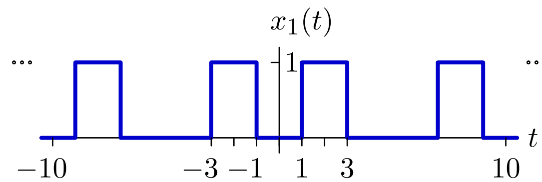
6.003 Homework #9 Solutions

Problems

1. Fourier varieties

a. Determine the Fourier series coefficients of the following signal, which is periodic in $T = 10$.



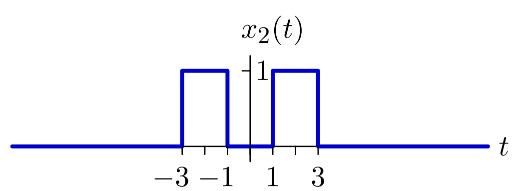
$$a_0 = \boxed{\frac{2}{5}}$$

$$a_k = \boxed{\frac{\sin\frac{3\pi k}{5} - \sin\frac{\pi k}{5}}{\pi k}} \quad \text{for } k \neq 0$$

$$a_k = \frac{1}{10}\int_{-3}^{-1} e^{-j\frac{2\pi k}{10}t}dt + \frac{1}{10}\int_{1}^{3} e^{-j\frac{2\pi k}{10}t}dt = \frac{1}{10}\left.\frac{e^{-j\frac{2\pi k}{10}t}}{-j\frac{2\pi k}{10}}\right|_{-3}^{-1} + \frac{1}{10}\left.\frac{e^{-j\frac{2\pi k}{10}t}}{-j\frac{2\pi k}{10}}\right|_{1}^{3}$$

$$= \frac{e^{j3\frac{2\pi k}{10}} - e^{j\frac{2\pi k}{10}}}{j2\pi k} + \frac{e^{-j\frac{2\pi k}{10}} - e^{-j3\frac{2\pi k}{10}}}{j2\pi k} = \frac{\sin\frac{3\pi k}{5} - \sin\frac{\pi k}{5}}{\pi k}$$

b. Determine the Fourier transform of the following signal, which is zero outside the indicated range.



$$X_2(j\omega) = \boxed{\frac{2\sin 3\omega - 2\sin\omega}{\omega}}$$

$$X_2(j\omega) = \int_{-3}^{-1} e^{-j\omega t}dt + \int_{1}^{3} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-3}^{-1} + \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{1}^{3}$$

$$= \frac{e^{j3\omega} - e^{j\omega}}{j\omega} + \frac{e^{-j\omega} - e^{-j3\omega}}{j\omega} = \frac{2\sin 3\omega - 2\sin\omega}{\omega}$$

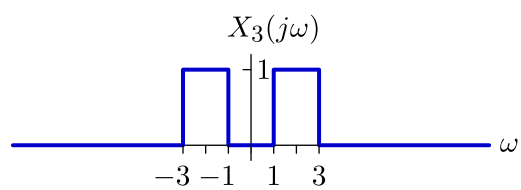
c. What is the relation between the answers to parts a and b? In particular, derive an expression for a_k (the solution to part a) in terms of $X_2(j\omega)$ (the solution to part b).

$$a_k = \boxed{\frac{1}{10} X_2\left(j\frac{2\pi k}{10}\right)}$$

The Fourier series coefficients a_k are

$$a_k = \frac{1}{10}\, X_2(j\omega)\big|_{\omega = \frac{2\pi k}{10}}$$

d. Determine the time waveform that corresponds to the following Fourier transform, which is zero outside the indicated range.



$$x_3(t) = \boxed{\frac{\sin 3t - \sin t}{\pi t}}$$

$$x_3(t) = \frac{1}{2\pi}\int_{-3}^{-1} e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{1}^{3} e^{j\omega t}d\omega = \left.\frac{e^{j\omega t}}{j2\pi t}\right|_{-3}^{-1} + \left.\frac{e^{j\omega t}}{j2\pi t}\right|_{1}^{3}$$
$$= \frac{e^{-jt} - e^{-j3t}}{j2\pi t} + \frac{e^{j3t} - e^{jt}}{j2\pi t} = \frac{\sin 3t - \sin t}{\pi t}$$

e. What is the relation between the answers to parts b and d? In particular, derive an expression for $x_3(t)$ (the solution to part d) in terms of $X_2(j\omega)$ (the solution to part b).

$$x_3(t) = \boxed{\frac{1}{2\pi} X_2\big(j(-t)\big) \quad \text{or} \quad \frac{1}{2\pi} X_2\big(j(t)\big)}$$

The relation between this answer and that of the previous part is **duality**.

$$x_3(t) = \frac{1}{2\pi}\, X_2(j\omega)\big|_{\omega = -t}$$

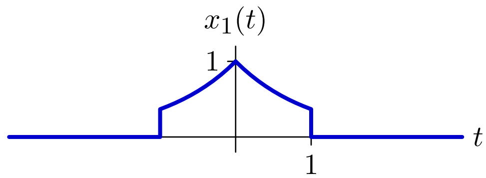
Since $X_2(j\omega)$ is real and even, $\frac{1}{2\pi}X_2(jt)$ would also work.

2. Fourier transform properties

Let $X(j\omega)$ represent the Fourier transform of

$$x(t) = \begin{cases} e^{-t} & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases} .$$

Express the Fourier Transforms of each of the following signals in terms of $X(j\omega)$.

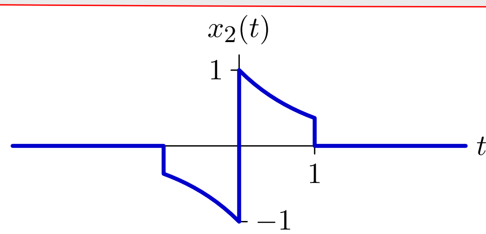


$X_1(j\omega) =$ $\boxed{X(j\omega) + X(-j\omega)}$

$$x_1(t) = x(t) + x(-t)$$

$$x(-t) \leftrightarrow \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-j\omega)$$

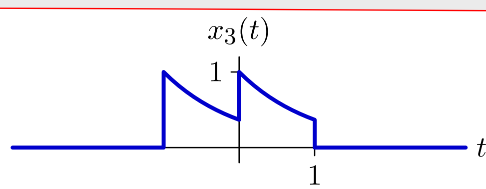
$$X_1(j\omega) = X(j\omega) + X(-j\omega)$$



$X_2(j\omega) =$ $\boxed{X(j\omega) - X(-j\omega)}$

$$x_2(t) = x(t) - x(-t)$$

$$X_2(j\omega) = X(j\omega) - X(-j\omega)$$



$X_3(j\omega) =$ $\boxed{(1 + e^{j\omega})X(j\omega)}$

$$x_3(t) = x(t) + x(t+1)$$

$$x(t+1) \leftrightarrow \int_{-\infty}^{\infty} x(t+1)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega(t-1)}dt = e^{j\omega}X(j\omega)$$

$$X_3(j\omega) = (1 + e^{j\omega})X(j\omega)$$

3. Fourier transforms

Find the Fourier transforms of the following signals.

a. $x_1(t) = e^{-|t|}\cos(2t)$

$$X_1(j\omega) = \boxed{\frac{1}{1+(\omega-2)^2} + \frac{1}{1+(\omega+2)^2}}$$

$$e^{-t}u(t) \leftrightarrow \frac{1}{1+j\omega}$$

$$e^{-|t|} \leftrightarrow \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}$$

$$\cos(2t) \leftrightarrow \pi\delta(\omega-2) + \pi\delta(\omega+2)$$

Therefore, by the multiplication property,

$$e^{-|t|}\cos(2t) \leftrightarrow \frac{1}{1+(\omega-2)^2} + \frac{1}{1+(\omega+2)^2}$$

b. $x_2(t) = \dfrac{\sin(2\pi t)}{\pi(t-1)}$

$$X_2(j\omega) = \boxed{e^{-j\omega}\left(u(\omega+2\pi) - u(\omega-2\pi)\right)}$$

$$\frac{\sin(2\pi t)}{\pi t} \leftrightarrow u(\omega+2\pi) - u(\omega-2\pi)$$

$$\frac{\sin(2\pi t)}{\pi(t-1)} = \frac{\sin(2\pi t - 2\pi)}{\pi(t-1)} = \frac{\sin(2\pi(t-1))}{\pi(t-1)} \leftrightarrow e^{-j\omega}\left(u(\omega+2\pi) - u(\omega-2\pi)\right)$$

c. $x_3(t) = \begin{cases} t^2 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

$$X_3(j\omega) = \boxed{\frac{j}{\omega}e^{-j\omega} + \frac{2}{\omega^2}e^{-j\omega} + \frac{2j}{\omega^3}\left(1 - e^{-j\omega}\right)}$$

$$x_3(t) = t^2(u(t) - u(t-1))$$

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$u(t) - u(t-1) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega) - e^{-j\omega}\frac{1}{j\omega} - e^{-j\omega}\pi\delta(\omega) = \frac{1 - e^{-j\omega}}{j\omega}$$

$$tf(t) \leftrightarrow j\frac{d}{d\omega}F(j\omega)$$

$$t^2 f(t) \leftrightarrow -\frac{d^2}{d\omega^2}F(j\omega)$$

$$t^2\left(u(t) - u(t-1)\right) \leftrightarrow -\frac{d^2}{d\omega^2}\left(\frac{1 - e^{-j\omega}}{j\omega}\right)$$

$$x_3(t) \leftrightarrow \frac{j}{\omega}e^{-j\omega} + \frac{2}{\omega^2}e^{-j\omega} + \frac{2j}{\omega^3}\left(1 - e^{-j\omega}\right)$$

d. $x_4(t) = (1 - |t|)\, u(t+1)u(1-t)$

$$X_4(j\omega) = \boxed{\frac{2(1 - \cos\omega)}{\omega^2}}$$

Let $p(t) = u(t + 0.5) - u(t - 0.5)$. Then

$$x_4(t) = p(t) * p(t)$$

and

$$X_4(j\omega) = P^2(j\omega)\,.$$

$$P(j\omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{2\sin\frac{\omega}{2}}{\omega}$$

$$X_4(j\omega) = \frac{4\sin^2\frac{\omega}{2}}{\omega^2} = \frac{2(1 - \cos\omega)}{\omega^2}$$

You could also solve this by differentiating twice in the time domain to get a sequence of delta functions, computing the transform, and multiplying twice by $\frac{1}{j\omega}$ [the delta functions in the integration property have zero weight].

Engineering Design Problem

4. Parseval's theorem

Parseval's theorem relates time- and frequency-domain methods for calculating the average energy of a signal as follows:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

where a_k represents the Fourier series coefficients of the periodic signal $x(t)$ with period T.

a. We can derive Parseval's theorem from the properties of CT Fourier series.

1. Let $y(t) = |x(t)|^2$. Find the Fourier series coefficients b_k of $y(t)$.
[Hint: $|x(t)|^2 = x(t)x^*(t)$.]

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-j\frac{2\pi}{T}kt}$$

$$\begin{aligned}
b_k &= \frac{1}{T}\int_T y(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T}\int_T x(t)x^*(t) e^{-j\frac{2\pi}{T}kt} dt \\
&= \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty} a_l e^{j\frac{2\pi}{T}lt} \sum_{m=-\infty}^{\infty} a_m^* e^{-j\frac{2\pi}{T}mt} e^{-j\frac{2\pi}{T}kt} dt \\
&= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_l a_m^* \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}(k-l+m)t} dt \\
&= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_l a_m^* \delta[k-l+m] \\
&= \sum_{m=-\infty}^{\infty} a_{k+m} a_m^*
\end{aligned}$$

2. Use the result from the previous part to derive Parseval's theorem.

$$\frac{1}{T}\int_T |x(t)|^2 dt = b_0 = \sum_{k=-\infty}^{\infty} a_k a_k^* = \sum_{k=-\infty}^{\infty} |a_k|^2$$

b. Let $x_1(t)$ represent the input to an LTI system, where

$$x_1(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{jk\frac{\pi}{4}t}$$

for $0 < \alpha < 1$. The frequency response of the system is

$$H(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise.} \end{cases}$$

What is the minimum value of W so that the average energy in the output signal will be at least 90% of that in the input signal.

The signal $x_1(t)$ is periodic with period $T = 8$ and has Fourier series coefficients $a_k = \alpha^{|k|}$. The average energy in the input signal is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k=-\infty}^{\infty} \alpha^{2|k|} = \frac{1}{1-\alpha^2} + \frac{1}{1-\alpha^2} - 1 = \frac{1+\alpha^2}{1-\alpha^2}$$

The lowpass filter passes some number K of the harmonic components of $x(t)$ so that the average energy in the output signals is

$$\frac{1}{T}\int_T |y(t)|^2 dt = \sum_{k=-K}^{K} \alpha^{2|k|} = \frac{1-\alpha^{2K+2}}{1-\alpha^2} + \frac{1-\alpha^{2K+2}}{1-\alpha^2} - 1 = \frac{1+\alpha^2-2\alpha^{2K+2}}{1-\alpha^2}$$

To make the energy in the output at least 90% of that in the input

$$\frac{1+\alpha^2-2\alpha^{2K+2}}{1-\alpha^2} \geq 0.9\left(\frac{1+\alpha^2}{1-\alpha^2}\right)$$

$$1+\alpha^2-2\alpha^{2K+2} \geq 0.9(1+\alpha^2)$$

$$2\alpha^{2K+2} \leq 0.1(1+\alpha^2)$$

$$(K+1)\log\alpha^2 \leq \log\frac{1+\alpha^2}{20}$$

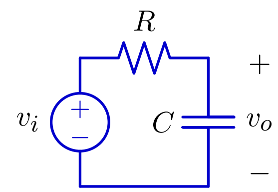
$$K > \frac{\log\frac{1+\alpha^2}{20}}{\log\alpha^2} - 1 \qquad \text{(inequality switches because the logs are negative)}$$

Because the harmonics are spaced at $\frac{2\pi}{T}$ intervals in frequency

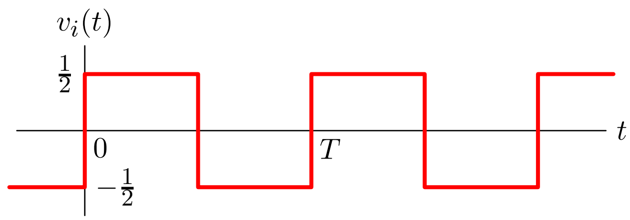
$$W > \left(\frac{\log\frac{1+\alpha^2}{20}}{\log\alpha^2} - 1\right)\frac{2\pi}{8}$$

5. Filtering

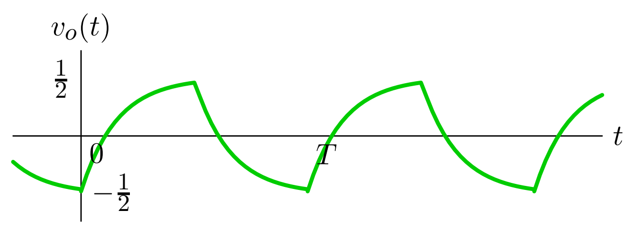
The point of this question is to understand how the magnitude of a filter affects the output and how the angle of a filter affects the output. Consider the following RC circuit as a "filter."



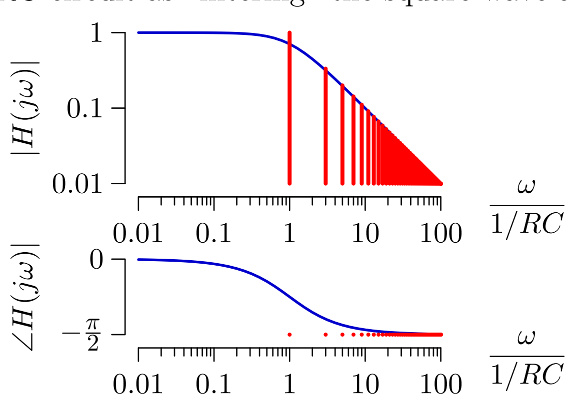
Assume that the input $v_i(t)$ is the following square wave.



If the fundamental frequency of the square wave ($\frac{2\pi}{T}$) is equal to the cutoff frequency of the RC circuit ($\frac{1}{RC}$) then the output $v_o(t)$ will have the following form.



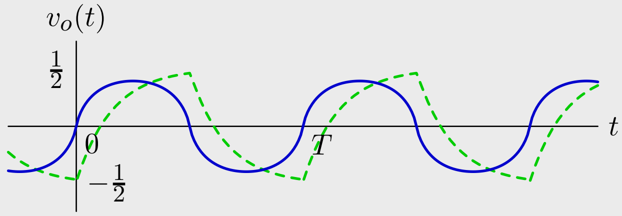
We can think of the RC circuit as "filtering" the square wave as shown below.



The RC filter has two effects: (1) The amplitudes of the Fourier components of the input (vertical red lines in upper panel) are multiplied by the magnitude of the frequency response ($|H(j\omega)|$). (2) The phase of the Fourier components (red dots in lower panel) are shifted by the phase of the frequency response ($\angle H(j\omega)$).

a. Determine (using whatever method you find convenient) the output that would result if $v_i(t)$ were passed through a filter whose magnitude is $|H(j\omega)|$ (as above) but whose phase function is 0 for all frequencies. Compare the result with $v_o(t)$ above.

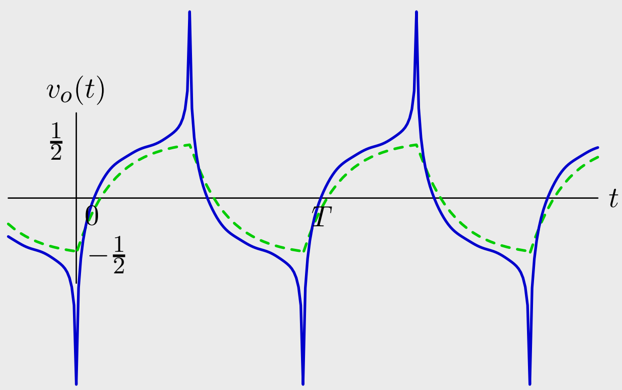
The following plot shows the sum of the first 46 terms of the series expansion for the square wave, with each term filtered by the magnitude (but not the phase) of the RC lowpass filter. The original output is also shown (dashed green) for reference.



The asymmetry in the "charging" and "discharging" portions of the RC response is gone. The effect of the filter is to reduce the magnitudes of the higher harmonics without adding phase delay.

b. Determine (using whatever method you find convenient) the output that would result if $v_i(t)$ were passed through a filter whose phase function is $\angle H(j\omega)$ (as above) but whose magnitude function is 1 for all frequencies. Compare the result with $v_o(t)$ above.

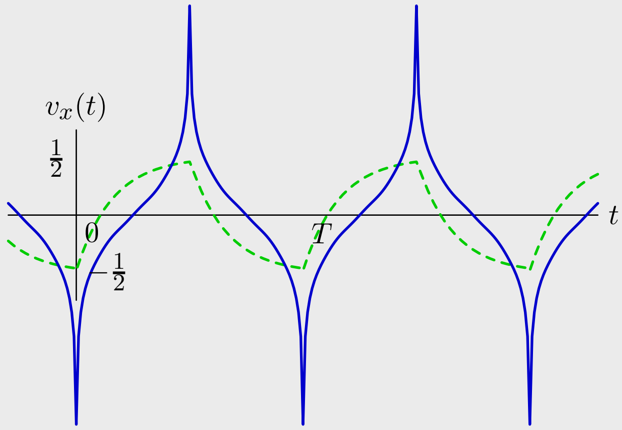
The following plot shows the sum of the first 46 terms of the series expansion for the square wave, with each term filtered by the phase (but not the magnitude) of the RC lowpass filter.



The asymmetry in the "charging" and "discharging" portions of the RC response is even more pronounced than before. Because the magnitudes of the higher harmonics are not attenuated, they now accumulate to make a substantial peak that was not seen in the RC response. We can understand the large overshoot as follows. Each harmonic component in the square wave is a sinusoid:

$$v_i(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{\pi k} \sin\left(\frac{2\pi k t}{T}\right) .$$

Except for the fundamental, the phase shifts are nearly a quarter cycle. Delaying each of the harmonic components by a quarter cycle aligns the peaks of each component at $t = \frac{T}{2}$ as shown below



where

$$v_x(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{\pi k} \sin\left(\frac{2\pi k t}{T} - \pi/2\right) .$$