6.003 (Fall 2011)

Final Examination *December 19, 2011*

Name:

Kerberos Username:

Please circle your section number:

Section	*Time*
2	11 am
3	1 pm
4	2 pm

Grades will be determined by the correctness of your answers (explanations are not required).

Partial credit will be given for ANSWERS that demonstrate some but not all of the important conceptual issues.

You have **three hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

This quiz is closed book, but you may use four 8.5 × 11 sheets of paper (eight sides total).

No calculators, computers, cell phones, music players, or other aids.

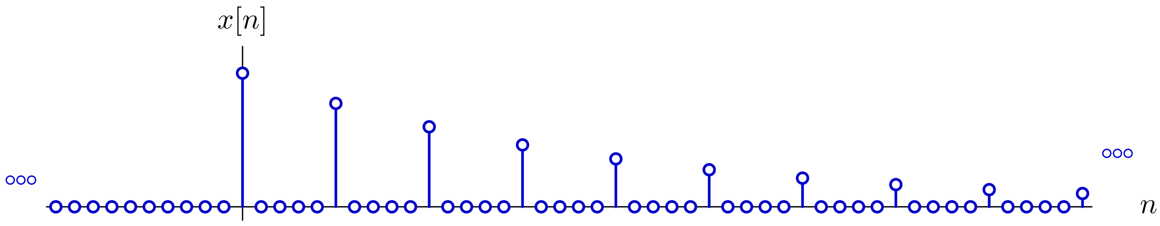
1	/14
2	/14
3	/12
4	/16
5	/16
6	/14
7	/14
Total	/100

1. Z Transform *[14 points]*

Determine $X(z)$, the Z transform of $x[n]$, where

$$x[n] \;=\; \sum_{k=0}^{\infty} a^k \delta[n-5k] \;=\; \delta[n] + a\delta[n-5] + a^2\delta[n-10] + a^3\delta[n-15] + \cdots$$

is plotted below.



Enter a closed-form expression for $X(z)$ in the box below.

$$X(z) = \boxed{\frac{z^5}{z^5 - a}}$$

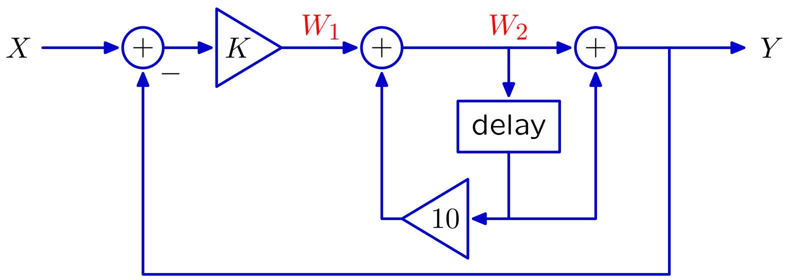
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} a^k \delta[n-5k]$$

The $\delta[n-5k]$ function is 1 if $n = 5k$ and zero otherwise.

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-5k} = \frac{1}{1-az^{-5}} = \frac{z^5}{z^5 - a}$$

2. DT Stability *[14 points]*

Determine the range of K for which the following discrete-time system is stable (and causal).



range of K: $K > 4.5$

$$W_2 = W_1 + 10\mathcal{R}W_2$$

$$Y = W_2 + \mathcal{R}W_2$$

$$\frac{Y}{W_1} = \frac{1+\mathcal{R}}{1-10\mathcal{R}}$$

$$\frac{Y}{X} = \frac{K\frac{1+\mathcal{R}}{1-10\mathcal{R}}}{1+K\frac{1+\mathcal{R}}{1-10\mathcal{R}}} = \frac{K(1+\mathcal{R})}{1-10\mathcal{R}+K+K\mathcal{R}} = \frac{K(z+1)}{z-10+Kz+K}$$

For stability, the pole must be inside the unit circle.

$$-1 < z = \frac{10-K}{1+K} < 1$$

$$K > 4.5$$

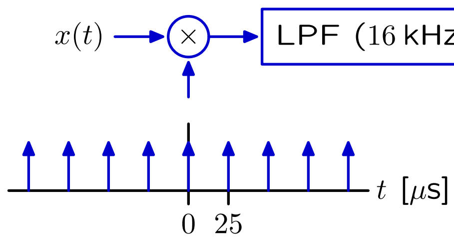
3. Harmonic Aliasing *[12 points]*

Let $x(t)$ represent a periodic signal with the following harmonics:

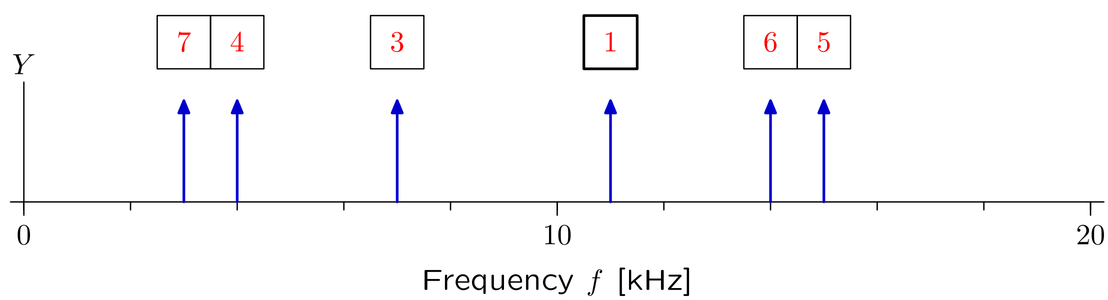
harmonic number	frequency [kHz]
1	11
2	22
3	33
4	44
5	55
6	66
7	77

Throughout this problem, frequencies (f) are expressed in cycles per second (Hz), which are related to corresponding radian frequencies (ω) by $f = \frac{\omega}{2\pi}$.

The signal $x(t)$ is multiplied by an infinite train of impulses separated by 25×10^{-6} seconds, and the result is passed through an ideal lowpass filter with a cutoff frequency of 16 kHz.

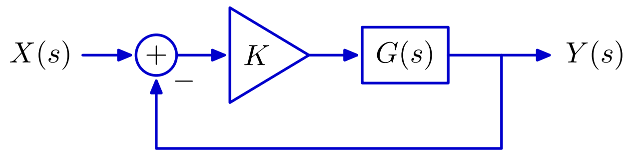


The plot below shows the Fourier transform Y of the output signal, for frequencies between 0 and 20 kHz. Write the number of the harmonic of $x(t)$ that produced each component of Y in the box above that component. If none of 1-7 could have produced this frequency, enter **X**.

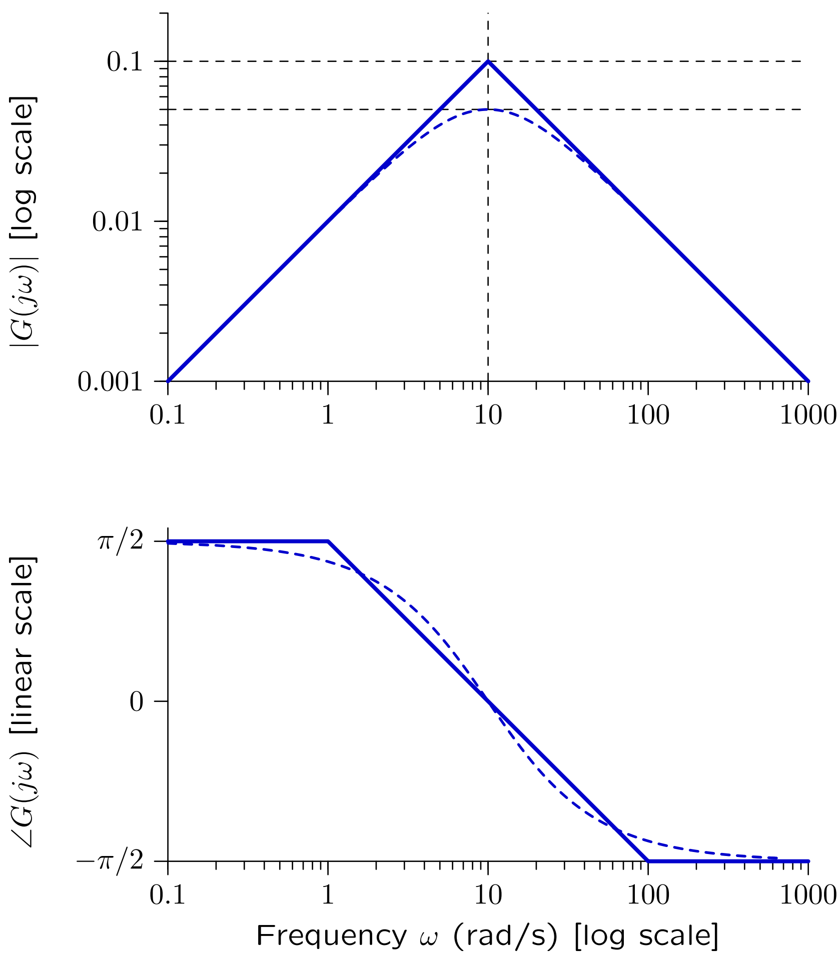


4. Feedback *[16 points]*

Let $H(s) = \dfrac{Y(s)}{X(s)}$ represent the system function of the following feedback system



where $G(s)$ represents a linear, time-invariant system. The frequency response of $G(s)$ is given by the following Bode plots (magnitude and frequency plotted on log scales).



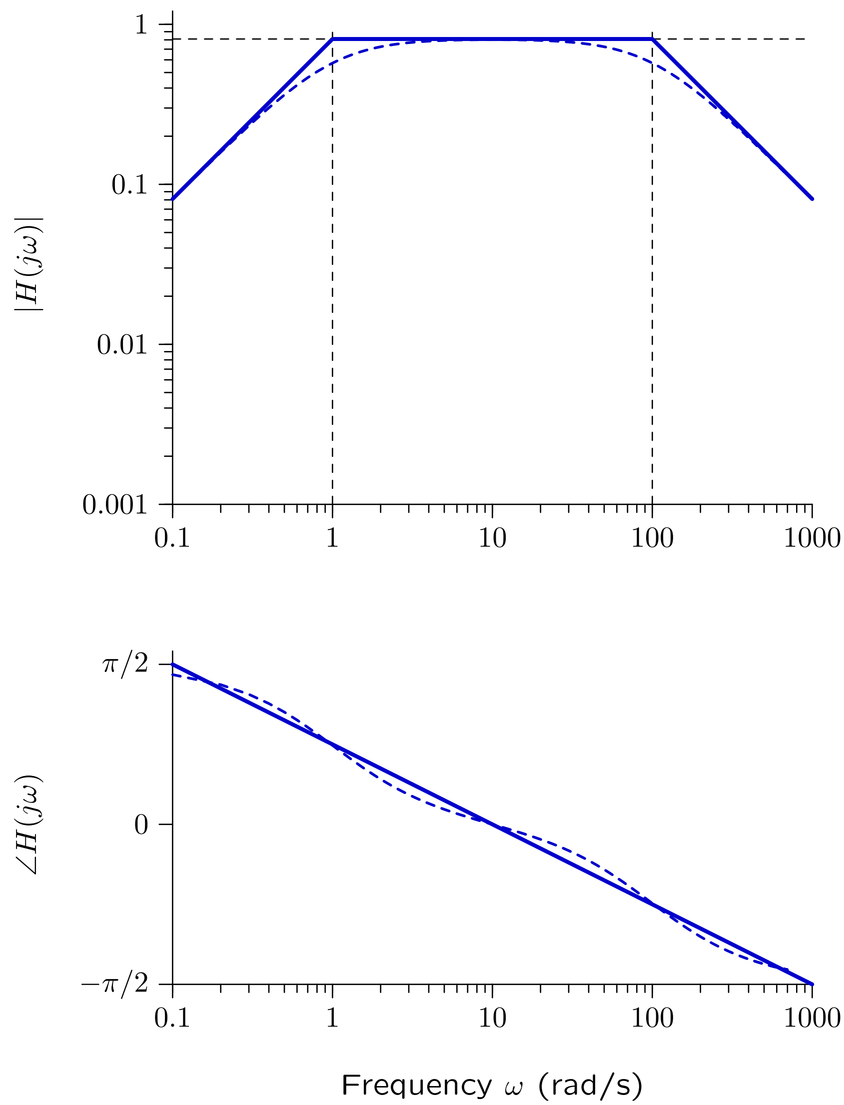
Part a. Determine a closed-form expression for $g(t)$, the impulse response of $G(s)$.

$g(t) =$ $(1 - 10t)e^{-10t}\, u(t)$

$$G(s) = \frac{s}{(s+10)^2} = \frac{1}{s+10} - \frac{10}{(s+10)^2}$$

$$g(t) = (1 - 10t)e^{-10t}\, u(t)$$

Part b. Sketch straight-line approximations (Bode plots) for the magnitude (log scale) and angle (linear scale) of $H(j\omega)$ when $K = 81$.
Clearly label all important magnitudes, angles, and frequencies.

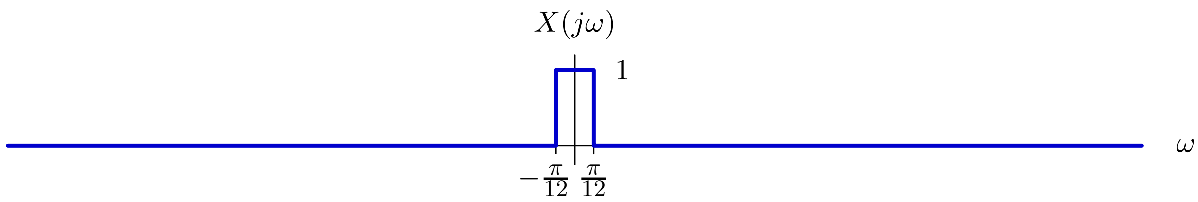


$$H(s) = \frac{KG(s)}{1+KG(s)} = \frac{K\frac{s}{(s+10)^2}}{1+K\frac{s}{(s+10)^2}}$$

$$= \frac{Ks}{s^2+20s+100+Ks} = \frac{81s}{s^2+101s+100} = \frac{81s}{(s+1)(s+100)}$$

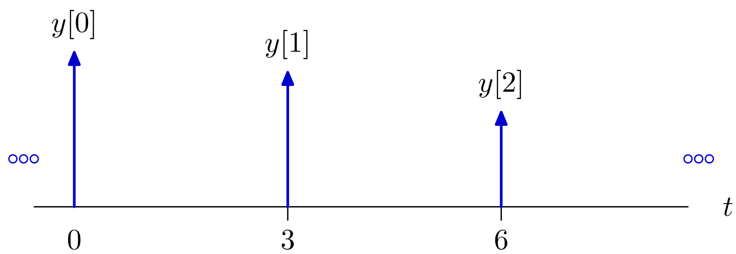
5. Triple reconstruction *[16 points]*

A CT signal $x(t)$ is sampled to produce a DT signal $y[n] = x(3n)$. The Fourier transform of $x(t)$ is given below.



We wish to compare two methods of using $y[n]$ to reconstruct approximations to $x(t)$.

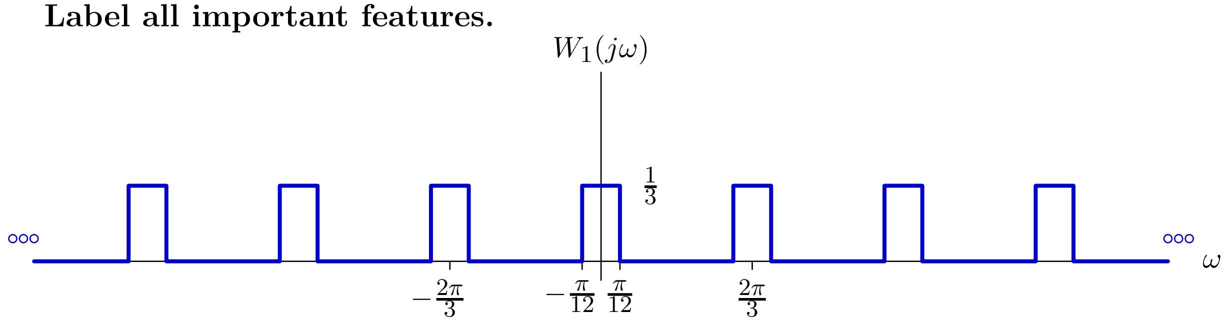
Part a. Let $w_1(t)$ represent a signal in which each sample of $y[n]$ is replaced by an impulse of area $y[n]$ located at $t = 3n$. Thus $w_1(t)$ has the following form



which can be represented mathematically as

$$w_1(t) = \sum_{n=-\infty}^{\infty} y[n]\,\delta(t - 3n)\,.$$

Sketch the Fourier transform of $w_1(t)$ on the axes below.
Label all important features.



Since $y[n] = x(3n)$, $w_1(t)$ is equal to $x(t)$ times and infinite train of unit impulses separated by $T = 3$ seconds:

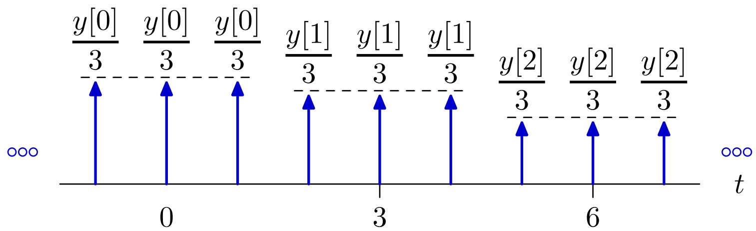
$$w_1(t) = \sum_{n=-\infty}^{\infty} y[n]\delta(t-3n) = \sum_{n=-\infty}^{\infty} x(3n)\delta(t-3n) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-3n)$$

where the last step follows from the fact that $\delta(t-3n)$ is zero except at $t = 3n$. Multiplication in time by an impulse train corresponds to convolution in frequency

$$W_1(j\omega) = X(j\omega) * P(j\omega)$$

where $P(j\omega)$ represents an infinite train of impulses, each of weight $2\pi/3$, and separated by $\omega = 2\pi/3$. (i.e., the Fourier transform of an infinite train of impulses spaced at $T = 3$ seconds). Thus the frequency content of $X(j\omega)$ is periodically replicated in frequency, spaced at $\Delta\omega = 2\pi/3$.

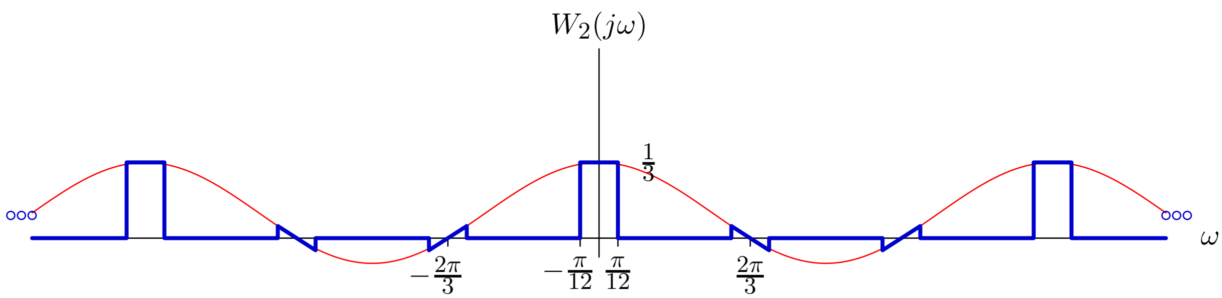
Part b. Let $w_2(t)$ represent a signal in which each sample of $y[n]$ is replaced by three impulses (one at $t = 3n-1$, one at $t = 3n$, and one at $t = 3n+1$), each with area $y[n]/3$. Thus $w_2(t)$ has the following form



which can be represented mathematically as

$$w_2(t) = \frac{1}{3}\sum_{n=-\infty}^{\infty} y[n]\left(\delta(t-3n-1) + \delta(t-3n) + \delta(t-3n+1)\right).$$

Sketch the Fourier transform of $w_2(t)$ on the axes below.
Label all important frequencies as well as the value of $W_2(j\omega)$ at $\omega = 0$.



The signal $w_2(t)$ can be derived by convolving $w_1(t)$ with a signal

$$s(t) = \frac{1}{3}\delta(t-1) + \frac{1}{3}\delta(t) + \frac{1}{3}\delta(t+1).$$

Thus $W_2(j\omega)$ can be determined by multiplying $W_1(j\omega)$ by the Fourier transform of $s(t)$

$$S(j\omega) = \frac{1}{3}e^{-j\omega} + \frac{1}{3} + \frac{1}{3}e^{j\omega} = \frac{1}{3} + \frac{2}{3}\cos\omega$$

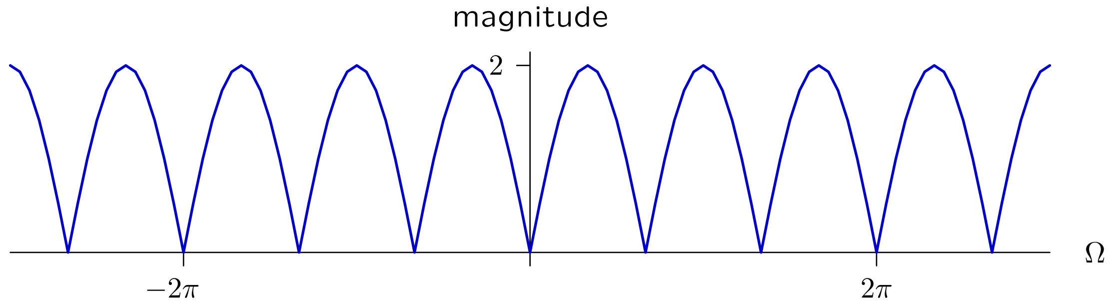
which is shown in red above.

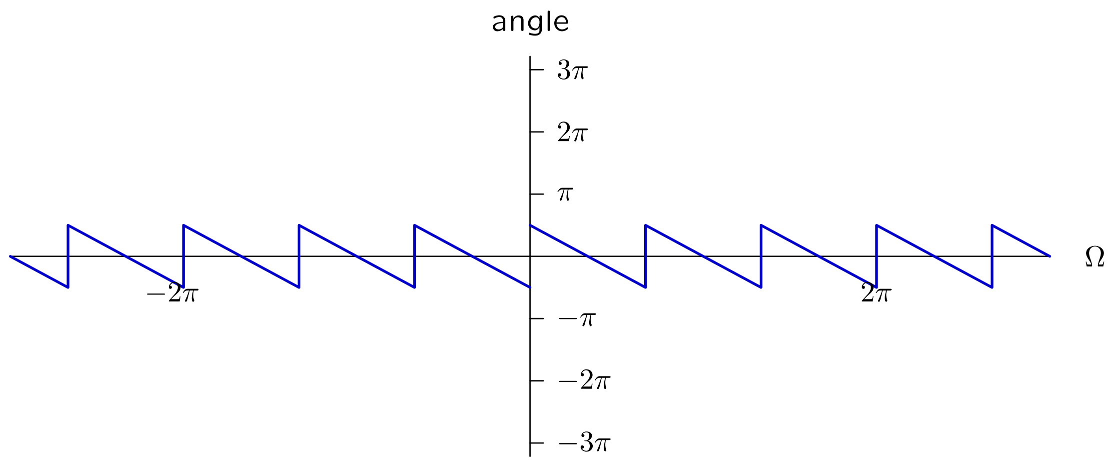
6. DT Filtering *[14 points]*

Sketch the magnitude and angle of the frequency response of a linear, time-invariant system with the following unit-sample response:

$$h[n] = \delta[n] - \delta[n-3]\,.$$

Label all important magnitudes, angles, and frequencies. All scales are linear.

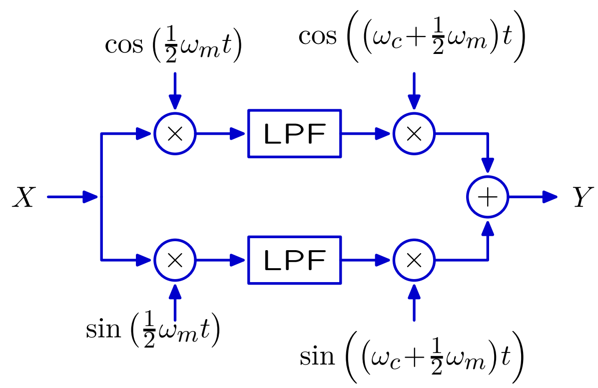




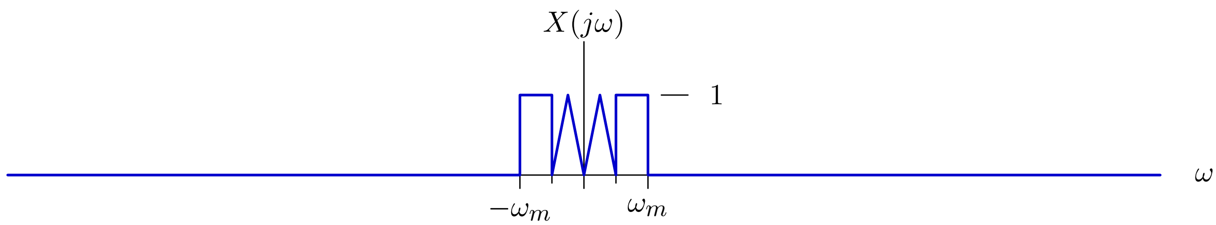
$$H(e^{j\Omega}) = 1 - e^{-j3\Omega} = e^{-j3\Omega/2}\left(e^{j3\Omega/2} - e^{-j3\Omega/2}\right) = j2e^{-j3\Omega/2}\sin\frac{3\Omega}{2}$$

7. Bandwidth Conservation *[14 points]*

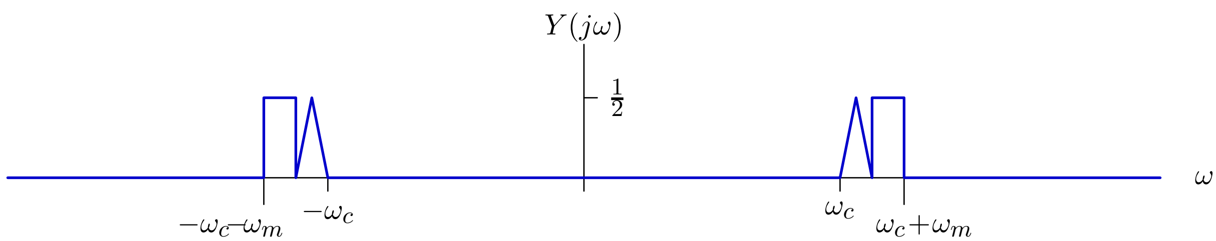
Consider the following modulation scheme, where $\omega_c >> \omega_m$.



Assume that each lowpass filter (LPF) is ideal, with cutoff frequency $\omega_m/2$. Also assume that the input signal has the following Fourier transform.



Sketch $Y(j\omega)$ on the following axes.
Label all important magnitudes and frequencies.



Worksheet (intentionally blank)

Worksheet (intentionally blank)

Worksheet (intentionally blank)