6.003: Signals and Systems

Laplace Transform

September 27, 2011

Mid-term Examination #1

Wednesday, October 5, 7:30-9:30pm, 26-310, 26-322, 26-328.

No recitations on the day of the exam.

Coverage: CT and DT Systems, Z and Laplace Transforms
Lectures 1–7
Recitations 1–7
Homeworks 1–4

Homework 4 will not collected or graded. Solutions will be posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

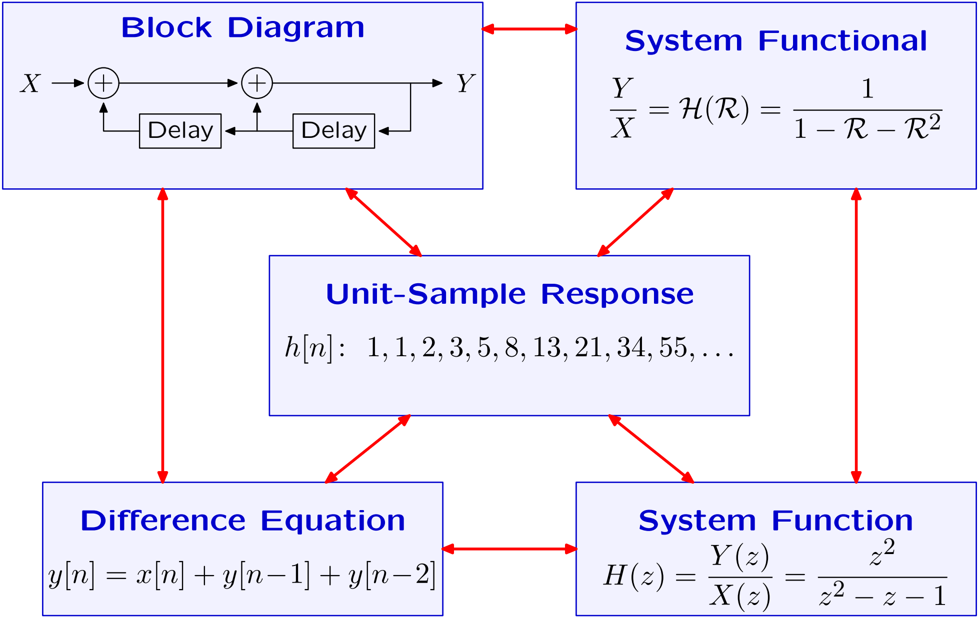
Review sessions during open office hours.

Conflict? Contact freeman@mit.edu before Friday, Sept. 30, 5pm.

Prior term midterm exams have been posted on the 6.003 website.

Concept Map for Discrete-Time Systems

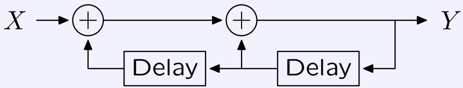
Last time: relations among representations of DT systems.



Concept Map for Discrete-Time Systems

Most important new concept from last time was the **Z transform**.

Block Diagram



System Functional

$$\frac{Y}{X} = \mathcal{H}(\mathcal{R}) = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \;\; 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Z transform

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

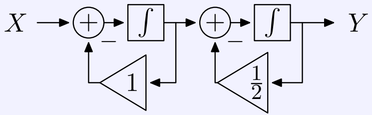
System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{z^2 - z - 1}$$

Concept Map for Continuous-Time Systems

Today: similar relations among representations of CT systems.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

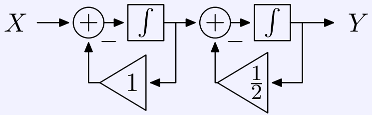
System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map for Continuous-Time Systems

Corresponding concept for CT is the **Laplace Transform**.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Laplace transform

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Laplace Transform: Definition

Laplace transform maps a function of time t to a function of s.

$$X(s) = \int x(t)e^{-st}dt$$

There are two important variants:

Unilateral (18.03)

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

Bilateral (6.003)

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

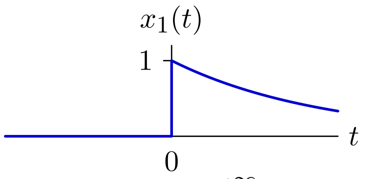
Both share important properties.

We will focus on bilateral version, and discuss differences later.

Laplace Transforms

Example: Find the Laplace transform of $x_1(t)$:

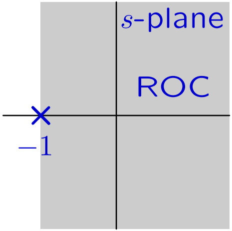
$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt = \int_0^{\infty} e^{-t} e^{-st} dt = \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^{\infty} = \frac{1}{s+1}$$

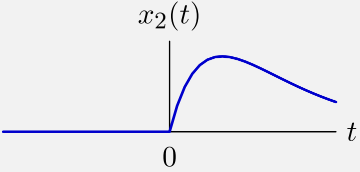
provided $\text{Re}(s+1) > 0$ which implies that $\text{Re}(s) > -1$.

$$\frac{1}{s+1} \; ; \quad \text{Re}(s) > -1$$



Check Yourself

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Which of the following is the Laplace transform of $x_2(t)$?

1. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}(s) > -1$
2. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}(s) > -2$
3. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\text{Re}(s) > -1$
4. $X_2(s) = \frac{s}{(s+1)(s+2)}$; $\text{Re}(s) > -2$
5. none of the above

Check Yourself

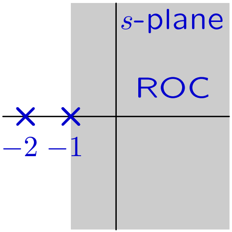
$$X_2(s) = \int_0^\infty (e^{-t} - e^{-2t})e^{-st}dt$$

$$= \int_0^\infty e^{-t}e^{-st}dt - \int_0^\infty e^{-2t}e^{-st}dt$$

$$= \frac{1}{s+1} - \frac{1}{s+2} = \frac{(s+2)-(s+1)}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)}$$

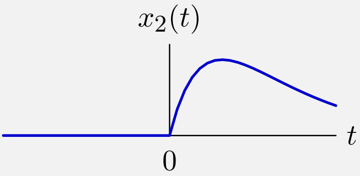
These equations converge if $\text{Re}(s+1) > 0$ and $\text{Re}(s+2) > 0$, thus $\text{Re}(s) > -1$.

$$\frac{1}{(s+1)(s+2)} \; ; \quad \text{Re}(s) > -1$$



Check Yourself

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Which of the following is the Laplace transform of $x_2(t)$? 1

1. $X_2(s) = \frac{1}{(s+1)(s+2)}$; $\text{Re}(s) > -1$
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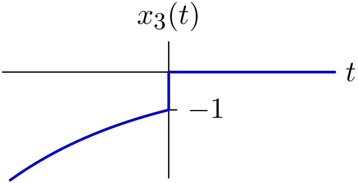
5. none of the above

Regions of Convergence

Left-sided signals have left-sided Laplace transforms (bilateral only).

Example:

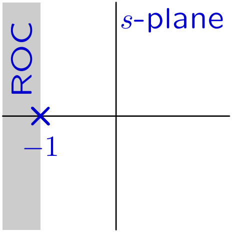
$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \le 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st}dt = \int_{-\infty}^{0} -e^{-t}e^{-st}dt = \frac{-e^{-(s+1)t}}{-(s+1)}\bigg|_{-\infty}^{0} = \frac{1}{s+1}$$

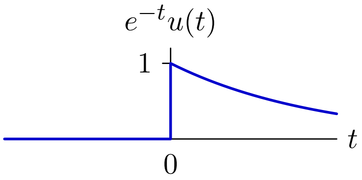
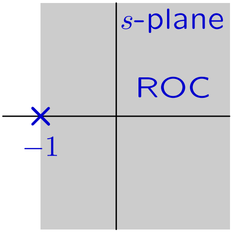
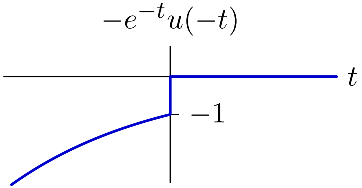
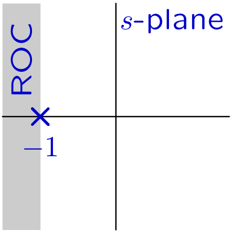
provided $\mathrm{Re}(s+1) < 0$ which implies that $\mathrm{Re}(s) < -1$.

$$\frac{1}{s+1}\ ; \quad \mathrm{Re}(s) < -1$$



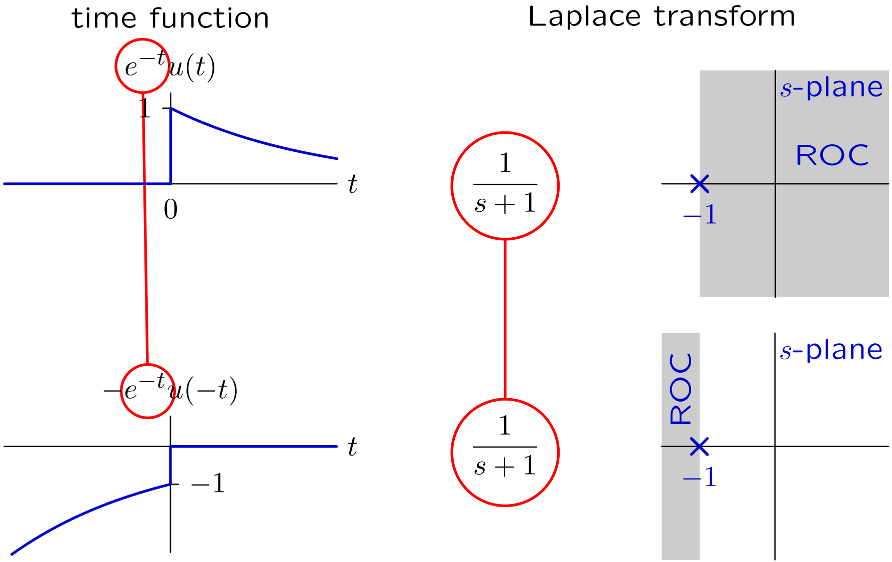
Left- and Right-Sided ROCs

Laplace transforms of left- and right-sided exponentials have the same form (except −); with left- and right-sided ROCs, respectively.

time function	Laplace transform
	$\dfrac{1}{s+1}$ 
	$\dfrac{1}{s+1}$ 

Left- and Right-Sided ROCs

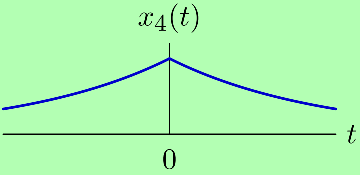
Laplace transforms of left- and right-sided exponentials have the same form (except $-$); with left- and right-sided ROCs, respectively.



Check Yourself

Find the Laplace transform of $x_4(t)$.

$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $-\infty < \text{Re}(s) < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $-1 < \text{Re}(s) < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $-\infty < \text{Re}(s) < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $-1 < \text{Re}(s) < 1$
5. none of the above

Check Yourself

$$X_4(s) = \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt$$

$$= \int_{-\infty}^{0} e^{(1-s)t} dt + \int_{0}^{\infty} e^{-(1+s)t} dt$$

$$= \left. \frac{e^{(1-s)t}}{(1-s)} \right|_{-\infty}^{0} + \left. \frac{e^{-(1+s)t}}{-(1+s)} \right|_{0}^{\infty}$$

$$= \underbrace{\frac{1}{1-s}}_{\text{Re}(s)<1} + \underbrace{\frac{1}{1+s}}_{\text{Re}(s)>-1}$$

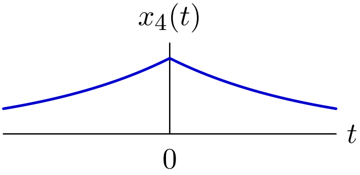
$$= \frac{1+s+1-s}{(1-s)(1+s)} = \frac{2}{1-s^2} \; ; \quad -1 < \text{Re}(s) < 1$$

The ROC is the intersection of $\text{Re}(s) < 1$ and $\text{Re}(s) > -1$.

Check Yourself

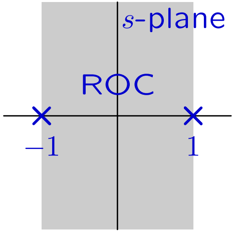
Laplace transform of a signal that is both-sided is a vertical strip.

$$x_4(t) = e^{-|t|}$$



$$X_4(s) = \frac{2}{1 - s^2}$$

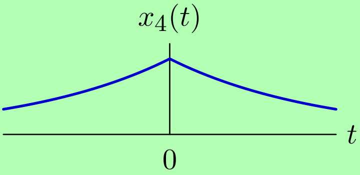
$$-1 < \text{Re}(s) < 1$$



Check Yourself

Find the Laplace transform of $x_4(t)$.   2

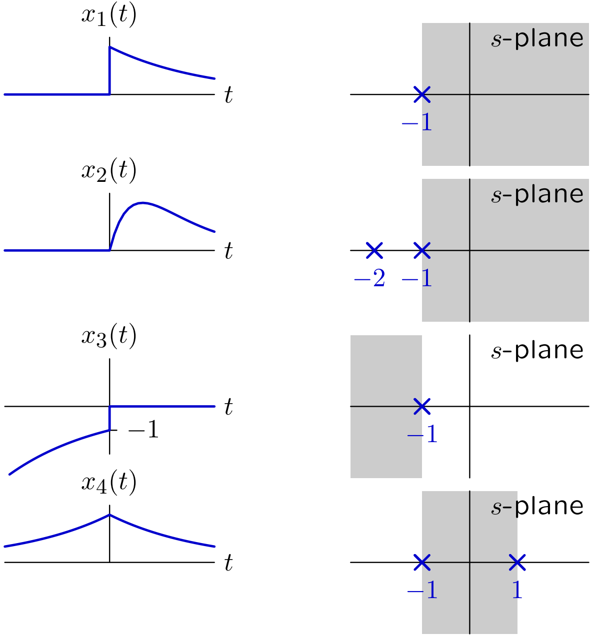
$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $-\infty < \mathrm{Re}(s) < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $-1 < \mathrm{Re}(s) < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $-\infty < \mathrm{Re}(s) < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $-1 < \mathrm{Re}(s) < 1$
5. none of the above

Time-Domain Interpretation of ROC

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$



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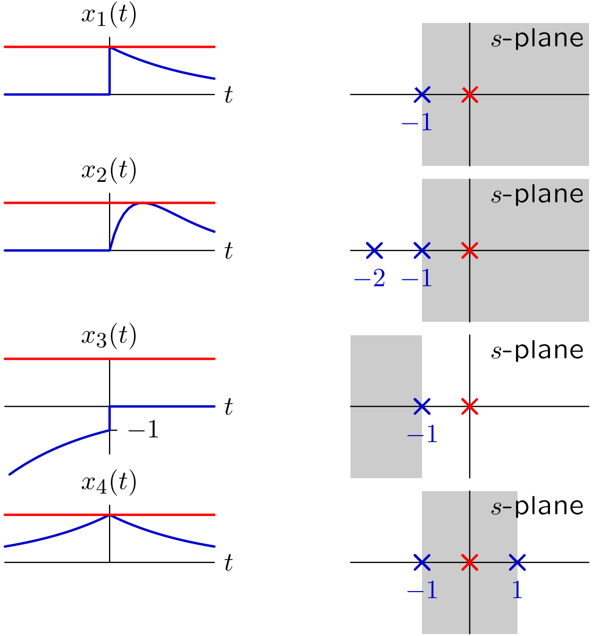

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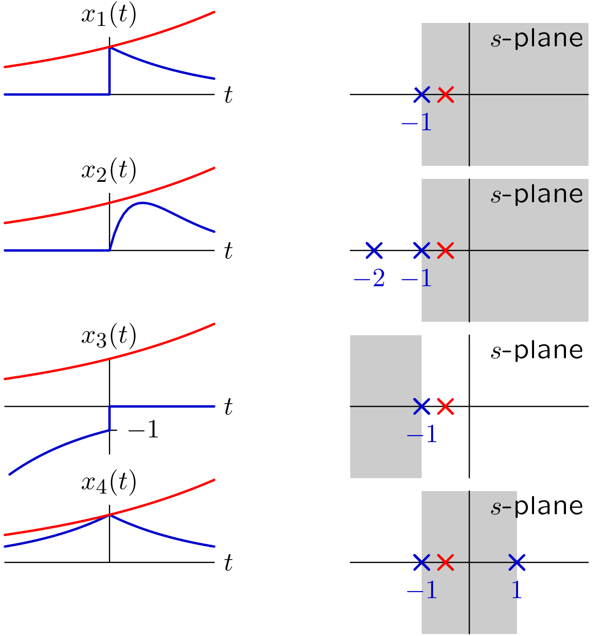

Time-Domain Interpretation of ROC

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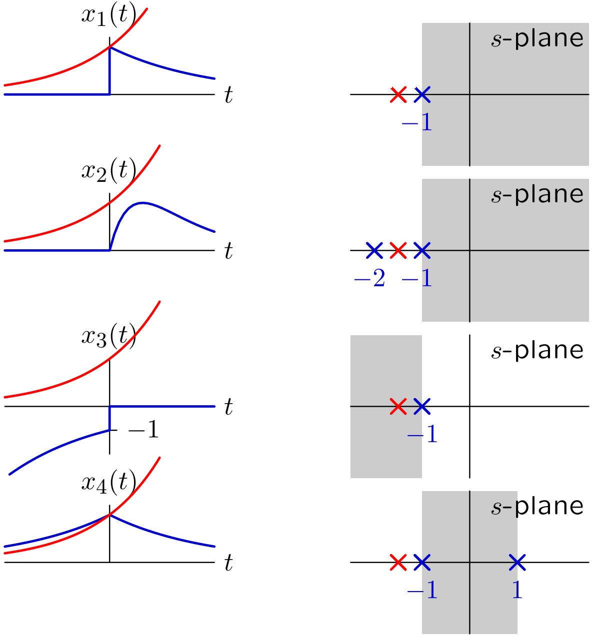


Check Yourself

The Laplace transform $\dfrac{2s}{s^2-4}$ corresponds to how many of the following signals?

1. $e^{-2t}u(t)+e^{2t}u(t)$
2. $e^{-2t}u(t)-e^{2t}u(-t)$
3. $-e^{-2t}u(-t)+e^{2t}u(t)$
4. $-e^{-2t}u(-t)-e^{2t}u(-t)$

Check Yourself

Expand with partial fractions:

$$\frac{2s}{s^2-4} = \underbrace{\frac{1}{s+2}}_{\text{pole at } -2} + \underbrace{\frac{1}{s-2}}_{\text{pole at } 2}$$

pole	function	right-sided; ROC	left-sided (ROC)
-2	e^{-2t}	$e^{-2t}u(t)$; $\operatorname{Re}(s) > -2$	$-e^{-2t}u(-t)$; $\operatorname{Re}(s) < -2$
2	e^{2t}	$e^{2t}u(t)$; $\operatorname{Re}(s) > 2$	$-e^{2t}u(-t)$; $\operatorname{Re}(s) < 2$

1.	$e^{-2t}u(t) + e^{2t}u(t)$	$\operatorname{Re}(s) > -2 \cap \operatorname{Re}(s) > 2$	$\operatorname{Re}(s) > 2$
2.	$e^{-2t}u(t) - e^{2t}u(-t)$	$\operatorname{Re}(s) > -2 \cap \operatorname{Re}(s) < 2$	$-2 < \operatorname{Re}(s) < 2$
3.	$-e^{-2t}u(-t) + e^{2t}u(t)$	$\operatorname{Re}(s) < -2 \cap \operatorname{Re}(s) > 2$	none
4.	$-e^{-2t}u(-t) - e^{2t}u(-t)$	$\operatorname{Re}(s) < -2 \cap \operatorname{Re}(s) < 2$	$\operatorname{Re}(s) < -2$

Check Yourself

The Laplace transform $\frac{2s}{s^2-4}$ corresponds to how many of the following signals? 3

1. $e^{-2t}u(t)+e^{2t}u(t)$
2. $e^{-2t}u(t)-e^{2t}u(-t)$
3. $-e^{-2t}u(-t)+e^{2t}u(t)$
4. $-e^{-2t}u(-t)-e^{2t}u(-t)$

Solving Differential Equations with Laplace Transforms

Solve the following differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation.

$$\mathcal{L}\left\{\dot{y}(t) + y(t)\right\} = \mathcal{L}\left\{\delta(t)\right\}$$

The Laplace transform of a sum is the sum of the Laplace transforms (prove this as an exercise).

$$\mathcal{L}\left\{\dot{y}(t)\right\} + \mathcal{L}\left\{y(t)\right\} = \mathcal{L}\left\{\delta(t)\right\}$$

What's the Laplace transform of a derivative?

Laplace Transform of a Derivative

Assume that $X(s)$ is the Laplace transform of $x(t)$:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Find the Laplace transform of $y(t) = \dot{x}(t)$.

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} \underbrace{\dot{x}(t)}_{\dot{v}}\underbrace{e^{-st}}_{u}dt$$

$$= \underbrace{x(t)}_{v}\underbrace{e^{-st}}_{u}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x(t)}_{v}(\underbrace{-se^{-st}}_{\dot{u}})dt$$

The first term must be zero since $X(s)$ converged. Thus

$$Y(s) = s\int_{-\infty}^{\infty} x(t)e^{-st}dt = sX(s)$$

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$\mathcal{L}\left\{\dot{y}(t)\right\} + \mathcal{L}\left\{y(t)\right\} = \mathcal{L}\left\{\delta(t)\right\}$$

Let $Y(s)$ represent the Laplace transform of $y(t)$.

Then $sY(s)$ is the Laplace transform of $\dot{y}(t)$.

$$sY(s) + Y(s) = \mathcal{L}\left\{\delta(t)\right\}$$

What's the Laplace transform of the impulse function?

Laplace Transform of the Impulse Function

Let $x(t) = \delta(t)$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ e^{-st}\Big|_{t=0} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ 1\ dt \\ &= 1 \end{aligned}$$

Sifting property:

Multiplying $f(t)$ by $\delta(t)$ and integrating over t **sifts** out $f(0)$.

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\} = 1$$

This is a simple algebraic expression.

Laplace transform **converts a differential equation**

$$\dot{y}(t) + y(t) = \delta(t)$$

to an equivalent **algebraic equation.**

$$sY + Y = 1$$

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\{\delta(t)\} = 1$$

This is a simple algebraic expression. Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

We've seen this Laplace transform previously.

$$y(t) = e^{-t}u(t)$$

Notice that we solved the differential equation $\dot{y}(t) + y(t) = \delta(t)$ without computing homogeneous and particular solutions.

Solving Differential Equations with Laplace Transforms

Back to the previous problem:

$$sY(s) + Y(s) = \mathcal{L}\left\{\delta(t)\right\} = 1$$

This is a simple algebraic expression. Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

We've seen this Laplace transform previously.

$$y(t) = e^{-t}u(t)$$ (why not $y(t) = -e^{-t}u(-t)$?)

Notice that we solved the differential equation $\dot{y}(t)+y(t) = \delta(t)$ without computing homogeneous and particular solutions.

Solving Differential Equations with Laplace Transforms

Summary of method.

Start with differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation:

$$sY(s) + Y(s) = 1$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

Take inverse Laplace transform (by recognizing form of transform):

$$y(t) = e^{-t}u(t)$$

Solving Differential Equations with Laplace Transforms

Recognizing the form ...

Is there a more systematic way to take an inverse Laplace transform?

Yes ... and no.

Formally,

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

but this integral is not generally easy to compute.

This equation can be useful to prove theorems.

We will find better ways (e.g., partial fractions) to compute inverse transforms for common systems.

Solving Differential Equations with Laplace Transforms

Example 2:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \delta(t)$$

Laplace transform:

$$s^2 Y(s) + 3sY(s) + 2Y(s) = 1$$

Solve:

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Inverse Laplace transform:

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

These forward and inverse Laplace transforms are easy if

- differential equation is linear with constant coefficients, and
- the input signal is an impulse function.

Properties of Laplace Transforms

Usefulness of Laplace transforms derives from its many properties.

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\frac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	shift R by $-\alpha$
Differentiate in t	$\frac{dx(t)}{dt}$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{X(s)}{s}$	$\supset \Big(R \cap \big(\text{Re}(s) > 0\big)\Big)$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$

and many others!

Initial Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$ then

$$x(0^+) = \lim_{s \to \infty} sX(s)\,.$$

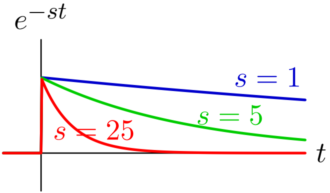
Initial Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$ then

$$x(0^+) = \lim_{s\to\infty} sX(s)\,.$$

Consider $\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{s\to\infty}\int_{0}^{\infty} x(t)\,se^{-st}dt$.

As $s \to \infty$ the function e^{-st} shrinks towards 0.



Area under e^{-st} is $\frac{1}{s}$ $\rightarrow$ area under se^{-st} is 1 $\rightarrow$ $\lim_{s\to\infty} se^{-st} = \delta(t)$!

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\int_{0}^{\infty} x(t)se^{-st}dt \rightarrow \int_{0}^{\infty} x(t)\delta(t)dt = x(0^+)$$

(the 0^+ arises because the limit is from the right side.)

Final Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$

$$x(\infty) = \lim_{s \to 0} sX(s)\,.$$

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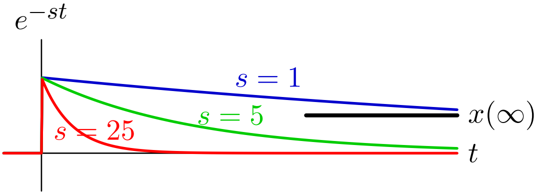
As $s \to 0$ the function e^{-st} flattens out. But again, the area under se^{-st} is always 1.

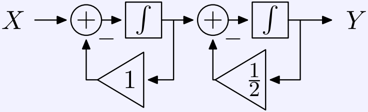


As $s \to 0$, area under se^{-st} monotonically shifts to higher values of t (e.g., the average value of se^{-st} is $\frac{1}{s}$ which grows as $s \to 0$).

In the limit, $\lim_{s\to 0} sX(s) \to x(\infty)\,.$

Summary: Relations among CT representations

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\, u(t)$$

Laplace transform

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Many others: e.g., Laplace transform of a circuit (see HW4)!