{n=0}^{N-1} e^{j\Omega_0 k n} e^{-j\Omega_0 l n} = \sum_{n=0}^{N-1} e^{j\Omega_0 (k-l) n} = \begin{cases} N & ; k = l \\
1 & ; k \neq l, \Omega_0 (k-l) = 0 \\
1 & ; k \neq l, \Omega_0 (k-l) = 0 \end{cases}
\]

\[ a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega_0 n} \]

We can use the orthogonality property of these complex exponentials to sift out the Fourier series coefficients, one at a time.

Assume \( x[n] = \sum_{k=0}^{N-1} a_k e^{j\Omega_0 n} \)

Multiply both sides by the complex conjugate of the \( l^{th} \) harmonic, and sum over time.

\[
\sum_{n=0}^{N-1} x[n] e^{-j\Omega_0 l n} = \sum_{n=0}^{N-1} a_k e^{j\Omega_0 n} e^{-j\Omega_0 l n} = \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j\Omega_0 n} e^{-j\Omega_0 l n} = \sum_{k=0}^{N-1} a_k N \delta[k-l] = N a_l
\]

\[ a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega_0 n} \]
**DT Fourier Series**

Since both $x[n]$ and $a_k$ are periodic in $N$, the sums can be taken over any $N$ successive indices.

**Notation.** If $f[n]$ is periodic in $N$, then

$$
\sum_{n=0}^{N-1} f[n] = \sum_{n=1}^N f[n] = \sum_{n=2}^{N+1} f[n] = \cdots = \sum_{n=\text{<N>}} f[n]
$$

**DT Fourier Series**

$$
a_k = a_{k+N} = \frac{1}{N} \sum_{n=\text{<N>}} x[n] e^{-j\Omega_0 n}; \quad \Omega_0 = \frac{2\pi}{N} \quad \text{("analysis" equation)}
$$

$$
x[n] = x[n+N] = \sum_{k=\text{<N>}} a_k e^{j\Omega_0 n} \quad \text{("synthesis" equation)}
$$

**DT Fourier series have simple matrix interpretations.**

$$
x[n] = x[n+4] = \sum_{k=\text{<4>}} a_k e^{j\Delta\Omega_0 n} = \sum_{k=\text{<4>}} a_k e^{j\frac{2\pi}{N} n} = \sum_{k=\text{<4>}} a_k e^{j\Omega n}
$$

$$
\begin{bmatrix}
  x[0] \\
  x[1] \\
  x[2] \\
  x[3]
\end{bmatrix} = 
\begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & j & -1 & -j \\
  1 & -1 & 1 & -1 \\
  1 & -j & 1 & j
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix}
$$

$$
a_k = a_{k+4} = \frac{1}{4} \sum_{n=\text{<4>}} x[n] e^{-j\Delta\Omega_0 n} = \frac{1}{4} \sum_{n=\text{<4>}} e^{-j\frac{2\pi}{N} n} = \frac{1}{4} \sum_{n=\text{<4>}} x[n] e^{-j\Omega n}
$$

$$
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} = 
\begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & -j & -1 & j \\
  1 & -1 & 1 & -1 \\
  1 & j & -1 & -j
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[1] \\
  x[2] \\
  x[3]
\end{bmatrix}
$$

These matrices are inverses of each other.

**All-Pass Filter: Effects of Phase**

$$
x[n] \xrightarrow{H(z)} \frac{1-az}{z-a} \quad y[n]
$$

$z$-plane

$|H(e^{j\Omega})|$ | $\angle H(e^{j\Omega})$
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$\frac{1}{2\pi}$</td>
<td>2$\pi$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>$\Omega$</td>
</tr>
</tbody>
</table>

33
All-Pass Filter: Effects of Phase

\[ x[n] \rightarrow H(z) = \frac{1 - az}{z - a} \rightarrow y[n] \]

All-Pass Filter: Effects of Phase


artificial speech synthesized by Robert Donovan
All-Pass Filter: Effects of Phase

\[ a_k = \sum x[n]e^{-j\Omega_0 n} \]
\[ b_k = \sum x[-n]e^{-j\Omega_0 n} = x[n]e^{j\Omega_0 n} = |a_k|e^{-j\angle a_k} \]

artificial speech synthesized by Robert Donovan

DT Fourier Series of Images

Magnitude

Angle

Uniform Angle
**DT Fourier Series of Images**

**Uniform Magnitude**

**Angle**

**DT Fourier Series of Images**

**Different Magnitude**

**Angle**

---

**DT Fourier Series: Summary**

Periodic DT signals can be represented by sums of sinusoids using DT Fourier series.

DT systems can be regarded as “filters” that modify the magnitude and angle of the frequency components of their inputs and outputs.

Thinking about signals by their frequency content and systems as filters has a large number of practical applications.
Relations among Fourier Representations

Different Fourier representations are related because they apply to signals that are related.

- DTFS (discrete-time Fourier series): periodic DT
- DTFT (discrete-time Fourier transform): aperiodic DT
- CTFS (continuous-time Fourier series): periodic CT
- CTFT (continuous-time Fourier transform): aperiodic CT

Duality of the Fourier transform

Because the forward and inverse Fourier transform relations are so similar, every transform pair has a dual.

If \( x(t) \leftrightarrow X(\omega) = g(\omega) \)

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega
\]

\[
g(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt
\]

Let \( \omega \to -t, \ t \to \omega \), and multiply and divide by \( 2\pi \):

\[
g(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{j\omega t} d\omega
\]

then \( g(-t) \leftrightarrow 2\pi x(\omega) \)
Duality of the Fourier transform

\[ x(t) \leftrightarrow g(\omega) \]

\[ \omega \rightarrow -t \quad t \rightarrow \omega; \quad 2\pi \]

\[ g(-t) \leftrightarrow 2\pi x(\omega) \]

\[ x(t) = \delta(t) \]

\[ H(j\omega) = g(\omega) = \]

\[ 2\pi \]

\[ x(t) = \cos \omega_0 t \]

\[ 1. \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) \]

\[ 2. \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \]

\[ 3. e^{j\omega_0} \]

\[ 4. \frac{j\omega}{\omega_0^2 - \omega^2} \]

\[ 5. \text{none of the above} \]

Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}; \quad w_0 = \frac{2\pi}{T} \]

Because the Fourier transform of \( e^{jkw_0 t} \) is \( 2\pi \delta(\omega - kw_0) \),

\[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - kw_0) \]

This expression shows the relation between the Fourier Series and Fourier transform for a periodic signal.

Check Yourself

Determine the Fourier transform of \( x(t) = \cos \omega_0 t \).

1. \( \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) \)
2. \( \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \)
3. \( e^{j\omega_0} \)
4. \( \frac{j\omega}{\omega_0^2 - \omega^2} \)
5. none of the above
Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

\[ x(t) = x(t + T) = \sum_{k=\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow \text{Fourier transform} \]

Check Yourself

What is the Fourier transform of the impulse train

\[ p(t) = \sum_{n=\infty}^{\infty} \delta(t - nT) \]

Relations among Fourier Representations

Start with an aperiodic CT signal. Determine its Fourier transform. Convert the signal so that it can be represented by alternate Fourier representations and compare.

\[ \text{periodic DT DTFS} \quad \rightarrow \quad N \rightarrow \infty \quad \rightarrow \quad \text{aperiodic DT DTFT} \]

\[ \downarrow \quad \text{periodic extension} \quad \downarrow \quad \text{interpolate} \quad \downarrow \quad \text{sample} \]

\[ \text{periodic CT CTFS} \quad \rightarrow \quad T \rightarrow \infty \quad \rightarrow \quad \text{aperiodic CT CTFT} \]

\[ \downarrow \quad \text{periodic extension} \quad \downarrow \quad \text{interpolate} \quad \downarrow \quad \text{sample} \]
Start with the CT Fourier Transform

Determine the Fourier transform of the following signal.

Could calculate Fourier transform from the definition.

Ugly. Alternatively, could calculate \( x(t) \) by convolution of two square pulses: \( x(t) = (y \ast y)(t) \).

Check Yourself

The Fourier transform of \( y(t) \) is a sinc function (i.e., of the form \( \frac{\sin \omega}{\omega} \)).

Determine \( C \) and \( \omega_0 \).

Start with the CT Fourier Transform

The Fourier transform of \( y(t) \) is a sinc function (i.e., of the form \( \frac{\sin \omega}{\omega} \)).

If \( x(t) = (y \ast y)(t) \) then \( X(j\omega) = Y(j\omega) \times Y(j\omega) \).
Relation between Fourier Transform and Series

Determine the Fourier transform of the periodic extension of \( x(t) \) to period \( T = 4 \).

\[
z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)
\]

Could calculate \( Z(j\omega) \) for the definition ... ugly.

Check Yourself

What is a good way to calculate the Fourier transform of \( z(t) \)?

\[
z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)
\]

Relation between Fourier Transform and Series

Multiply the Fourier transform of \( x(t) \) times the Fourier transform of \( p(t) \).

\[
X(j\omega)
\]

\[
P(j\omega)
\]

\[
Z(j\omega)
\]
Relation between Fourier Transform and Series

The Fourier transform of a periodically extended function is a discrete function of frequency $\omega$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$

Relation between Fourier Transform and Series

The weight (area) of each impulse in the Fourier transform of a periodically extended function is $2\pi$ times the corresponding Fourier series coefficient.

$$Z(j\omega)$$

Relation between Fourier Transform and Series

The effect of periodic extension of $x(t)$ to $z(t)$ is to sample the frequency representation.
**Relation between Fourier Transform and Series**

Periodic extension of a CT signal produces a discrete function of frequency.

Periodic extension

- convolving with impulse train in time
- multiplying by impulse train in frequency
- sampling in frequency

**Relations between CT and DT transforms**

Sampling a CT signal generates a DT signal.

\[ x[n] = x(nT) \]

Take \( T = \frac{1}{2} \).

**Relations between CT and DT transforms**

We can generate a signal with the same shape by multiplying \( x(t) \) by an impulse train with \( T = \frac{1}{2} \).

\[ x_p(t) = x(t) \times p(t) \]

where

\[ p(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT) \]
Relations between CT and DT transforms

Multiplying $x(t)$ by an impulse train in time is equivalent to convolving $X(j\omega)$ by an impulse train in frequency (then $\div 2\pi$).

The Fourier transform of the “sampled” signal $x_p(t)$ is periodic in $\omega$ with period $4\pi$.

The Fourier transform of the “sampled” signal $x_p(t)$ has the same shape as the DT Fourier transform of $x[n]$.
DT Fourier transform

The CT Fourier transform of a “sampled” signal \( x_p(t) \) is equal to the DT Fourier transform of the samples \( x[n] \) where \( \Omega = \omega T \), i.e., \( X(j\omega) = X(e^{j\Omega}) \) where \( \Omega = \omega T \).

The DT Fourier transform

\[
X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi n \Omega} \quad \text{ (“analysis” equation)}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j2\pi n \Omega} d\Omega \quad \text{ (“synthesis” equation)}
\]

CT Fourier transform

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{ (“analysis” equation)}
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) e^{j\omega t} d\omega \quad \text{ (“synthesis” equation)}
\]

Relation between CT and DT Fourier transforms

The CT Fourier transform of a “sampled” signal \( x_p(t) \) is equal to the DT Fourier transform of the samples \( x[n] \) where \( \Omega = \omega T \), i.e., \( X(j\omega) = X(e^{j\Omega}) \) where \( \Omega = \omega T \).
Relation Between DT Fourier Transform and Series

Periodic extension of a DT signal is equivalent to convolution of the signal with an impulse train.

\[ x[n] \]

\[ p[n] \]

\[ x_p[n] = (x * p)[n] \]

Relation Between DT Fourier Transform and Series

Convolution by an impulse train in time is equivalent to multiplication by an impulse train in frequency.

\[ X(e^{j\Omega}) \]

\[ P(e^{j\Omega}) \]

\[ X_p(e^{j\Omega}) \]

Relation Between DT Fourier Transform and Series

Periodic extension of a discrete signal \((x[n])\) results in a signal \((x_p[n])\) that is both periodic and discrete. Its transform \((X_p(e^{j\Omega}))\) is also periodic and discrete.

\[ x_p[n] = (x * p)[n] \]

\[ X_p(e^{j\Omega}) \]
**Relation Between DT Fourier Transform and Series**

The weight of each impulse in the Fourier transform of a periodically extended function is $2\pi$ times the corresponding Fourier series coefficient.

\[
X_p(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} a_k e^{-j\frac{2\pi}{N}k}\Omega
\]

**Relation between Fourier Transforms and Series**

The effect of periodic extension was to sample the frequency representation.

\[
X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] \delta(\Omega - n\Omega_0)
\]

**Relation between Fourier Transforms and Series**

Periodic extension of a DT signal produces a discrete function of frequency.

Periodic extension

\[
\text{convolving with impulse train in time} \rightarrow \text{sampling in frequency}
\]

\[
\begin{array}{c}
\text{periodic DT} \\
\text{DTFS}
\end{array}
\rightarrow
\begin{array}{c}
\text{aperiodic DT} \\
\text{DTFT}
\end{array}
\]

Interpolate \quad Sample

\[
\begin{array}{c}
\text{periodic CT} \\
\text{CTFS}
\end{array}
\rightarrow
\begin{array}{c}
\text{aperiodic CT} \\
\text{CTFT}
\end{array}
\]

\[
N \rightarrow \infty \quad \text{periodic extension (sampling in frequency)}
\]

\[
T \rightarrow \infty \quad \text{periodic extension}
\]
Relations among Fourier Representations

Different Fourier representations are related because they apply to signals that are related.

- DTFS (discrete-time Fourier series): periodic DT
- DTFT (discrete-time Fourier transform): aperiodic DT
- CTFS (continuous-time Fourier series): periodic CT
- CTFT (continuous-time Fourier transform): aperiodic CT

![Diagram showing relationships between DTFS, DTFT, CTFS, and CTFT]

- $N \to \infty$ for periodic extension
- $T \to \infty$ for periodic extension

Interpolate and sample between periodic and aperiodic representations.
Sampling

Sampling refers to conversion of a continuous-time signal to discrete time.

We have used sampling extensively throughout this subject. Sampling has many uses.

Sampling

Sampling allows the use of modern digital electronics to process, record, transmit, store, and retrieve CT signals.

1. audio: MP3, CD, cell phone
2. pictures: digital camera, inkjet printer
3. video: DVD
4. everything on the web
Sampling

Sampling is pervasive.
Example: digital cameras record sampled images.

Photographs in newsprint are “half-tone” images. Each point is black or white and the average conveys brightness.

Zoom in to see the binary pattern.
**Sampling**

Even high-quality photographic paper records discrete images. When AgBr crystals (0.04 – 1.5µm) are exposed to light, some of the Ag is reduced to metal. During “development” the exposed grains are completely reduced to metal and unexposed grains are removed.

---

**Sampling**

Every image that we see is sampled by the retina, which contains ≈100 million rods and 6 million cones (average spacing ≈3µm) which act as discrete sensors.

![Retina diagram](http://webvision.med.utah.edu/imageswv/sagschem.jpeg)

---

**Check Yourself**

Your retina is sampling this slide, which is composed of 1024 x 768 pixels.

Is the spatial sampling done by your rods and cones adequate to resolve individual pixels in this slide?
Sampling

Today, we will use Fourier representations to determine how sampling affects the information contained in a signal.

Sampling

We would like to sample in a way that preserves information, which may not seem possible.

\[ x(t) \]

Information between samples is lost. Therefore, the same samples can represent multiple signals.

\[ \cos \frac{7\pi}{3} n \quad \cos \frac{\pi}{3} n \]

Sampling

A simple sampling scheme is uniform sampling (sampling interval \( T \)).

\[ x[n] = x(nT) \]

To determine the effect of sampling, compare \( x(t) \) to \( x_p(t) \), a signal that is reconstructed from \( x[n] \).

A simple reconstruction scheme is impulse reconstruction.

\[ x_p(t) = \sum_n x[n]\delta(t - nT) \]
**Sampling**

Impulse reconstruction produces a signal $x_p(t)$ that is equal to the original signal $x(t)$ multiplied by an impulse train.

\[
x_p(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT)
\]

\[
= \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)
\]

\[
= \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT)
\]

\[
= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

\[
= x(t)p(t) ; \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

**Sampling**

Convolution by an impulse train in frequency introduces multiple copies of $X(j\omega)$.

\[
X(j\omega)
\]

\[
\begin{align*}
\omega
\end{align*}
\]

\[
P(j\omega)
\]

\[
\begin{align*}
\omega
\end{align*}
\]

\[
X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega)
\]

\[
\begin{align*}
-\omega_s & \quad \omega_s
\end{align*}
\]

\[
\omega_5 = \frac{2\pi}{T}
\]

**Check Yourself**

Sketch the DTFT of $x[n] = x(nT)$ where $x(t)$ has the following Fourier transform.

\[
X(j\omega)
\]

\[
\begin{align*}
\omega
\end{align*}
\]

DTFT of $x[n] : \quad X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$
**Sampling**

The high frequency copies can be removed with a low-pass filter.

\[ X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega) \]

The result must also be multiplied by \( T \) to undo the amplitude scaling of the impulse train in frequency.

Impulse reconstruction followed by ideal low-pass filtering is called **bandlimited reconstruction**.

**The Sampling Theorem**

If a signal is bandlimited, you can sample without losing information.

If \( x(t) \) is bandlimited so that

\[ X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_m \]

then \( x(t) \) is uniquely determined by its samples \( x(nT) \) if

\[ \omega_s = \frac{2\pi}{T} > 2\omega_m. \]

The minimum sampling frequency, \( 2\omega_m \), is called the “Nyquist rate.”

**Summary**

Sampling

\[ x(t) \rightarrow x[n] = x(nT) \]

Bandlimited Reconstruction

\[ x[n] \xrightarrow{\text{Impulse Reconstruction}} x_p(t) = \sum x[n]\delta(t - nT) \xrightarrow{\text{LPF}} x_r(t) \]

Sampling Theorem: If \( X(j\omega) = 0 \forall |\omega| > \frac{\omega_s}{2} \) then \( x_r(t) = x(t) \).
Check Yourself

We can hear sounds with frequency components between 20 Hz and 20 kHz.

What is the maximum sampling interval $T$ that can be used to sample a signal without loss of audible information?

1. 100 $\mu$s
2. 50 $\mu$s
3. 25 $\mu$s
4. 100$\pi$ $\mu$s
5. 50$\pi$ $\mu$s
6. 25$\pi$ $\mu$s

CT Model of Sampling and Reconstruction

Sampling followed by bandlimited reconstruction is equivalent to multiplying by an impulse train and then low-pass filtering.

$X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega)$

Aliasing

What happens if $X$ contains frequencies $|\omega| > \frac{\pi}{T}$?

$X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega)$
Aliasing

The effect of aliasing is to wrap frequencies.

\[ \frac{1}{2} \left( \frac{\omega}{\omega_s} - \frac{\omega}{\omega_s} \right) = \frac{1}{2} \left( \frac{\omega}{\omega_s} - \frac{1}{T} \right) \]

Input frequency

Output frequency

\[ X(j\omega) \]

Check Yourself

A periodic signal with a period of 0.1 ms is sampled at 44 kHz.

To what frequency does the eighth harmonic alias?

1. 18 kHz 2. 16 kHz 3. 14 kHz 4. 8 kHz 5. 6 kHz 6. none of the above

Aliasing

High frequency components of complex signals also wrap.

\[ X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega) \]
**Aliasing**

Aliasing increases as the sampling rate decreases.

\[
X(j\omega) \quad \omega
\]

\[
P(j\omega) \quad \omega
\]

\[
X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega)
\]

**Aliasing Demonstration**

Sampling Music

\[\omega_s = \frac{2\pi}{T} = 2\pi f_s\]

- \(f_s = 44.1\) kHz
- \(f_s = 22\) kHz
- \(f_s = 11\) kHz
- \(f_s = 5.5\) kHz
- \(f_s = 2.8\) kHz

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

**Anti-Aliasing Filter**

To avoid aliasing, remove the frequency components that alias before sampling.
Aliasing increases as the sampling rate decreases.

\[ X_p(j\omega) = \frac{1}{2\pi} (X \ast P)(j\omega) \]

\[ \omega_s^2 \frac{1}{T} \omega P(j\omega) \]

\[ -\omega_s \omega_s \frac{2\pi}{T} \]

Anti-aliasing Demonstration

Sampling Music
\[ \omega_s = \frac{2\pi}{T} = 2\pi f_s \]

- \( f_s = 11 \text{ kHz without Anti-aliasing} \)
- \( f_s = 11 \text{ kHz with Anti-aliasing} \)
- \( f_s = 5.5 \text{ kHz without Anti-aliasing} \)
- \( f_s = 5.5 \text{ kHz with Anti-aliasing} \)
- \( f_s = 2.8 \text{ kHz without Anti-aliasing} \)
- \( f_s = 2.8 \text{ kHz with Anti-aliasing} \)

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Sampling: Summary

Loss of information during sampling is easy to visualize using Fourier representations.

Signals that are bandlimited in frequency (e.g., \(-W < \omega < W\)) can be sampled without loss of information.

The minimum sampling frequency for sampling without loss of information is called the Nyquist rate. The Nyquist rate is twice the highest frequency contained in a bandlimited signal.

Sampling at frequencies below the Nyquist rate causes aliasing.

Aliasing can be eliminated by pre-filtering to remove frequency components that would otherwise alias.
Sampling

Last time, we discussed the use of “sampling” to convert continuous-time signals to discrete time and “bandlimited reconstruction” to convert back.

Sampling
\[ x(t) \rightarrow x[n] = x(nT) \]

Bandlimited Reconstruction

\[ x_p(t) = \sum x[n] \delta(t - nT) \]

\[ \text{LPF} \]

Sampling Theorem: If \( X(j\omega) = 0 \) \( \forall |\omega| > \frac{\omega_s}{2} \) then \( x_r(t) = x(t) \).

CT Model of Sampling and Reconstruction

We discussed time-domain representations of sampling and reconstruction.

Sampling followed by bandlimited reconstruction is equivalent to multiplying by an impulse train and then low-pass filtering.

\[ x(t) \rightarrow x_p(t) \]

\[ \text{LPF} \]

\[ p(t) = \text{“sampling function”} \]
Aliasing

We also discussed Fourier representations, which are useful for understanding the "sampling theorem" and aliasing.

\[
X(j\omega) = \frac{1}{2\pi} (X * P)(j\omega)
\]

\[
\omega_s = \frac{2\pi}{T}
\]

Anti-Aliasing Filter

The frequency representation also made it clear how to avoid aliasing: use an anti-aliasing filter.

Anti-aliasing Demonstration

Sampling Music

\[
\omega_s = \frac{2\pi}{T} = 2\pi f_s
\]

- \(f_s = 11\) kHz without Anti-aliasing
- \(f_s = 11\) kHz with Anti-aliasing
- \(f_s = 5.5\) kHz without Anti-aliasing
- \(f_s = 5.5\) kHz with Anti-aliasing
- \(f_s = 2.8\) kHz without Anti-aliasing
- \(f_s = 2.8\) kHz with Anti-aliasing

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin
Quantization

Analog to digital conversion requires not only sampling in time but also quantization in amplitude.

Bit rate = (# bits/sample) × (# samples/sec)

e.g., CD quality audio: bit rate =
(16 bits/sample) × (44,100 samples/sec) ≈ 0.7 Mbits/sec

Check Yourself

The original Milstein violin piece is taken from a CD. The sampling rate was 44.1 kHz and each sample was quantized to 16 bits.

If we re-quantize it to fewer bits, how few bits can we use without causing significant problems for lecture demos?

Quantizing Images

Converting an image from a continuous representation to a discrete representation involves the same sort of issues.

This image has 189 × 189 pixels, with brightness quantized to 8 bits.
Check Yourself
What is the most objectionable artifact of coarse quantization?

8 bit image 3 bit image

Check Yourself
When dithering, one adds a small amount (±1 quantum) of random noise to the image before quantizing. Since the noise is different for each pixel in the band, the noise causes some of the pixels to quantize to a higher value and some to a lower. But the average value of the brightness is preserved.

Check Yourself
What is the most objectionable artifact of dithering?

3 bit image 3 bit dithered image
Check Yourself

When dithering, one adds a small amount (±1 quantum) of random noise to the image before quantizing.

In Robert's technique, one then subtracts that same amount of random noise from the quantized value before displaying the result.

Quantizing Images with Robert's Method

3 bits with dither 3 bits with Robert's method

Quantizing Images: 3 bits

8 bits 3 bits

dither Robert's
Discrete-time Sampling

You can sample a DT signal much as you would a CT signal.

\[ x[n] \rightarrow \times \rightarrow x_p[n] \]

\[ p[n] = \sum_k \delta[n - kN] \]

\[ X_p(e^{j\Omega}) = \sum_n x_p[n]e^{-j\Omega n} = \sum_k x_p[k]e^{-j\Omega k/3} = X_p(e^{j\Omega/3}) \]
Discrete-time Sampling

But the shorter sequence has a wider frequency representation.

\[
X_b(e^{j\Omega}) = X_p(e^{j\Omega/3})
\]
Discrete-time Sampling: Progressive Refinement

Discrete-time Fourier representations are useful for coding digital images.
Example: JPEG (“Joint Photographic Experts Group”) encodes images by a sequence of transformations:
• color encoding
• DCT (discrete cosine transform): a kind of Fourier series
• quantization to achieve perceptual compression (lossy)
• Huffman encoding: lossless information theoretic coding

We will focus on the DCT and quantization of its components.
• the image is broken into 8 x 8 pixel blocks
• each block is represented by its 8 x 8 DCT coefficients
• each DCT coefficient is quantized, using higher resolutions for coefficients with greater perceptual importance

The discrete cosine transform (DCT) is similar to a Fourier series, but high-frequency components of the DCT are typically smaller in magnitude than those of a Fourier series.
Imagine coding the following 8 x 8 block.

For a two-dimensional transform, take the transforms of all of the rows, assemble those results into an image and then take the transforms of all of the columns of that image.
Periodically extend a row so that we can represent it with a Fourier series.

\[ x[n] = x[n + 8] \]

There are 8 distinct Fourier series coefficients.

\[ a_k = \frac{1}{8} \sum_{n=0}^{7} x[n] e^{j 2 \pi k \frac{n}{8}} ; \quad \phi_0 = \frac{2\pi}{8} \]

The DCT is based on a different periodic representation, shown below.

\[ y[n] = y[n + 16] \]

Which signal has greater high frequency content?

\[ x[n] = x[n + 8] \]
\[ y[n] = y[n + 16] \]
Periodic extension of an 8 × 8 pixel block can lead to a discontinuous function even when the “block” was taken from a smooth image.

Periodic extension of the type done for JPEG generates a continuous function from a smoothly varying image.

Although periodic in \( N = 16 \), \( y[n] \) can be represented by just 8 distinct DCT coefficients.

This results because \( y[n] \) is symmetric about \( n = -\frac{1}{2} \), and this symmetry introduces redundancy in the Fourier series representation.

Notice also that the DCT of a real-valued signal is real-valued.
The magnitudes of the higher order DCT coefficients are smaller than those of the Fourier series.

Humans are less sensitive to small deviations in high frequency components of an image than they are to small deviations at low frequencies. Therefore, the DCT coefficients are quantized more coarsely at high frequencies.

Divide coefficient \( b[m,n] \) by \( q[m,n] \) and round to nearest integer.

\[
q[m,n] \quad m \rightarrow \\
\begin{array}{cccccccc}
16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\
12 & 12 & 14 & 26 & 58 & 60 & 55 \\
14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\
\downarrow & 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\
24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\
49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\
72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \\
\end{array}
\]

Check Yourself

Which of the following tables of \( q[m,n] \) (top or bottom) will result in higher “quality” images?

\[
q[m,n] \quad n \rightarrow \\
\begin{array}{cccccccc}
16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\
12 & 12 & 14 & 26 & 58 & 60 & 55 \\
14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\
\downarrow & 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\
24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\
49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\
72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \\
\end{array}
\]

\[
q[m,n] \quad n \rightarrow \\
\begin{array}{cccccccc}
32 & 22 & 20 & 32 & 48 & 80 & 102 & 122 \\
24 & 24 & 28 & 38 & 52 & 116 & 120 & 110 \\
28 & 26 & 32 & 48 & 80 & 114 & 139 & 112 \\
\downarrow & 28 & 34 & 44 & 58 & 102 & 174 & 160 & 124 \\
48 & 70 & 110 & 129 & 162 & 208 & 226 & 194 \\
98 & 128 & 156 & 174 & 206 & 256 & 240 & 202 \\
144 & 184 & 190 & 196 & 224 & 200 & 206 & 198 \\
\end{array}
\]
Finally, encode the DCT coefficients for each block using “run-length” encoding followed by an information theoretic (lossless) “Huffman” scheme, in which frequently occurring patterns are represented by short codes.

The “quality” of the image can be adjusted by changing the values of $q[m,n]$. Large values of $q[m,n]$ result in large “runs” of zeros, which compress well.

**JPEG: Results**

1%: 1432 bytes 10%: 2457 bytes 20%: 3533 bytes
40%: 4795 bytes 80%: 8345 bytes 100%: 26k bytes

**Summary**

Today we looked at a number of applications of sampling.

- Sampling and Quantizing Music
- Sampling and Quantizing Images
- Upsampling and Downsampling Images
- JPEG Image Coding
Communications Systems

Today we will look at applications of signals and systems in communication systems.

Example: Transmit voice via telephone wires (copper)

- Mic
- Amp
- Telephone wire
- Amp
- Speaker

Works well: basis of local land-based telephones.

Wireless Communication

In cellular communication systems, signals are transmitted via electromagnetic (E/M) waves.

- Mic
- Amp
- E/M wave
- Amp
- Speaker

For efficient transmission and reception, the antenna length should be comparable to one-quarter of the wavelength.

Telephone-quality speech contains frequencies between 200 Hz and 3000 Hz.

How long should the antenna be?
Check Yourself

What frequency E/M wave is well matched to an antenna with a length of 4 cm (about 1.5 inches)?

1. 200 kHz 2. 2 MHz 3. 20 MHz
4. 200 MHz 5. 2 GHz 6. 20 GHz

Wireless Communication

Speech is not well matched to the wireless medium.
Many applications require the use of signals that are not well matched to the required media.

<table>
<thead>
<tr>
<th>signal</th>
<th>applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>audio</td>
<td>telephone, radio, phonograph, CD, cell phone, MP3</td>
</tr>
<tr>
<td>video</td>
<td>television, cinema, HDTV, DVD</td>
</tr>
</tbody>
</table>

Also, consider all of the different kinds of media used for internet signals: coaxial cable, twisted pair, cable TV, DSL, optical fiber, E/M.
We can often modify the signals to obtain a better match.
Today we will introduce simple matching strategies based on modulation.

Check Yourself

Construct a signal $Y$ that codes the audio frequency information in $X$ using frequency components near 2 GHz.

$|X(j\omega)|$

$\omega$

$|Y(j\omega)|$

$\omega_c$

Determine an expression for $Y$ in terms of $X$. 
Amplitude Modulation

Multiplying a signal by a sinusoidal carrier signal is called amplitude modulation. Amplitude modulation shifts the frequency components of $X$ by $\pm \omega_c$.

\[ x(t) \times y(t) \cos \omega_c t \]

How could you recover $x(t)$ from $y(t)$?
**Frequency-Division Multiplexing**

Multiple transmitters can co-exist, as long as the frequencies that they transmit do not overlap.

\[ x_1(t) \xrightarrow{\cos \omega_1 t} z_1(t) \]
\[ x_2(t) \xrightarrow{\cos \omega_2 t} z_2(t) \]
\[ x_3(t) \xrightarrow{\cos \omega_3 t} z_3(t) \]

**Frequency-Division Multiplexing**

Multiple transmitters simply sum (to first order).

\[ x_1(t) \xrightarrow{\cos \omega_1 t} z_1(t) \]
\[ x_2(t) \xrightarrow{\cos \omega_2 t} z_2(t) \]
\[ x_3(t) \xrightarrow{\cos \omega_3 t} z_3(t) \]

**Frequency-Division Multiplexing**

The receiver can select the transmitter of interest by choosing the corresponding demodulation frequency.

\[ Z(j\omega) \]
\[ X_3(j\omega) \]
\[ \omega_1, \omega_2, \omega_3 \]
**Broadcast Radio**

“Broadcast” radio was championed by David Sarnoff. He had previously worked at Marconi Wireless Telegraphy Company (point-to-point).

- envisioned “radio music boxes”
- analogous to newspaper, but at speed of light
- receiver must be cheap (as with newsprint)
- transmitter can be expensive (as with printing press)

![Sarnoff (left) and Marconi (right)]

**Check Yourself**

The problem with making an inexpensive radio receiver is that you must know the carrier signal exactly!

\[ z(t) = x(t) \cos(\omega_c t) \]

What happens if there is a phase shift \( \phi \) between the signal used to modulate and that used to demodulate?

**AM with Carrier**

One way to synchronize the sender and receiver is to send the carrier along with the message.

\[ z(t) = x(t) \cos(\omega_c t) + C \cos(\omega_c t) = x(t) + C \cos(\omega_c t) \]

Adding carrier is equivalent to shifting the DC value of \( x(t) \). If we shift the DC value sufficiently, the message is easy to decode: it is just the envelope (minus the DC shift).
**Inexpensive Radio Receiver**

If the carrier frequency is much greater than the highest frequency in the message, AM with carrier can be demodulated with a peak detector.

In AM radio, the highest frequency in the message is 5 kHz and the carrier frequency is between 500 kHz and 1500 kHz. This circuit is simple and inexpensive.

But there is a problem.

**Inexpensive Radio Receiver**

AM with carrier requires more power to transmit the carrier than to transmit the message!

Speech sounds have high crest factors (peak value divided by rms value). The DC offset $C$ must be larger than $x_p$ for simple envelope detection to work. The power needed to transmit the carrier can be $35^2 \approx 1000x$ that needed to transmit the message. Okay for broadcast radio (WBZ: 50 kwatts). Not for point-to-point (cell phone batteries wouldn’t last long!).

**Inexpensive Radio Receiver**

Envelope detection also cannot separate multiple senders.
Superheterodyne Receiver

Edwin Howard Armstrong invented the superheterodyne receiver, which made broadcast AM practical.

Amplitude, Phase, and Frequency Modulation

There are many ways to recode a signal to a different frequency band. Here are three.

Amplitude Modulation (AM): $y_1(t) = x(t) \cos \omega_c t$
Phase Modulation (PM): $y_2(t) = \cos(\omega_c t + kx(t))$
Frequency Modulation (FM): $y_3(t) = \cos(\omega_c t + \int_{-\infty}^{t} x(\tau)d\tau)$

Frequency Modulation

In FM, the instantaneous frequency of the carrier is modulated by the signal.

FM: $y_3(t) = \cos(\omega_c t + \int_{-\infty}^{t} x(\tau)d\tau)$

$\omega_i(t) = \frac{d}{dt}\phi(t) = \omega_c + kx(t)$
Frequency Modulation

Compare AM to FM for $x(t) = \cos \omega_m t$.

**AM:** $y_1(t) = (\cos \omega_m t + 1.1) \cos \omega_c t$

**FM:** $y_1(t) = \cos(\omega_c t + m \sin \omega_m t)$

Advantages: constant power, no need to transmit carrier (unless DC important), bandwidth?

---

Frequency Modulation

Early investigators thought that narrowband FM could have arbitrarily narrow bandwidth, allowing more channels than AM. Wrong!

$y_3(t) = \cos(\omega_c t + k \int_{-\infty}^{t} x(\tau)d\tau)$

$\approx \cos \omega_c t - \sin \omega_c t \left( k \int_{-\infty}^{t} x(\tau)d\tau \right)$

If $k \to 0$ then

- $\cos \left( k \int_{-\infty}^{t} x(\tau)d\tau \right) \to 1$
- $\sin \left( k \int_{-\infty}^{t} x(\tau)d\tau \right) \to k \int_{-\infty}^{t} x(\tau)d\tau$

Narrowband FM is very similar to AM! (integration does not change bandwidth)

---

Frequency Modulation

Wideband FM is useful because it is robust to noise.

**AM:** $y_1(t) = (\cos \omega_m t + 1.1) \cos \omega_c t$

**FM:** $y_1(t) = \cos(\omega_c t + m \sin \omega_m t)$

FM generates a very redundant signal, which is resilient to additive noise.