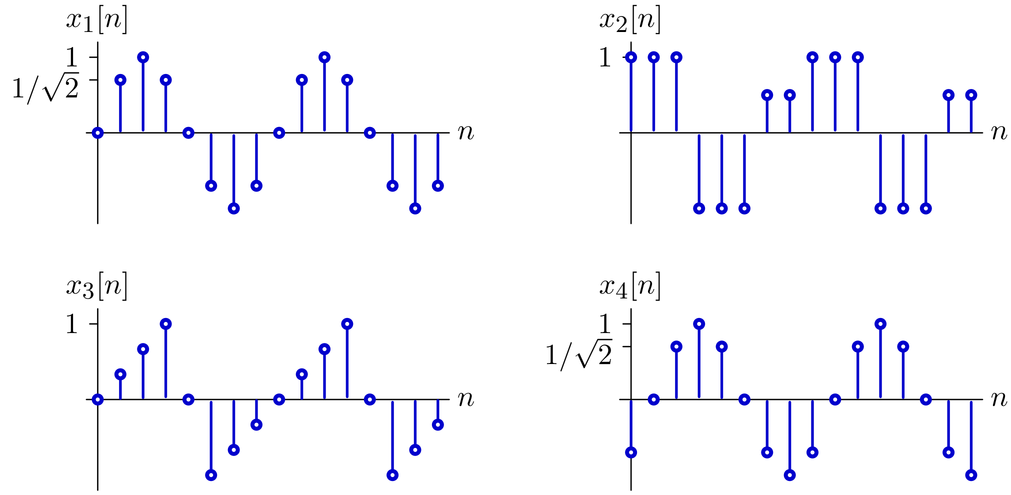
6.003 Homework #10 Solutions

Problems

1. DT Fourier Series

Determine the Fourier Series coefficients for each of the following DT signals, which are periodic in $N = 8$.



$$x_1[n] = \sin\frac{\pi n}{4} = \frac{e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}}}{j2} = \sum_{<N>} a_k e^{j\frac{2\pi}{N}kn}$$

$$a_k = a_{k+8} = \begin{cases} \frac{1}{j2} & k = 1 \\ -\frac{1}{j2} & k = -1 \\ 0 & |k| = 0, 2, 3, 4 \end{cases}$$

$$b_k = b_{k+8} = \frac{1}{N}\sum_{<N>} x[n]e^{-j\frac{2\pi}{N}kn}$$
$$= \frac{1}{8}\left(1 + e^{-j\frac{k\pi}{4}} + e^{-j\frac{k\pi}{2}} - e^{-j\frac{3k\pi}{4}} - e^{-j\pi k} - e^{-j\frac{5k\pi}{4}} + \frac{1}{2}e^{-j\frac{3k\pi}{2}} + \frac{1}{2}e^{-j\frac{7k\pi}{4}}\right)$$

$$c_k = c_{k+8} = \frac{1}{N}\sum_{<N>} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{e^{-j\frac{k\pi}{4}} - e^{j\frac{k\pi}{4}}}{24} + \frac{e^{-j\frac{k\pi}{2}} - e^{j\frac{k\pi}{2}}}{12} + \frac{e^{-j\frac{3k\pi}{4}} - e^{j\frac{3k\pi}{4}}}{8}$$
$$= -\frac{j}{12}\sin\frac{k\pi}{4} - \frac{j}{6}\sin\frac{k\pi}{2} - \frac{j}{4}\sin\frac{3k\pi}{4}$$

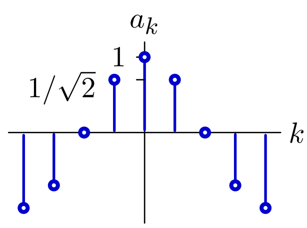
Because $x_3[n]$ is real-valued and an odd function of n, the series is purely imaginary.

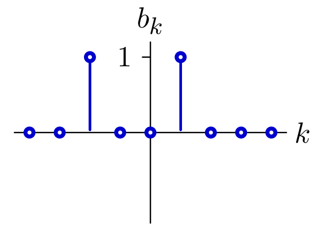
$$x_4[n] = x_1[n-1]$$

$$d_k = d_{k+8} = e^{-j\frac{2\pi}{N}}a_k = d_k = \begin{cases} \frac{1}{j2}e^{-j\pi/4} & k = 1 \\ -\frac{1}{j2}e^{-j\pi/4} & k = -1 \\ 0 & |k| = 0, 2, 3, 4 \end{cases}$$

2. Inverse DT Fourier Series

Determine the DT signals with the following Fourier series coefficients. Assume that the signals are periodic in $N = 8$.





$$a_k = \cos\frac{\pi k}{4} = \frac{e^{\frac{j2\pi k}{8}} + e^{\frac{-j2\pi k}{8}}}{2} = \frac{1}{N}\sum_{<N>} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$x_1[n] = 4\delta[n-1] + 4\delta[n+1]$$

for $|n| < 5$. Since $x_1[n]$ is periodic, a more general expression is

$$x_1[n] = \sum_{k=-\infty}^{\infty} 4\delta[n-1+8k] + 4\delta[n+1+8k]$$

$$x_2[n] = \sum_{k=<N>} b_k e^{j\frac{2\pi}{N}kn} = e^{j\frac{2\pi}{8}n} + e^{-j\frac{2\pi}{8}2n}$$

Notice that $x_2[n]$ has imaginary components, because the Fourier series coefficients are not conjugate symmetric ($b_{-k} \neq b_k^*$).

3. Impulsive Input

Let the following periodic signal

$$x(t) = \sum_{m=-\infty}^{\infty} \delta(t-3m) + \delta(t-1-3m) - \delta(t-2-3m)$$

be the input to an LTI system with system function

$$H(s) = e^{s/4} - e^{-s/4}\,.$$

Let b_k represent the Fourier series coefficients of the resulting output signal $y(t)$. Determine b_3.

The period of $x(t)$ is $T = 3$. Therefore the period of $y(t)$ is also $T = 3$. The fundamental frequency of $x(t)$ (and $y(t)$) is $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$.

Let a_k represent the Fourier series coefficients for $x(t)$. Then

$$a_k = \frac{1}{3}\int_0^3 \left(\delta(t) + \delta(t-1) - \delta(t-2)\right) e^{-j\frac{2\pi}{3}kt} dt = \frac{1}{3}\left(1 + e^{-j\frac{2\pi}{3}k} - e^{-j\frac{2\pi}{3}k2}\right)$$

$$a_3 = \frac{1}{3}$$

The frequency response of the system is given by

$$H(j\omega) = e^{j\omega/4} - e^{-j\omega/4} = j2\sin\frac{\omega}{4}$$

Therefore

$$b_k = H\left(j\frac{2\pi}{3}k\right) a_k = \left(j2\sin\frac{2\pi k}{12}\right) a_k$$

and

$$b_3 = \left(j2\sin\frac{\pi}{2}\right)\frac{1}{3} = j\frac{2}{3}\,.$$

4. Fourier transform

Part a. Find the Fourier transform of

$$x_1(t) = e^{-|t|}\,.$$

$$\begin{aligned}
X_1(j\omega) &= \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t}dt + \int_{0}^{\infty} e^{(-1-j\omega)t}dt = \left.\frac{e^{(1-j\omega)t}}{1-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{(-1-j\omega)t}}{-1-j\omega}\right|_{0}^{\infty} \\
&= \frac{1}{1-j\omega} - \frac{1}{-1-j\omega} = \frac{2}{1+\omega^2}
\end{aligned}$$

Part b. Find the Fourier transform of

$$x_2(t) = \frac{1}{1+t^2}\,.$$

Hint: Try duality.

By duality, if

$$e^{-|t|} \leftrightarrow \frac{2}{1+\omega^2}$$

then

$$\frac{2}{1+t^2} \leftrightarrow 2\pi e^{-|\omega|}$$

Therefore

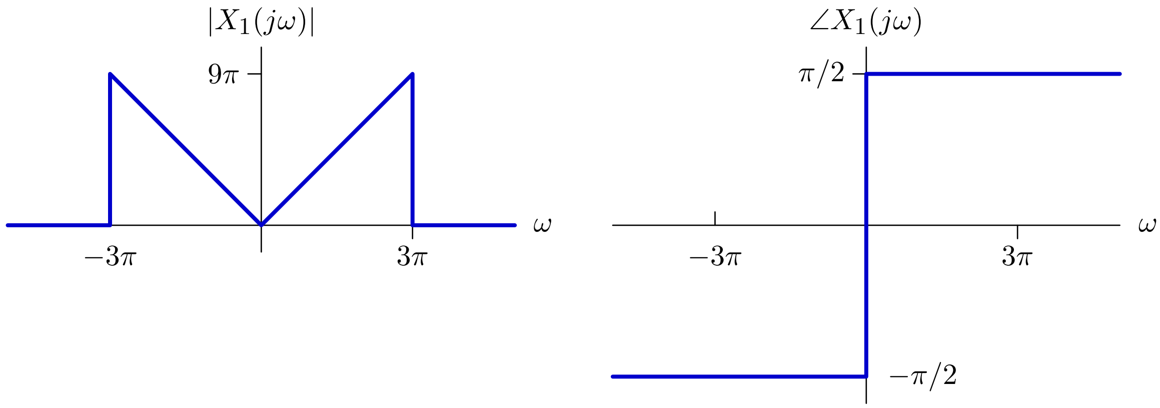
$$X_2(j\omega) = \pi e^{-|\omega|}$$

We can check this result by inverse transforming $X_2(j\omega)$:

$$\begin{aligned}
x_2(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi e^{-|\omega|}e^{j\omega t}d\omega = \frac{1}{2}\int_{-\infty}^{0} e^{\omega}e^{j\omega t}d\omega + \frac{1}{2}\int_{0}^{\infty} e^{-\omega}e^{j\omega t}d\omega \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{(1+jt)\omega}d\omega + \frac{1}{2}\int_{0}^{\infty} e^{(-1+jt)\omega}d\omega = \frac{1}{2}\left.\frac{e^{(1+jt)\omega}}{1+jt}\right|_{-\infty}^{0} + \frac{1}{2}\left.\frac{e^{(-1+jt)\omega}}{-1+jt}\right|_{0}^{\infty} \\
&= \frac{\frac{1}{2}}{1+jt} - \frac{\frac{1}{2}}{-1+jt} = \frac{1}{1+t^2}
\end{aligned}$$

5. Fourier transform

Part a. Determine $x_1(t)$, whose Fourier transform $X_1(j\omega)$ has the following magnitude and angle.



Express $x_1(t)$ as a closed-form and sketch this function of time.

$$X_1(j\omega) = |X_1(j\omega)|e^{j\angle X_1(j\omega)} = \begin{cases} 3j\omega & |\omega| < 3\pi \\ 0 & \text{otherwise} \end{cases}$$

Notice that $X_1(j\omega)$ is $3j\omega$ times $X_{1a}(j\omega)$ defined as

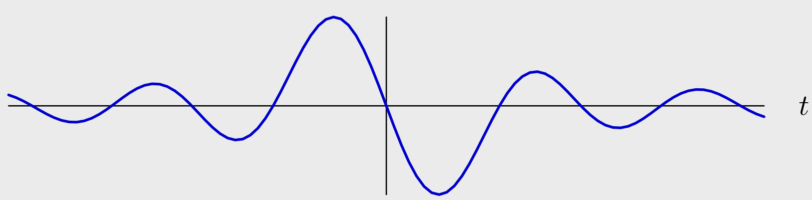
$$X_{1a}(j\omega) = \begin{cases} 1 & |\omega| < 3\pi \\ 0 & \text{otherwise} \end{cases}$$

Thus $x_1(t) = 3\frac{d}{dt}x_{1a}(t)$ where

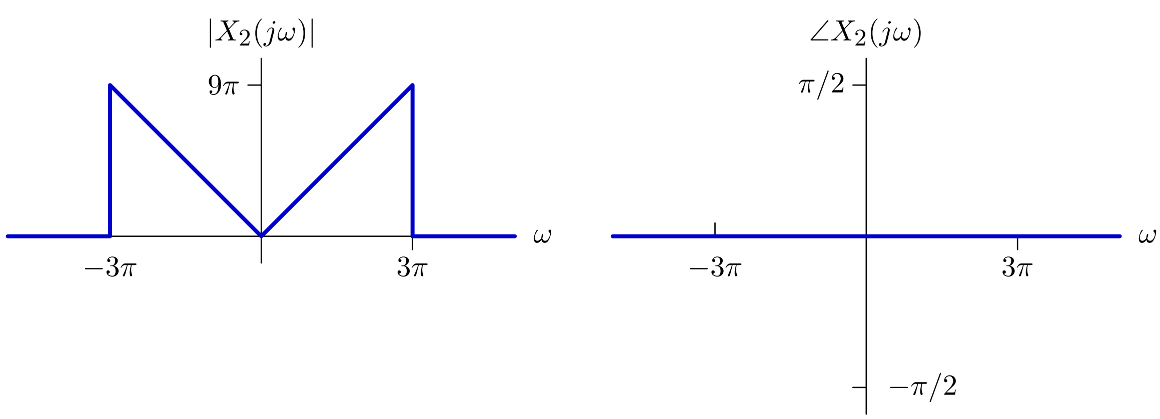
$$x_{1a}(t) = \frac{\sin 3\pi t}{\pi t}\,.$$

Thus

$$x_1(t) = \frac{3}{\pi t^2}\left(3\pi t \cos 3\pi t - \sin 3\pi t\right)$$

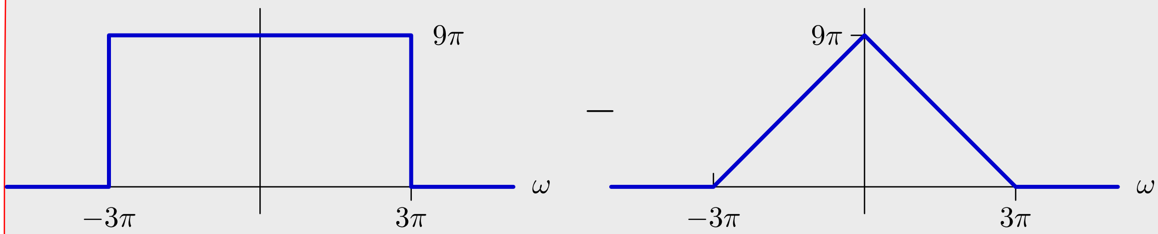


Part b. Determine $x_2(t)$, whose Fourier transform $X_2(j\omega)$ has the following magnitude and angle.



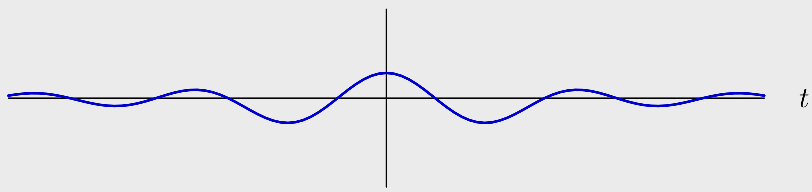
Express $x_2(t)$ as a closed-form and sketch this function of time.

$X_2(j\omega)$ can be expressed as a difference:



Thus

$$x_2(t) = 9\pi\frac{\sin 3\pi t}{\pi t} - 6\pi\frac{\sin^2 \frac{3\pi}{2}t}{\pi^2 t^2}$$



Part c. What are important similarities and differences between $x_1(t)$ and $x_2(t)$? How do those similarities and differences manifest in their Fourier transforms?

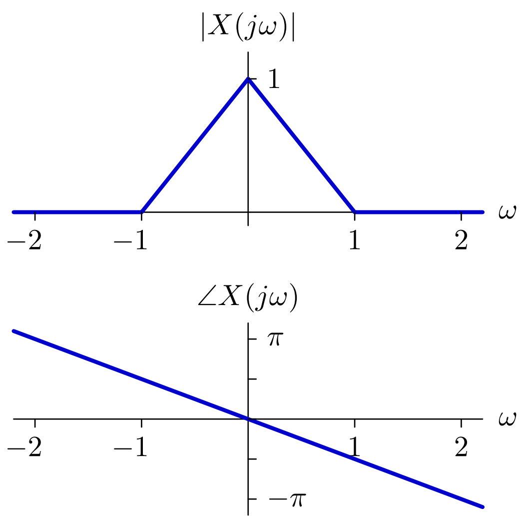
Both $x_1(t)$ and $x_2(t)$ are real functions of time. However, $x_1(t)$ is an odd function of time and $x_2(t)$ is an even function of time. Taken together, these features mean that $X_1(j\omega)$ is an odd function of ω that is purely imaginary, and $X_2(\omega)$ is an even function of ω that is purely real.

Both $X_1(j\omega)$ and $X_2(j\omega)$ are zero for $|\omega| > 3\pi$. Therefore, both $x_{(}t)$ and $x_2(t)$ have infinite extents in time.

Both $X_1(j\omega)$ and $X_2(j\omega)$ are discontinuous functions of ω. Thus, the magnitudes of $x_1(t)$ and $x_2(t)$ both decrease as $\frac{1}{t}$ for large t.

6. Fourier Transforms

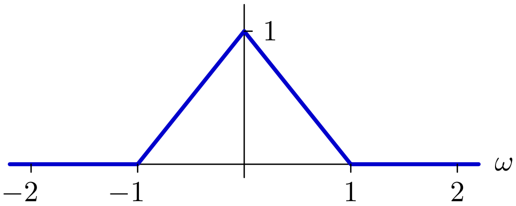
The magnitude and angle of the Fourier transform of a signal $x(t)$ are given in the following plots.



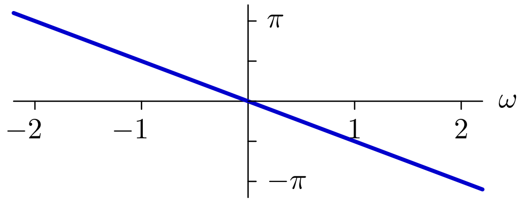
Five signals are derived from $x(t)$ as shown in the left column of the following table. Six magnitude plots (M1-M6) and six angle plots (A1-A6) are shown on the next page. Determine which of these plots is associated with each of the derived signals and place the appropriate label (e.g., M1 or A3) in the following table. Note that more than one derived signal could have the same magnitude or angle.

signal	magnitude	angle
$\frac{dx(t)}{dt}$	M5	A4
$(x * x)(t)$	M3	A2
$x\left(t - \frac{\pi}{2}\right)$	M1	A2
$x(2t)$	M4	A3
$x^2(t)$	M6	A1

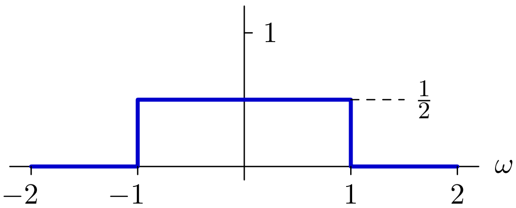
M1



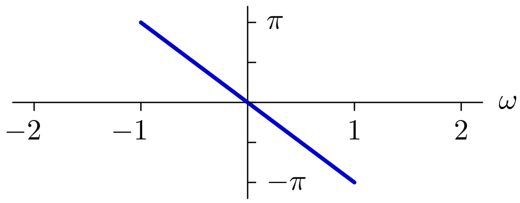
A1



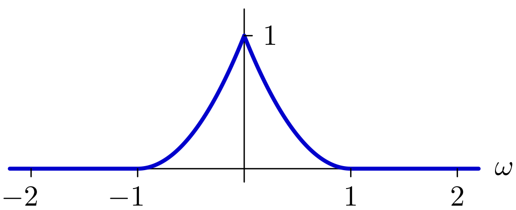
M2



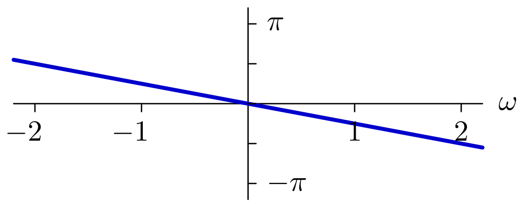
A2



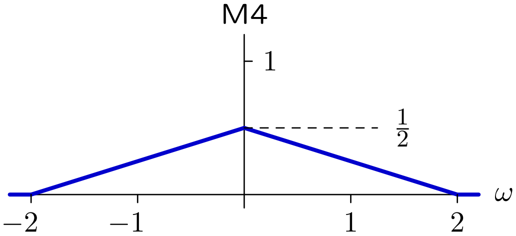
M3



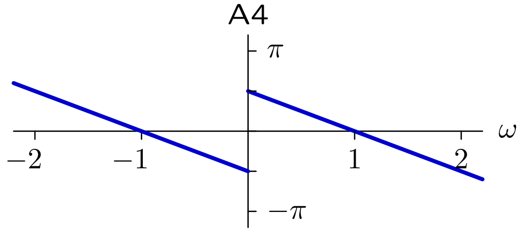
A3



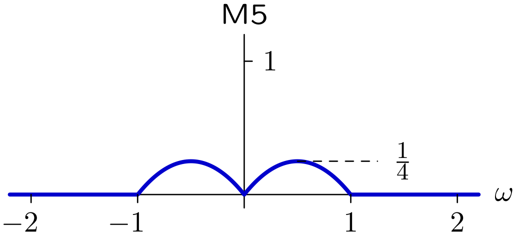
M4



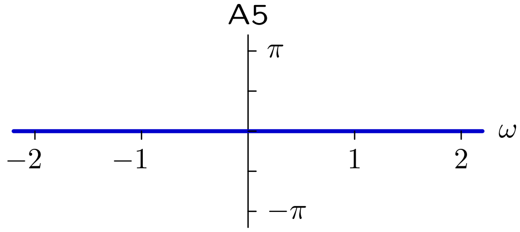
A4



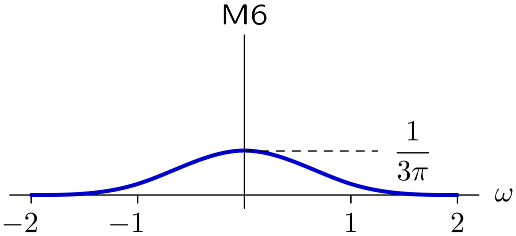
M5



A5



M6



A6