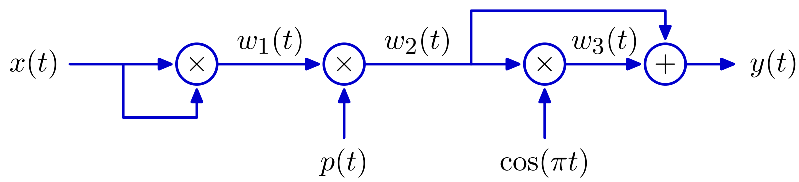
6.003 Homework #13 Solutions

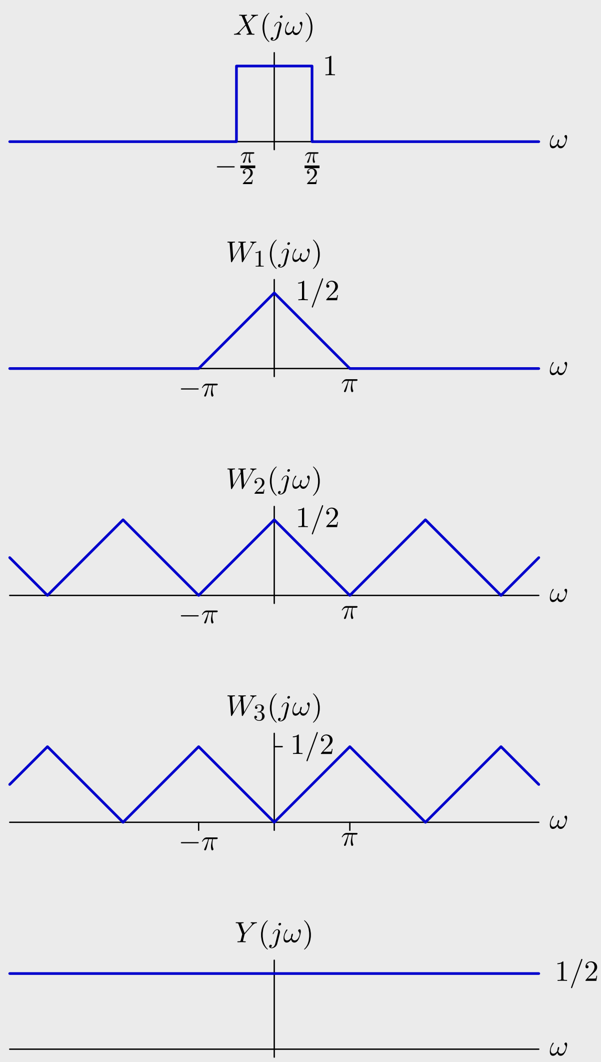
Problems

1. Transformation

Consider the following transformation from $x(t)$ to $y(t)$:



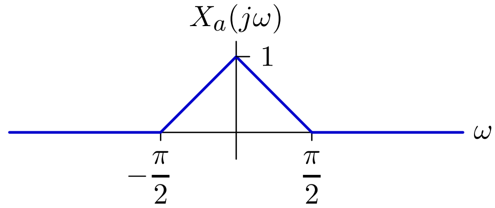
where $p(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$. Determine an expression for $y(t)$ when $x(t) = \sin(\pi t/2)/(\pi t)$.



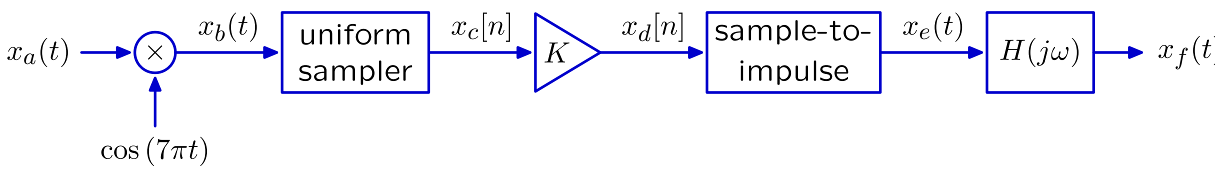
$$y(t) = \frac{1}{2}\delta(t)$$

2. Multiplied Sampling

The Fourier transform of a signal $x_a(t)$ is given below.



This signal passes through the following system



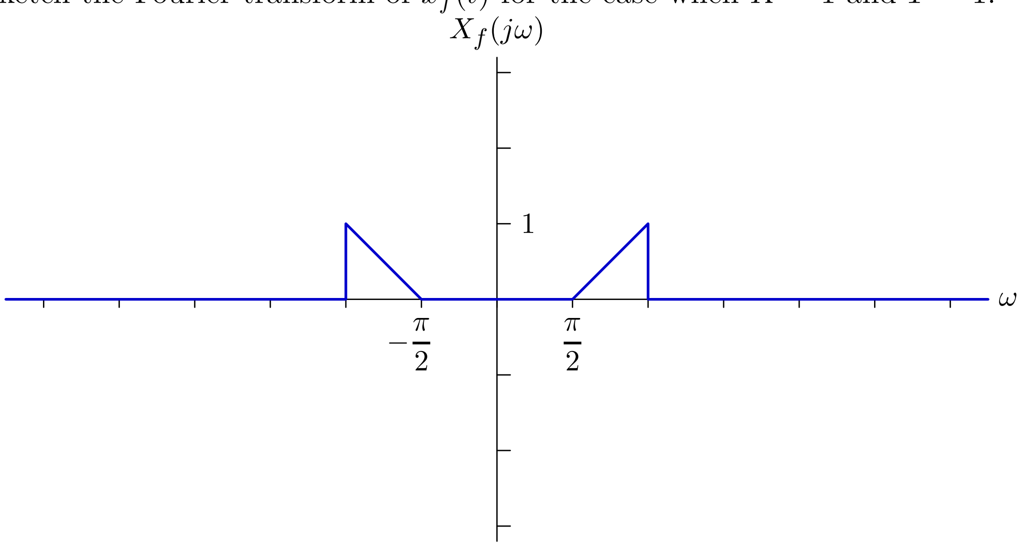
where $x_c[n] = x_b(nT)$ and

$$x_e(t) = \sum_{n=-\infty}^{\infty} x_d[n]\delta(t - nT)$$

and

$$H(j\omega) = \begin{cases} T & \text{if } |\omega| < \frac{\pi}{T} \\ 0 & \text{otherwise}. \end{cases}$$

a. Sketch the Fourier transform of $x_f(t)$ for the case when $K = 1$ and $T = 1$.



Use your sketch to determine an expression for $X_f(j\omega)$ for the following intervals:

$0 < \omega < \pi/2$: 0

$\pi/2 < \omega < \pi$: $2\omega/\pi - 1$

$\pi < \omega < 3\pi/2$: 0

$3\pi/2 < \omega < 2\pi$: 0

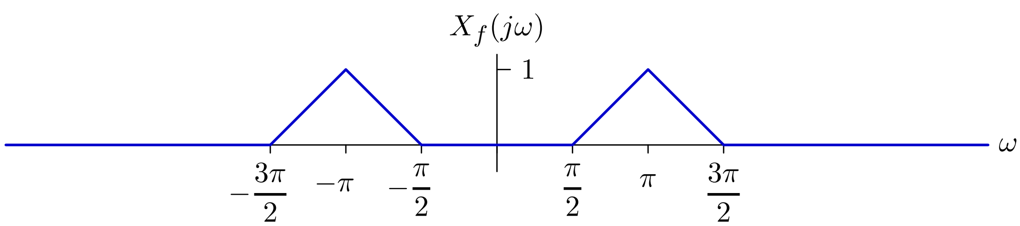
b. Is it possible to adjust T and K so that $x_f(t) = x_a(t)$?

If yes, specify a value T and the corresponding value of K (there may be multiple solutions, you need only specify one of them). If no, write **none.**

$$T = \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{8}{7}, \frac{10}{7}, \frac{12}{7}, \text{or} 2$$

$$K = 1$$

c. Is it possible to adjust T and K so that the Fourier transform of $x_f(t)$ is equal to the following, and is zero outside the indicated range?



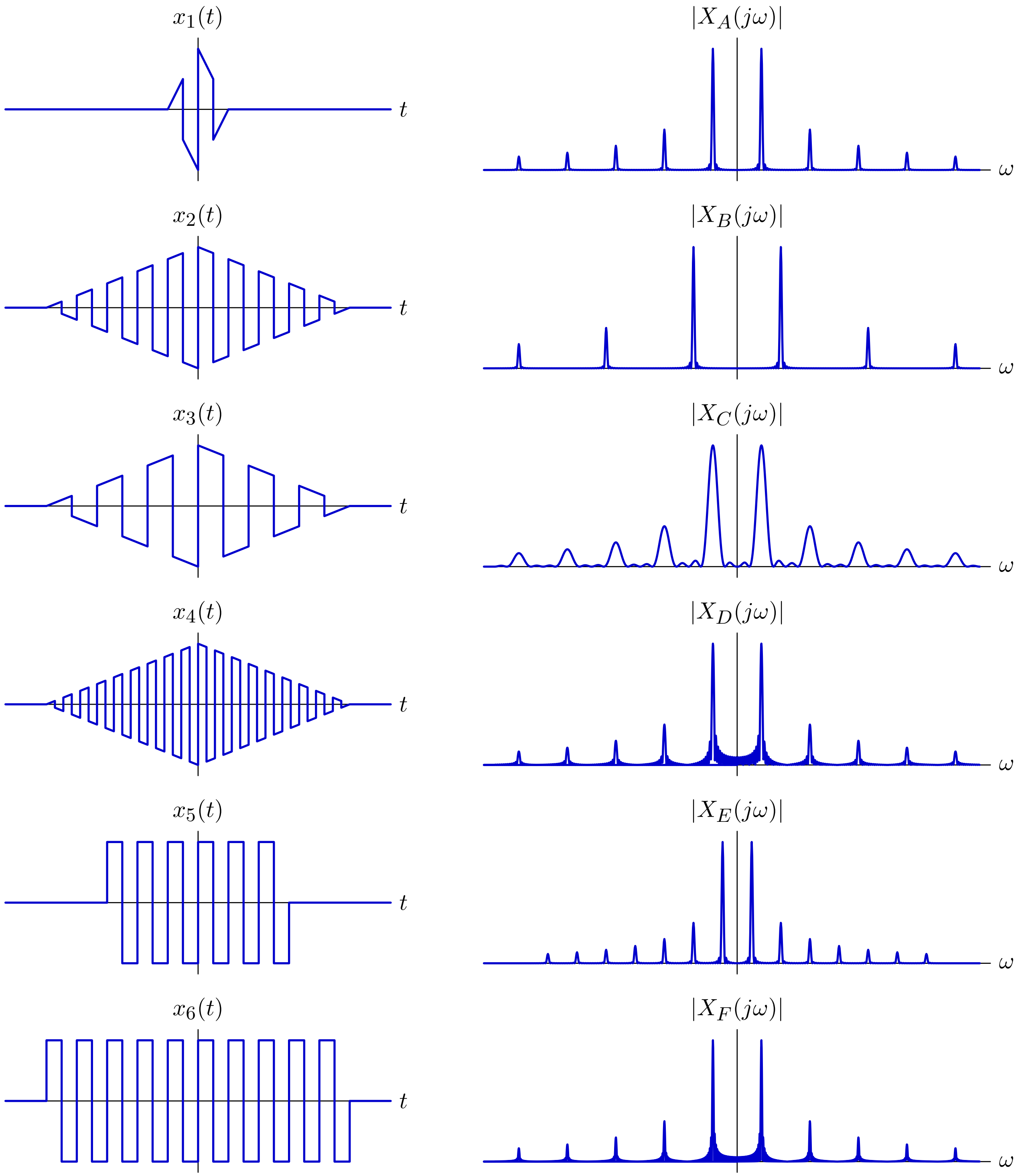
If yes, specify all possible pairs of T and K that work in the table below. If there are more rows in the table than are needed, leave the remaining entries blank. If no, enter **none.**

T	K
$\frac{1}{3}$	2
$\frac{2}{3}$	2
$\frac{1}{4}$	2
$\frac{1}{2}$	2

3. Patterns

The time waveforms for six signals are shown in the left panels below. The right panels show the magnitudes of the Fourier transforms of $x_1(t)$ to $x_6(t)$, however, the order has been shuffled. For each panel on the left, find the corresponding panel on the right.

All of the time functions are plotted on the same time scale. Similarly, all of the frequency functions are plotted on the same frequency scale.

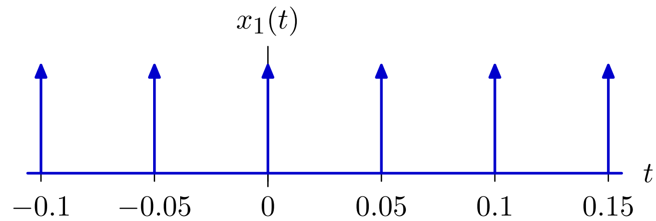


1 C
2 A
3 E
4 B
5 D
6 F

4. Inputs and Outputs

A **causal, stable** LTI system with frequency response $H(j\omega)$ has input $x(t)$ and output $y(t)$. The problem is to determine which of the following inputs can or cannot give rise to the output $y(t) = \sin(2\pi \cdot 100 \cdot t)$. For each part of the problem, determine if the statement is True (T) or False (F) and give an explanation.

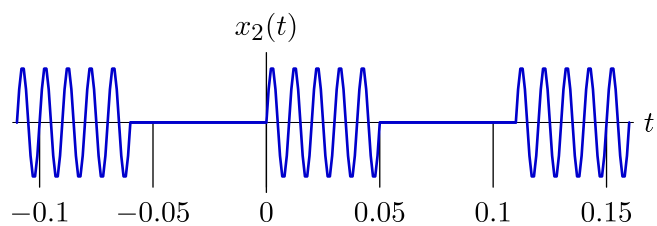
Part a. $x_1(t)$ is a periodic impulse train of period 0.05 s.



(T or F) $x_1(t)$ **can** generate the response $y(t) = \sin(2\pi \cdot 100 \cdot t)$.

True. Since $x(t)$ has a period of 0.05 seconds, the fundamental frequency is 20 Hz (= $2\pi \times 20$ radians/second). The Fourier transform of $x(t)$ is an impulse train with impulses at integer multiples of 20 Hz. Thus, the fifth harmonic occurs at 100 Hz. Thus, $H(j\omega)$ could be a narrowband filter centered at 100 Hz with a phase shift of $-\pi/2$ at 100 Hz. That would produce the desired output. However, this filter is not causal. An alternative is to construct a filter with zeros at all the unwanted frequencies.

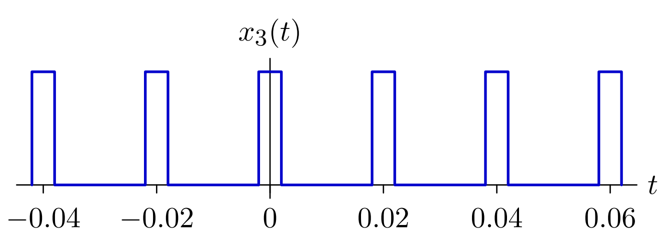
Part b. $x_2(t)$ is a periodic function of period 0.11 s. Each period consists of five cycles of a sinewave of the form $\sin(2\pi \cdot 100 \cdot t)$.



(T or F) $x_2(t)$ **can** generate the response $y(t) = \sin(2\pi \cdot 100 \cdot t)$.

True. Since the period of $x_2(t)$ is 0.11s, the fundamental frequency is 1/0.11 Hz. Thus, $x_2(t)$ will contain impulses at the frequencies $f = k/0.11$ Hz and the 11th harmonic will appear at the frequency $f = 100$ Hz. The only remaining issue is whether the area of this impulse is non-zero. The pulse of sinusoid has a spectrum which is a sinc function centered on 100 Hz. Hence, its value at 100 Hz is non-zero and so the amplitude of the impulse at 100 Hz is also non-zero. The filter can be chosen as indicated in part a.

Part c. $x_3(t)$ is a periodic pulse train of period 0.02 s. Each pulse has duration 0.004 s.

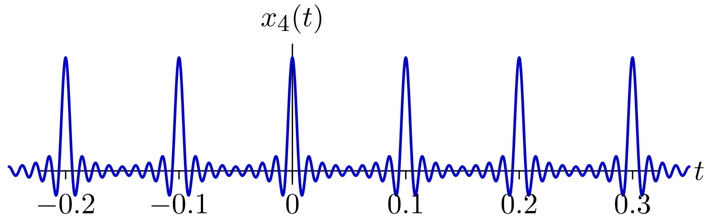


(T or F) $x_3(t)$ **can** generate the response $y(t) = \sin(2\pi \cdot 100 \cdot t)$.

True. $x_3(t)$ can be represented by a uniform impulse train of period 0.02 s convolved with a rectangular pulse of duration 0.004 s. Thus, the Fourier transform of $x_3(t)$ is a uniform impulse train, whose period in frequency is $1/0.02 = 50$ Hz, multiplied by a sinc function. Thus, there are clearly impulses at the frequencies ± 100, the only issue is whether the sinc function has a zero at 100 Hz. Since the duration of the rectangular pulse is 0.004 s, the first zero of the sinc function is at 250 Hz. The filter can be chosen as indicated in part a.

Part d. $x_4(t)$ is a periodic sinc pulse train of period 0.1 s. Each sinc pulse has the formula

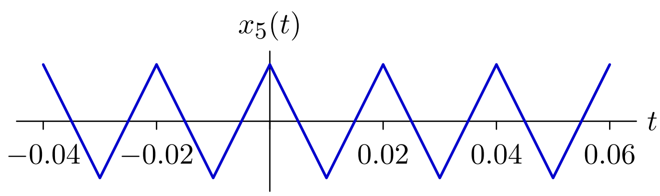
$$\frac{\sin(\pi \cdot \frac{t}{0.006})}{\pi \cdot \frac{t}{0.006}}$$



(T or F) $x_4(t)$ **can** generate the response $y(t) = \sin(2\pi \cdot 100 \cdot t)$.

False. $x_4(t)$ is a uniform impulse train of period 0.1 s convolved with a sinc function. Hence, the Fourier transform of $x_4(t)$ is the product of an impulse train in frequency with an ideal lowpass filter. Since the period of $x_4(t)$ is 0.1 s, the fundamental frequencey is 10 Hz and there will be an impulse at 100 Hz in the Fourier transform of the impulse train. But, the total width of the ideal lowpass filter is $1/0.006 = 166.66$ Hz so that the passband is from -83.33 to $+83.33$ Hz. Thus, there is no component at 100 Hz at the input to the filter $H(j\omega)$ and $x_4(t)$ cannot generate the output $y(t)$.

Part e. $x_5(t)$ is a periodic triangular wave of period 0.02 s.

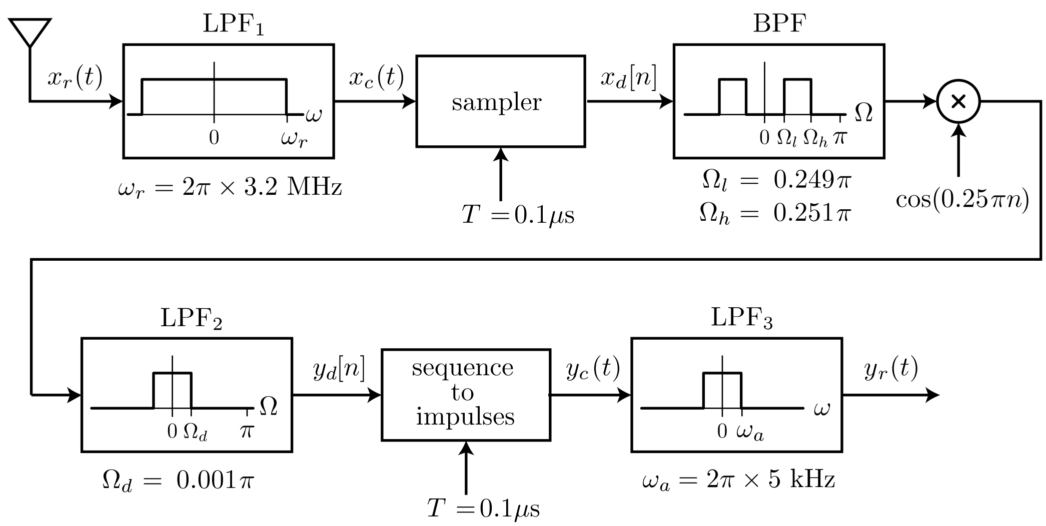


(T or F) $x_5(t)$ **can** generate the response $y(t) = \sin(2\pi \cdot 100 \cdot t)$.

False. $x_5(t)$ is a triangular wave which has half-wave symmetry and therefore its even harmonics are zero. Since the fundamental frequency is 50 Hz, the component at 100 Hz is the second harmonic and its magnitude must be zero. An alternative approach is to recognize that the triangular wave is the convolution of a periodic impulse train in time with a triangular pulse. The triangular pulse has duration 0.02 s and can be generated by convolving a square pulse of duration 0.01 s with itself. Thus, the Fourier transform is the squared sinc function. But the sinc function has a zero at 100 Hz. Hence, the component at 100 Hz at the input to the filter has a magnitude of zero, and $x_5(t)$ cannot give rise to $y(t)$.

5. DT Radio Demodulation

Commercial AM radio stations broadcast radio frequencies within a limited range: $2\pi(f_c - 5\text{ kHz}) < \omega < 2\pi(f_c + 5\text{ kHz})$, where $f_c = \omega_c/(2\pi) = n \times 10$ kHz and n is an integer between 54 and 160. The system shown below is intended to decode one of the AM radio signals using DT signal processing methods. Assume that all of the filters are ideal.



sampler: $x_d[n] = x_c(nT)$

sequence to impulses: $y_c(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT)$

Part a. Determine the center frequency f_c for the AM station that this receiver will detect.

$$\omega = \frac{\Omega}{T} = \frac{\frac{1}{4}\pi}{10^{-7}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{8} \times 10^7 = 1.25\,\text{MHz}$$

Part b. Which of the following statement(s) is/are correct?

b1. Increasing the cutoff frequency ω_r of LPF$_1$ by a factor of 1.5 will cause aliasing.

b2. Decreasing the cutoff frequency ω_r of LPF$_1$ by a factor of 2 will have no effect on the output $y_r(t)$.

b3. Halving the sampling interval T would have no effect on the output $y_r(t)$.

b4. Doubling the sampling interval T would have no effect on the output $y_r(t)$.

b2

Part c. Which of the following statement(s) is/are correct?

c1. Increasing the cutoff frequency Ω_d of LPF$_2$ will change $y_r(t)$ by adding signals from unwanted radio stations.

c2. Increasing the cutoff frequency Ω_d of LPF$_2$ will change $y_r(t)$ because aliasing will occur.

c3. Doubling the cutoff frequency Ω_d of LPF_2 will have no effect on $y_r(t)$.

c4. Halving the cutoff frequency Ω_d of LPF_2 will have no effect on $y_r(t)$.

c3