6.003: Signals and Systems

CT Feedback and Control

October 20, 2011

Mid-term Examination #2

Wednesday, October 26, 7:30-9:30pm, Walker (50-340)

No recitations on the day of the exam.

Coverage: Lectures 1–12
             Recitations 1–12
             Homeworks 1–7

Homework 7 will not be collected or graded. Solutions will be posted.

Closed book: 2 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

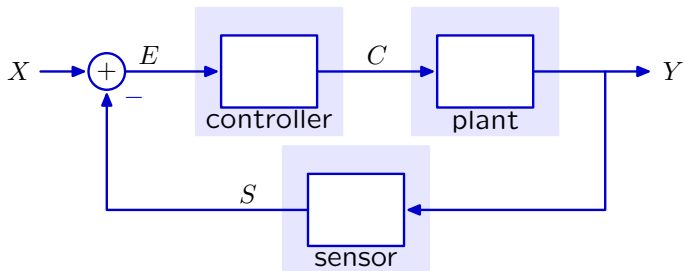No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.
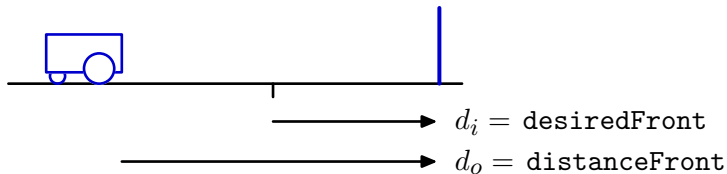
Review sessions during open office hours.

Conflict? Contact freeman@mit.edu before Friday, Oct. 21, 5pm.

Feedback and Control

Feedback: simple, elegant, and robust framework for control.



Last time: robotic driving.

Feedback and Control

Using feedback to enhance performance.

Examples:

- increasing speed and bandwidth
- controlling position instead of speed
- reducing sensitivity to parameter variation
- reducing distortion
- stabilizing unstable systems
  - magnetic levitation
  - inverted pendulum

Op-amps

An "ideal" op-amp has many desireable characteristics.

$$V_o = K\left(V_+ - V_-\right)$$

with inputs V_+ and V_- and gain K.
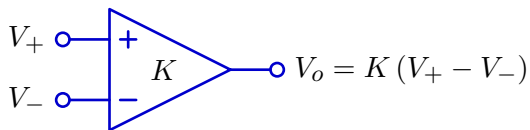
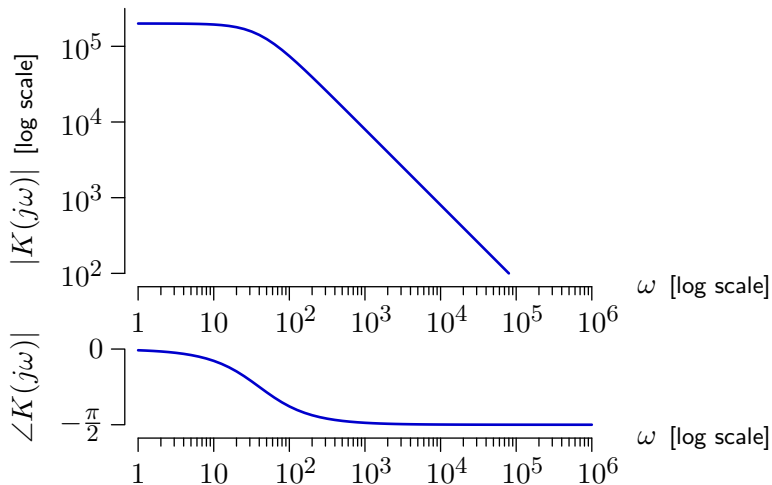- high speed
- large bandwidth
- high input impedance
- low output impedance
- ...

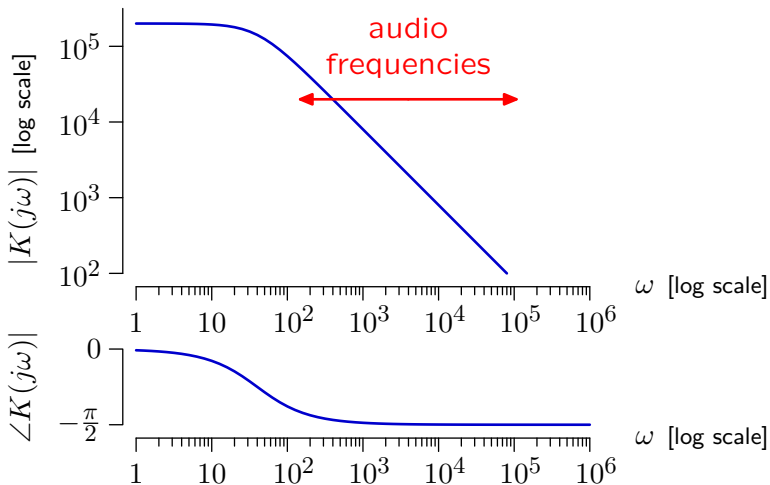It is difficult to build a circuit with all of these features.

Op-Amp

The gain of an op-amp depends on frequency.

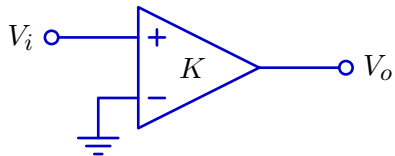
Frequency dependence of LM741 op-amp.

Op-Amp

Low-gain at high frequencies limits applications.

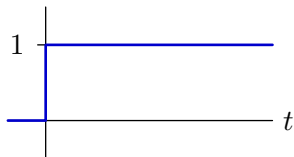Unacceptable frequency response for an audio amplifier.
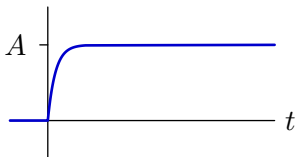
Op-Amp

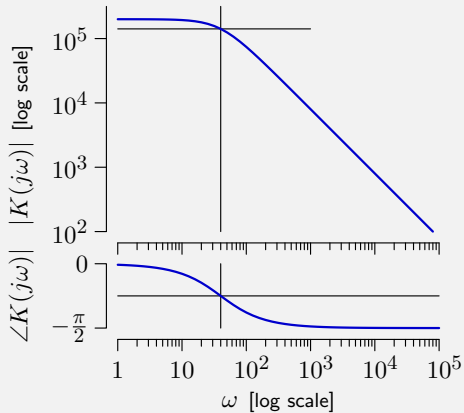An ideal op-amp has fast time response.

Step response:

Check Yourself

Determine τ for the unit-step response $s(t)$ of an LM741.
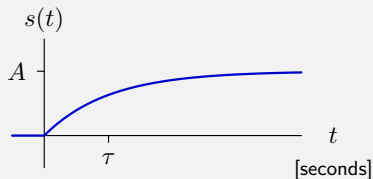
1. $40\,\text{s}$ 2. $\frac{40}{2\pi}\,\text{s}$ 3. $\frac{1}{40}\,\text{s}$ 4. $\frac{2\pi}{40}\,\text{s}$ 5. $\frac{1}{2\pi\times40}\,\text{s}$

0. none of the above

Check Yourself

Determine the step response of an LM741.

System function:
$$K(s) = \frac{\alpha K_0}{s + \alpha}$$

Impulse response:
$$h(t) = \alpha K_0 e^{-\alpha t} u(t)$$

Step response:
$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \int_{0}^{t} \alpha K_0 e^{-\alpha \tau} d\tau = \left. \frac{\alpha K_0 e^{-\alpha \tau}}{-\alpha} \right|_{0}^{t} = K_0(1 - e^{-\alpha t})u(t)$$
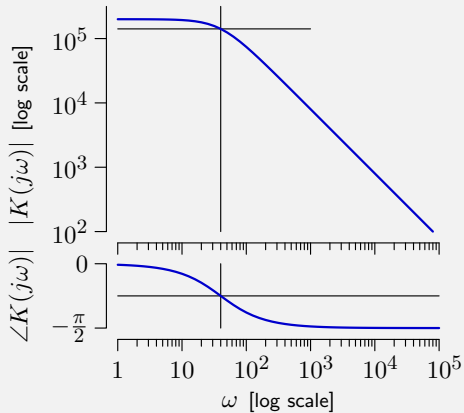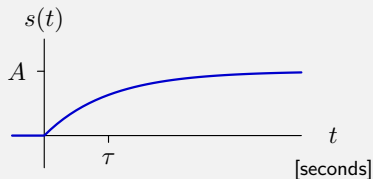
Parameters:
$$A = K_0 = 2 \times 10^5$$
$$\tau = \frac{1}{\alpha} = \frac{1}{40} \, \text{s}$$

Check Yourself

Determine τ for the unit-step response $s(t)$ of an LM741.

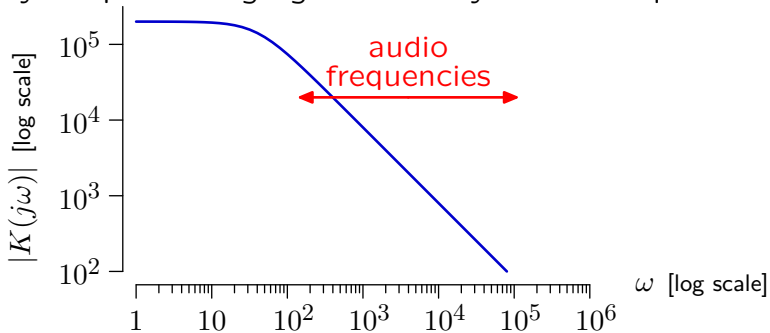1. 40 s   2. $\frac{40}{2\pi}$ s   3. $\frac{1}{40}$ s   4. $\frac{2\pi}{40}$ s   5. $\frac{1}{2\pi \times 40}$ s
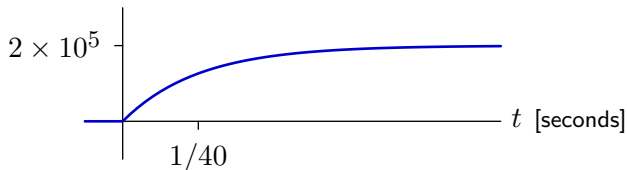
0. none of the above

Op-Amp

Performance parameters for real op-amps fall short of the ideal.

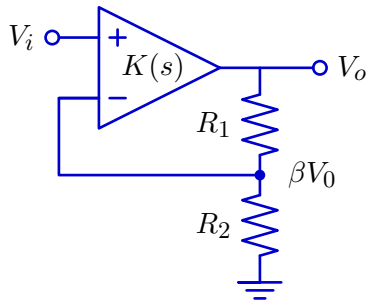Frequency Response: high gain but only at low frequencies.

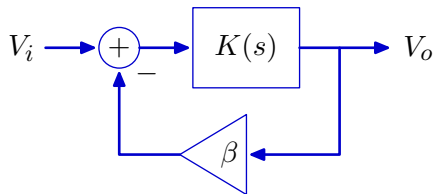Step Response: slow by electronic standards.

Op-Amp

We can use feedback to improve performance of op-amps.

circuit

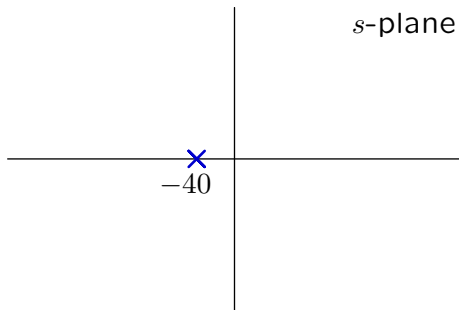6.003 model

$$V_- = \beta V_o = \left(\frac{R_2}{R_1 + R_2} \right) V_o$$

$$\frac{V_o}{V_i} = \frac{K(s)}{1 + \beta K(s)}$$

Dominant Pole

Op-amps are designed to have a dominant pole at low frequencies:
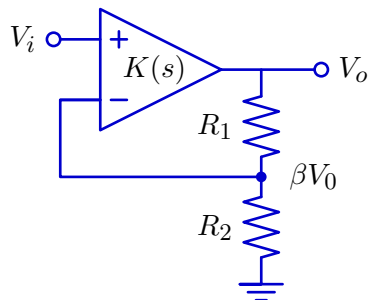$\rightarrow$ simplifies the application of feedback.

s-plane

$$\alpha = 40 \, \text{rad/s} = \frac{40 \, \text{rad/s}}{2\pi \, \text{rad/cycle}} \approx 6.4 \, \text{Hz}$$
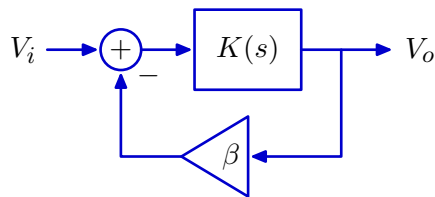
Improving Performance

Using feedback to improve performance parameters.

circuit

6.003 model

$$V_- = \beta V_o = \left(\frac{R_2}{R_1 + R_2} \right) V_o$$

$$\frac{V_o}{V_i} = \frac{K(s)}{1 + \beta K(s)}$$

$$= \frac{\frac{\alpha K_0}{s + \alpha}}{1 + \beta \frac{\alpha K_0}{s + \alpha}}$$

$$= \frac{\alpha K_0}{s + \alpha + \alpha \beta K_0}$$

Check Yourself

What is the most negative value of the closed-loop pole that can be achieved with feedback?

s-plane

? $-\alpha$

1. $-\alpha(1 + \beta)$ 2. $-\alpha(1 + \beta K_0)$
3. $-\alpha(1 + K_0)$ 4. $-\infty$
5. none of the above

Improving Performance

Using feedback to improve performance parameters.

circuit

6.003 model

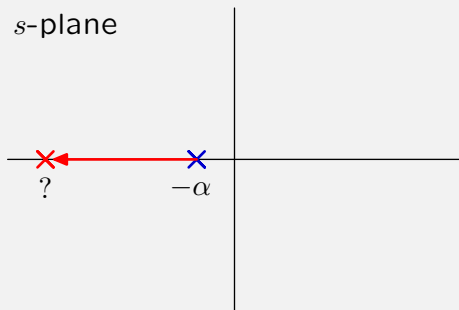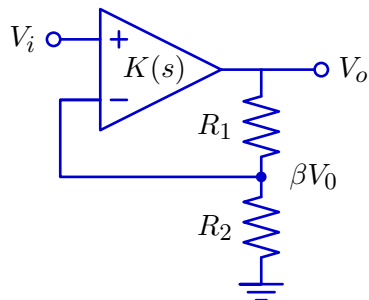$$V_- = \beta V_o = \left(\frac{R_2}{R_1 + R_2} \right) V_o$$

$$\frac{V_o}{V_i} = \frac{K(s)}{1 + \beta K(s)}$$

$$= \frac{\frac{\alpha K_0}{s + \alpha}}{1 + \beta \frac{\alpha K_0}{s + \alpha}}$$

$$= \frac{\alpha K_0}{s + \alpha + \alpha \beta K_0}$$

Check Yourself

What is the most negative value of the closed-loop pole that can be achieved with feedback?

Open loop system function: $\dfrac{\alpha K_0}{s + \alpha}$

$\rightarrow$ pole: $s = -\alpha$.

Closed-loop system function: $\dfrac{\alpha K_0}{s + \alpha + \alpha\beta K_0}$

$\rightarrow$ pole: $s = -\alpha(1 + \beta K_0)$.

The feedback constant is $0 \le \beta \le 1$.

$\rightarrow$ most negative value of the closed-loop pole is $s = -\alpha(1 + K_0)$.

What is the most negative value of the closed-loop pole that can be achieved with feedback?   3

s-plane

$-\alpha(1+K_0)$ $-\alpha$

1. $-\alpha(1+\beta)$ 2. $-\alpha(1+\beta K_0)$
3. $-\alpha(1+K_0)$ 4. $-\infty$
              5. none of the above

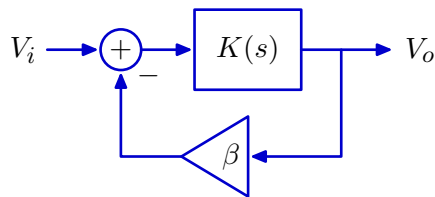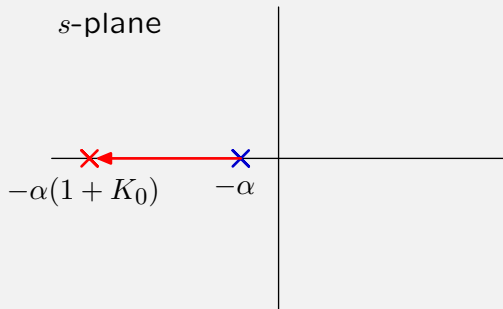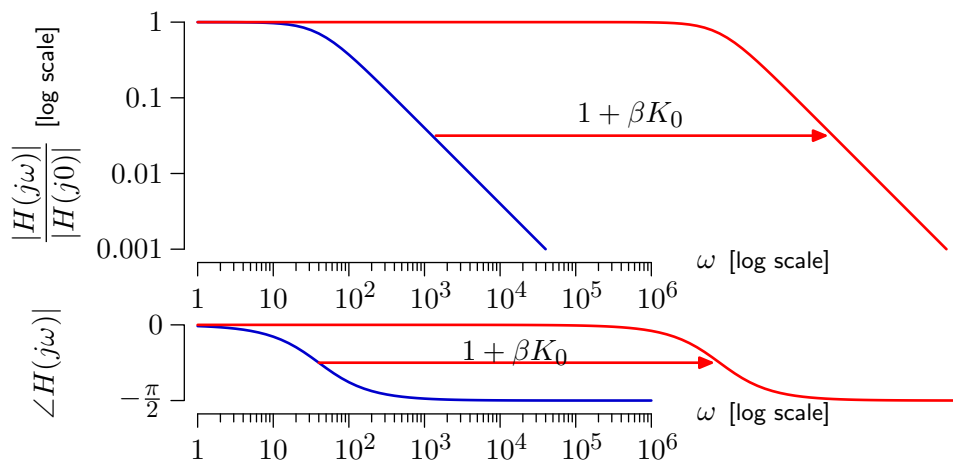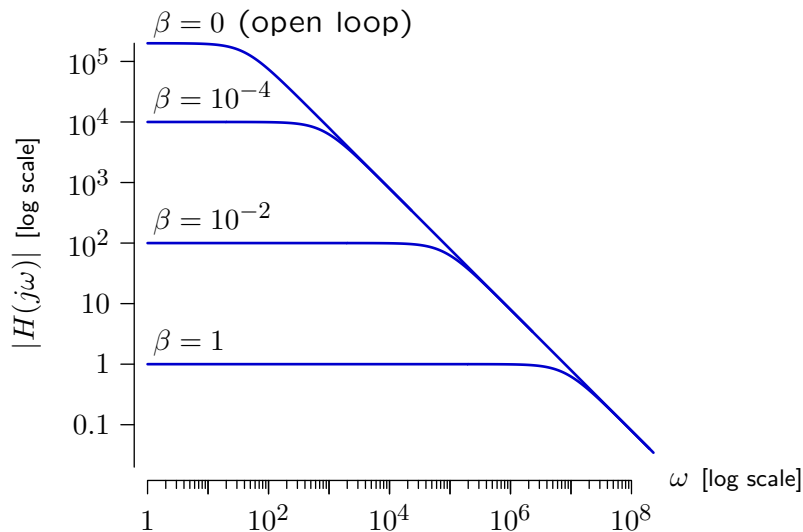## Improving Performance

Feedback extends frequency response by a factor of $1 + \beta K_0$
($K_0 = 2 \times 10^5$).

Improving Performance

Feedback produces higher bandwidths by **reducing** the gain at low frequencies. It trades gain for bandwidth.
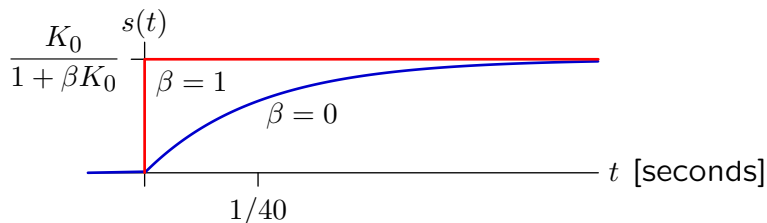
Improving Performance

Feedback makes the time response faster by a factor of $1 + \beta K_0$ ($K_0 = 2 \times 10^5$).

Step response

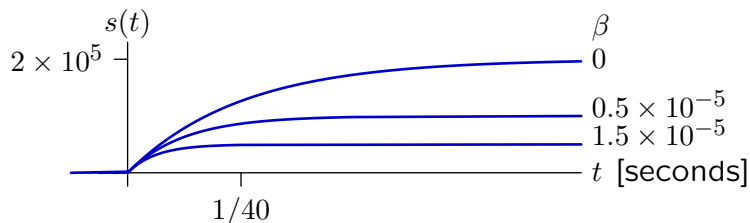$$s(t) = \frac{K_0}{1 + \beta K_0}(1 - e^{-\alpha(1 + \beta K_0)t})u(t)$$

Improving Performance

Feedback produces faster responses by **reducing** the final value of the step response. It trades gain for speed.

Step response

$$s(t) = \frac{K_0}{1 + \beta K_0}(1 - e^{-\alpha(1+\beta K_0)t})u(t)$$

The maximum rate of voltage change $\left. \dfrac{ds(t)}{dt} \right|_{t=0+}$ is not increased.

Improving Performance

Feedback improves performance parameters of op-amp circuits.

- can extend frequency response
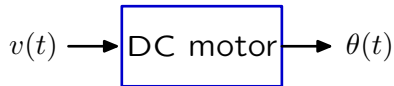- can increase speed

Performance enhancements are achieved through a reduction of gain.

Motor Controller

We wish to build a robot arm (actually its elbow). The input should
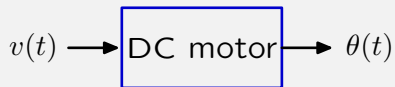be voltage $v(t)$, and the output should be the elbow angle $\theta(t)$.

We wish to build the robot arm with a DC motor.

This problem is similar to the head-turning servo in 6.01!

Check Yourself

What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

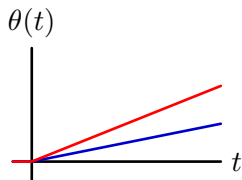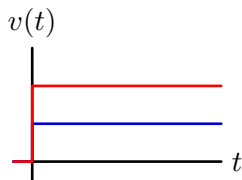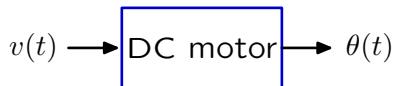$$v(t) \longrightarrow \boxed{\text{DC motor}} \longrightarrow \theta(t)$$

1. $\theta(t) \propto v(t)$
2. $\cos\theta(t) \propto v(t)$
3. $\theta(t) \propto \dot{v}(t)$
4. $\cos\theta(t) \propto \dot{v}(t)$
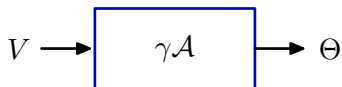5. none of the above

Check Yourself

What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

To first order, the rotational speed $\dot{\theta}(t)$ of a DC motor is proportional to the input voltage $v(t)$.
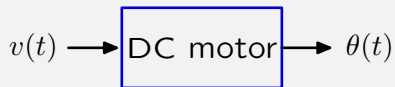

First-order model: integrator

Check Yourself

What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

$$v(t) \longrightarrow \boxed{\text{DC motor}} \longrightarrow \theta(t)$$

1. $\theta(t) \propto v(t)$
2. $\cos\theta(t) \propto v(t)$
3. $\theta(t) \propto \dot{v}(t)$
4. $\cos\theta(t) \propto \dot{v}(t)$
5. none of the above

Motor Controller

Use proportional feedback to control the angle of the motor's shaft.

amplifier   DC motor

$v(t) \longrightarrow \boxed{+ \atop -} \longrightarrow \triangleright\alpha \longrightarrow \boxed{\gamma\mathcal{A}} \longrightarrow \theta(t)$
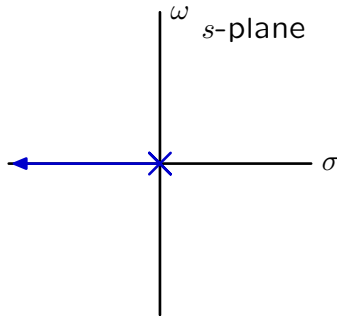
$\triangleleft\beta$

feedback
(potentiometer)
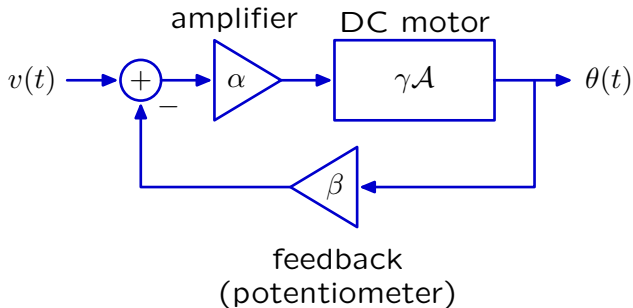
$$\frac{\Theta}{V} = \frac{\alpha\gamma\mathcal{A}}{1 + \alpha\beta\gamma\mathcal{A}} = \frac{\alpha\gamma\frac{1}{s}}{1 + \alpha\beta\gamma\frac{1}{s}} = \frac{\alpha\gamma}{s + \alpha\beta\gamma}$$

Motor Controller

The closed loop system has a single pole at $s = -\alpha\beta\gamma$.

$$\frac{\Theta}{V} = \frac{\alpha\gamma}{s + \alpha\beta\gamma}$$

As α increases, the closed-loop pole becomes increasingly negative.

Motor Controller

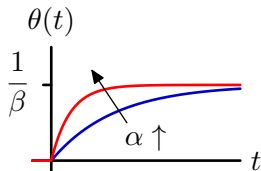Find the impulse and step response.

The system function is
$$\frac{\Theta}{V} = \frac{\alpha\gamma}{s + \alpha\beta\gamma}.$$

The impulse response is
$$h(t) = \alpha\gamma e^{-\alpha\beta\gamma t} u(t)$$

and the step response is therefore
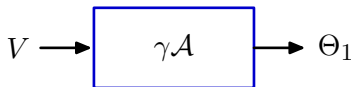$$s(t) = \frac{1}{\beta}\left(1 - e^{-\alpha\beta\gamma t}\right) u(t).$$

The response is faster for larger values of α.
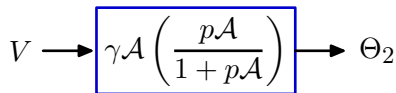
Try it: Demo.

Motor Controller

The speed of a DC motor does not change instantly if the voltage is stepped. There is lag due to rotational inertia.
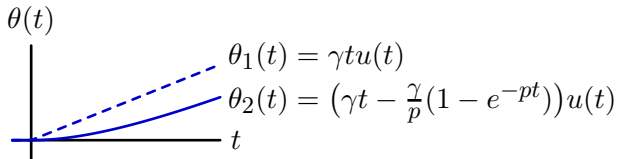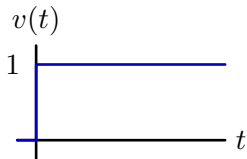
First-order model
integrator

$$V \longrightarrow \boxed{\gamma \mathcal{A}} \longrightarrow \Theta_1$$

Second-order model
integrator with lag

$$V \longrightarrow \boxed{\gamma \mathcal{A} \left(\frac{p\mathcal{A}}{1 + p\mathcal{A}} \right)} \longrightarrow \Theta_2$$

Step response of the models:

$$\theta_1(t) = \gamma t u(t)$$
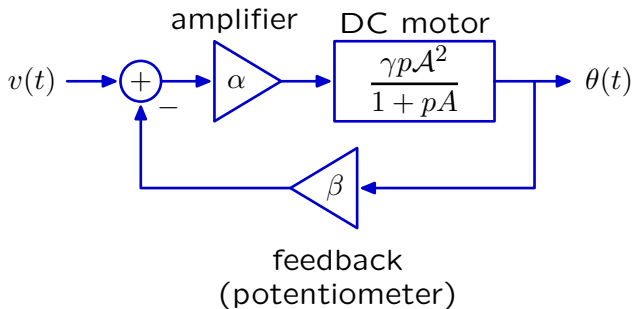$$\theta_2(t) = \left(\gamma t - \frac{\gamma}{p}(1 - e^{-pt})\right) u(t)$$

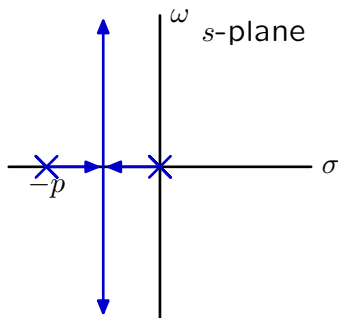## Motor Controller

Analyze second-order model.

$$\frac{\Theta}{V} = \frac{\frac{\alpha\gamma p\mathcal{A}^2}{1+pA}}{1 + \frac{\alpha\beta\gamma p\mathcal{A}^2}{1+p\mathcal{A}}} = \frac{\alpha\gamma p\mathcal{A}^2}{1 + pA + \alpha\beta\gamma p\mathcal{A}^2} = \frac{\alpha\gamma p}{s^2 + ps + \alpha\beta\gamma p}$$

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - \alpha\beta\gamma p}$$
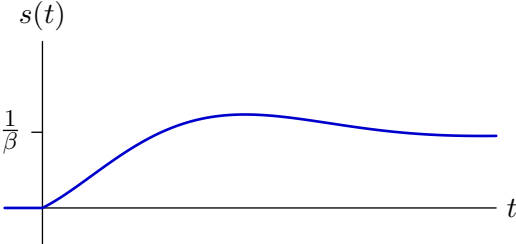
Motor Controller

For second-order model, increasing α causes the poles at 0 and $-p$ to approach each other, collide at $s = -p/2$, then split into two poles with imaginary parts.

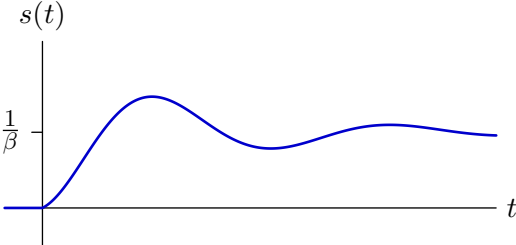Increasing the gain α does not increase speed of convergence.
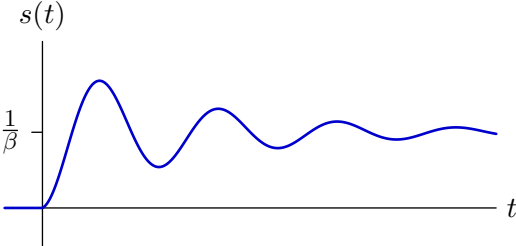
Motor Controller

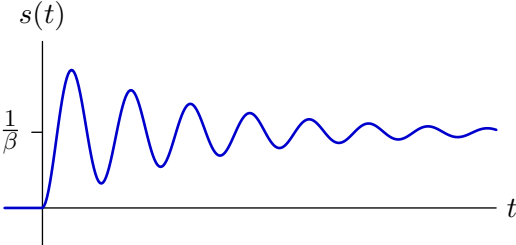Step response.

Motor Controller

Step response.

Motor Controller
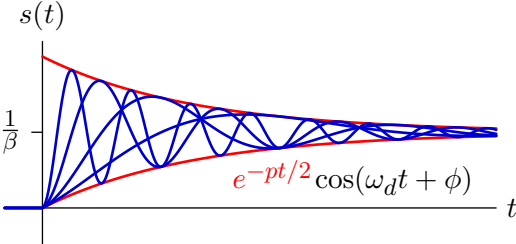
Step response.

Motor Controller

Step response.

Motor Controller

Step response.

Feedback and Control: Summary

CT feedback is useful for many reasons. Today we saw two:

- increasing speed and bandwidth
- controlling position instead of speed

Next time we will look at several others:

- reduce sensitivity to parameter variation
- reduce distortion
- stabilize unstable systems
  - magnetic levitation
  - inverted pendulum