6.003: Signals and Systems

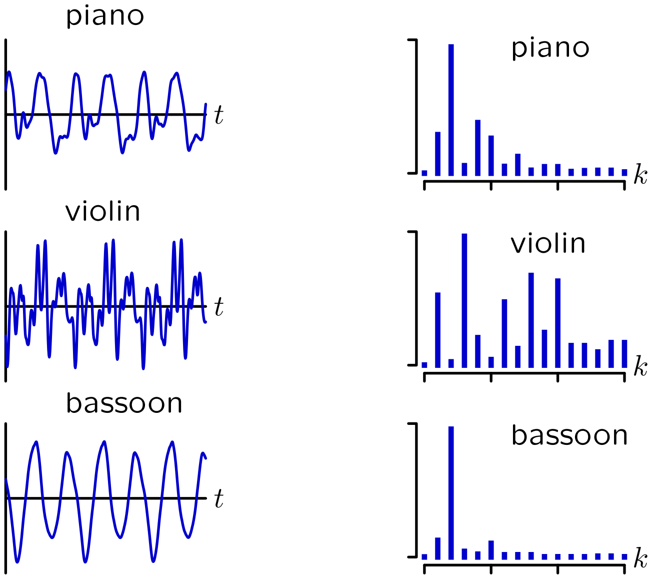
Fourier Series

November 1, 2011

Last Time: Describing Signals by Frequency Content

Harmonic content is natural way to describe some kinds of signals.

Ex: musical instruments (http://theremin.music.uiowa.edu/MIS)



Last Time: Fourier Series

Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

("analysis" equation)

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

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We can think of Fourier series as an **orthogonal decomposition**.

Orthogonal Decompositions

Vector representation of 3-space: let $\bar{r}$ represent a vector with components $\{x$, y, and $z\}$ in the $\{\hat{x}$, $\hat{y}$, and $\hat{z}\}$ directions, respectively.

$$x = \bar{r} \cdot \hat{x}$$
$$y = \bar{r} \cdot \hat{y} \qquad \text{(“analysis” equations)}$$
$$z = \bar{r} \cdot \hat{z}$$

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad \text{(“synthesis” equation)}$$

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Orthogonal Decompositions

Integrating over a period **sifts** out the k^{th} component of the series.

Sifting as a dot product:

$$x = \bar{r} \cdot \hat{x} \equiv |\bar{r}||\hat{x}| \cos\theta$$

Sifting as an inner product:

$$a_k = e^{j\frac{2\pi}{T}kt} \cdot x(t) \equiv \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

where

$$a(t) \cdot b(t) = \frac{1}{T}\int_T a^*(t)b(t)dt\,.$$

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The complex conjugate (*) makes the inner product of the k^{th} and m^{th} components equal to 1 iff $k = m$:

$$\frac{1}{T}\int_T \left(e^{j\frac{2\pi}{T}kt}\right)^* \left(e^{j\frac{2\pi}{T}mt}\right) dt = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt} e^{j\frac{2\pi}{T}mt} dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$

Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$?

1. $\cos 2\pi t \perp \sin 2\pi t$?
2. $\cos 2\pi t \perp \cos 4\pi t$?
3. $\cos 2\pi t \perp \sin \pi t$?
4. $\cos 2\pi t \perp e^{j2\pi t}$?

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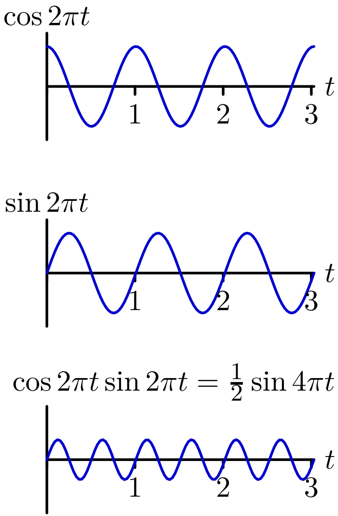

$\int_0^3 dt = 0$ therefore YES

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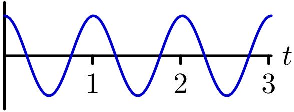
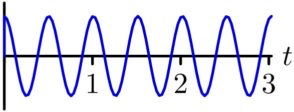
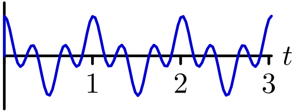


$\cos 4\pi t$



$\cos 2\pi t \cos 4\pi t = \frac{1}{2} \cos 6\pi t + \frac{1}{2} \cos 2\pi t$



$\int_0^3 dt = 0$ therefore YES

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$?

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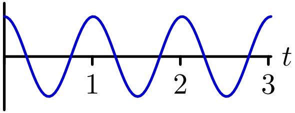
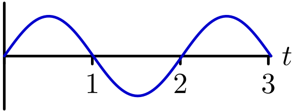
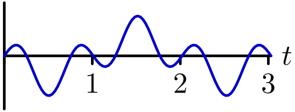


$\sin \pi t$



$$\cos 2\pi t \sin \pi t = \frac{1}{2} \sin 3\pi t - \frac{1}{2} \sin \pi t$$



$\int_0^3 dt \neq 0$ therefore NO

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp e^{2\pi t}$?

Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp e^{2\pi t}$?

$$e^{2\pi t} = \cos 2\pi t + j \sin 2\pi t$$

$\cos 2\pi t \perp \sin 2\pi t$ but not $\cos 2\pi t$

Therefore NO

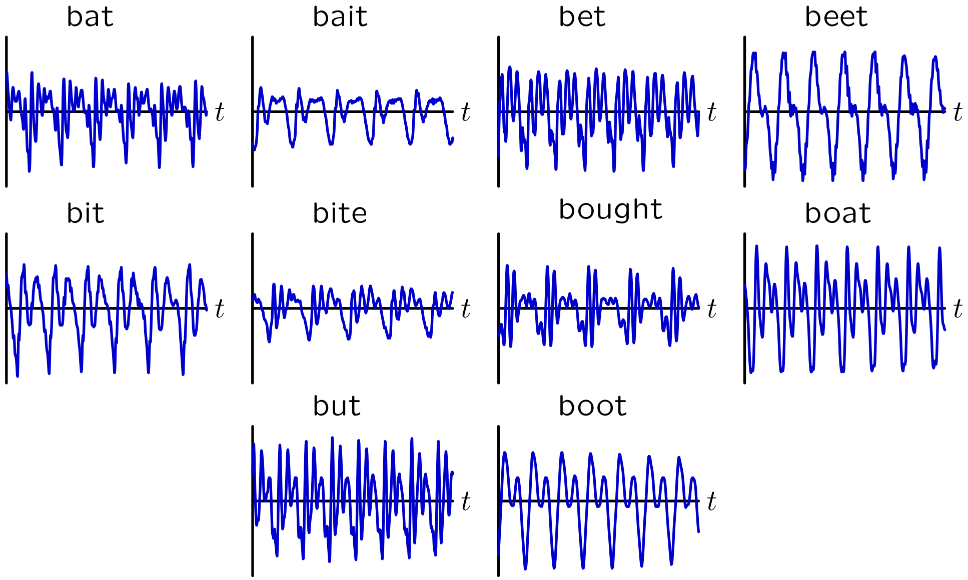
Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$? 2

1. $\cos 2\pi t \perp \sin 2\pi t$? $\surd$
2. $\cos 2\pi t \perp \cos 4\pi t$? $\surd$
3. $\cos 2\pi t \perp \sin \pi t$? X
4. $\cos 2\pi t \perp e^{j2\pi t}$? X

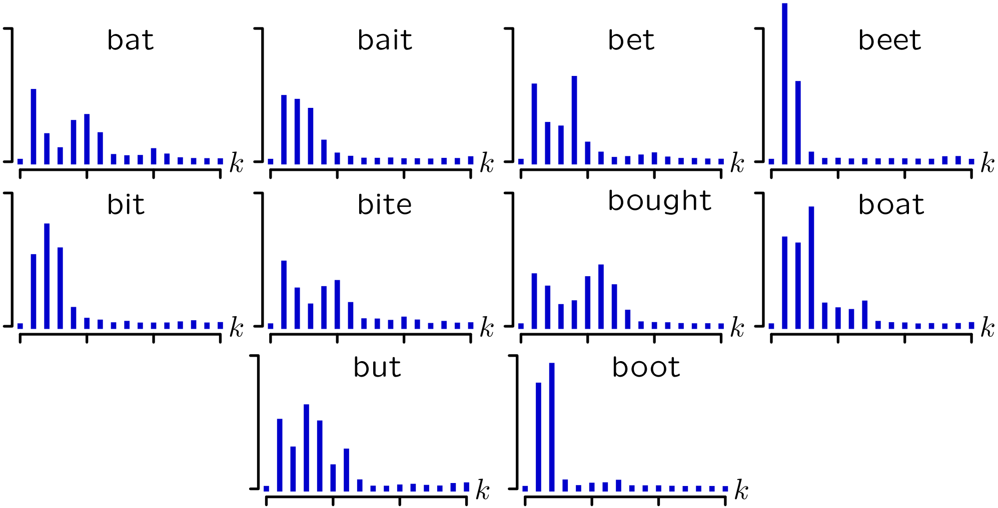
Speech

Vowel sounds are quasi-periodic.



Speech

Harmonic content is natural way to describe vowel sounds.



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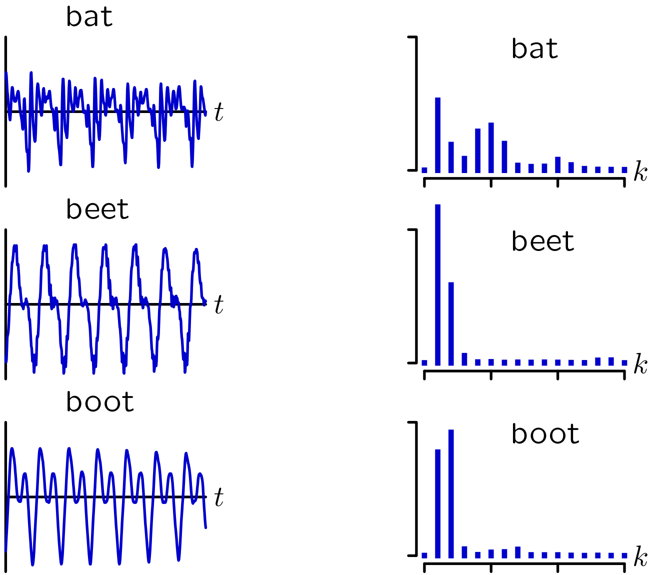
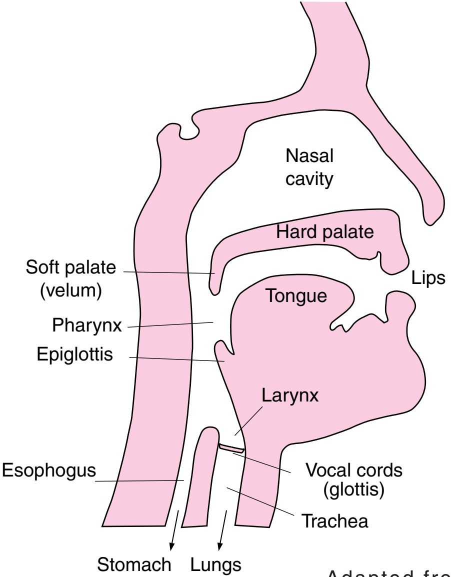


Speech Production

Speech is generated by the passage of air from the lungs, through the vocal cords, mouth, and nasal cavity.

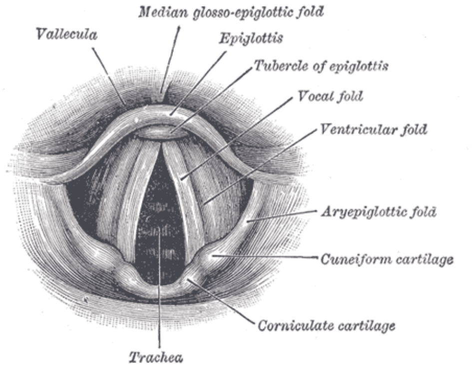


Adapted from T.F. Weiss

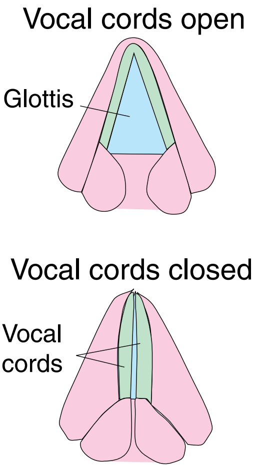
Speech Production

Controlled by complicated muscles, vocal cords are set in vibration by the passage of air from the lungs.

Looking down the throat:



Gray's Anatomy



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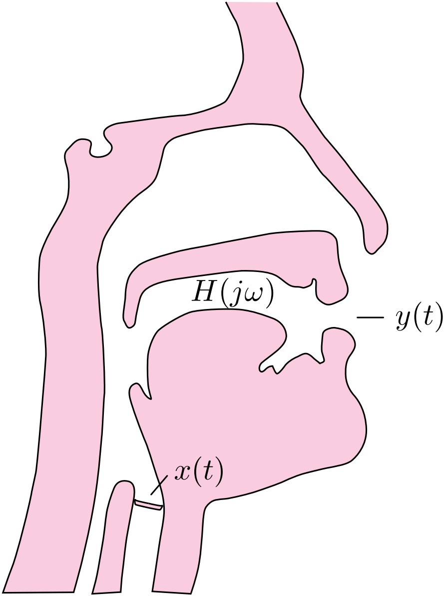
Speech Production

Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.



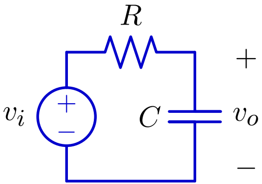
Filtering

Notion of a filter.

LTI systems

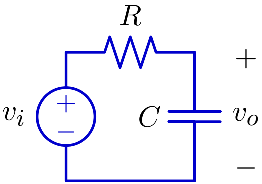
- cannot create new frequencies.
- can only scale magnitudes & shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

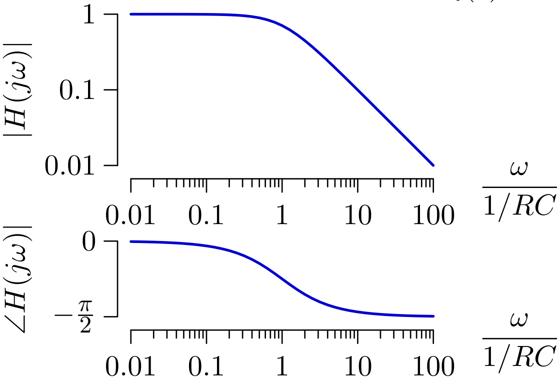


Lowpass Filter

Calculate the frequency response of an RC circuit.

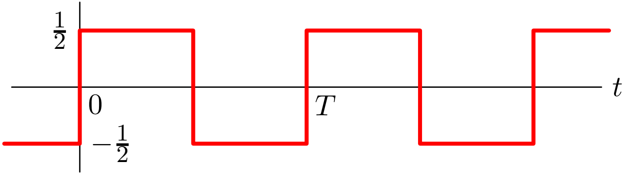


$$
\begin{aligned}
\text{KVL:} \quad & v_i(t) = Ri(t) + v_o(t) \\
\text{C:} \quad & i(t) = C\dot{v}_o(t) \\
\text{Solving:} \quad & v_i(t) = RC\dot{v}_o(t) + v_o(t) \\
& V_i(s) = (1 + sRC)V_o(s) \\
& H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

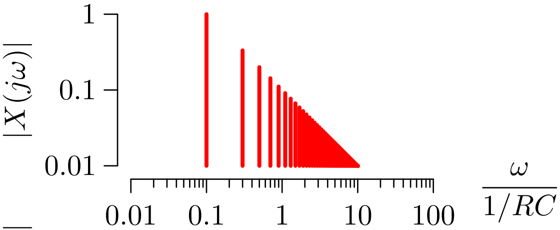


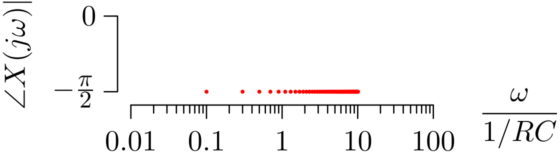
Lowpass Filtering

Let the input be a square wave.



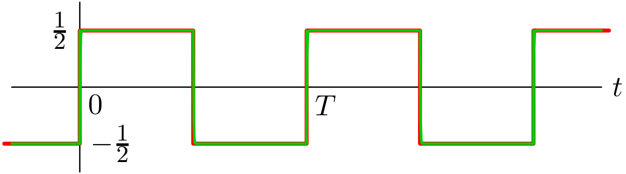
$$x(t) = \sum_{k\ \text{odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



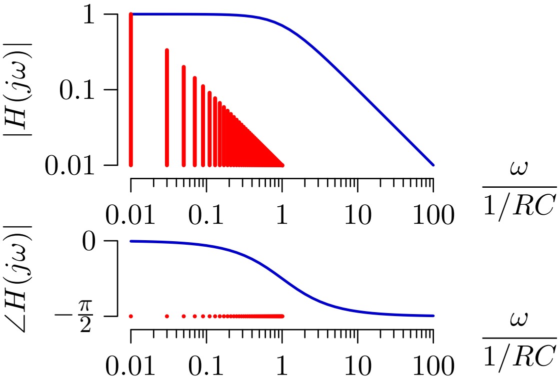


Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

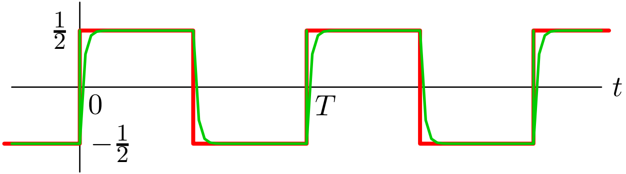


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

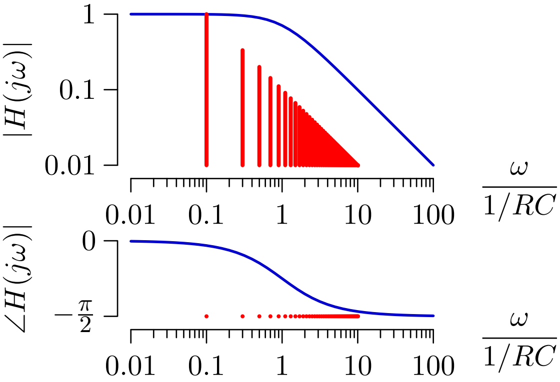


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

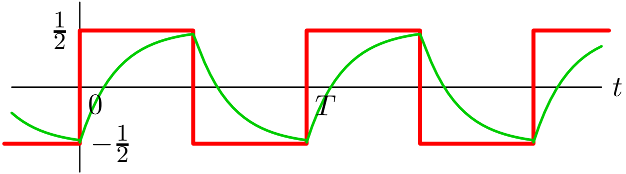


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.



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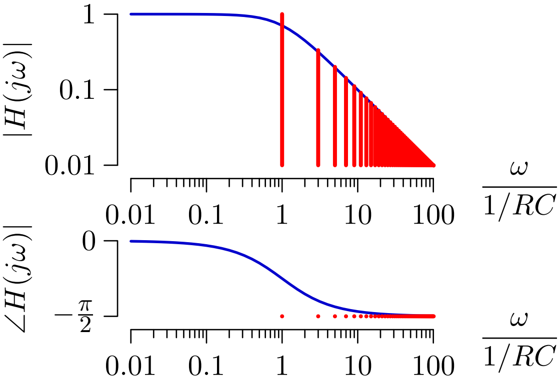
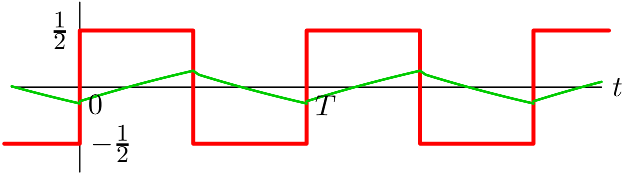


Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.



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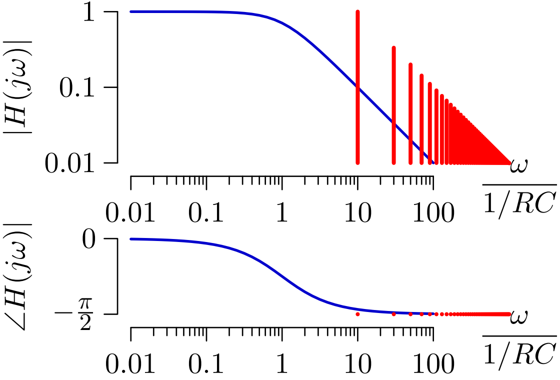
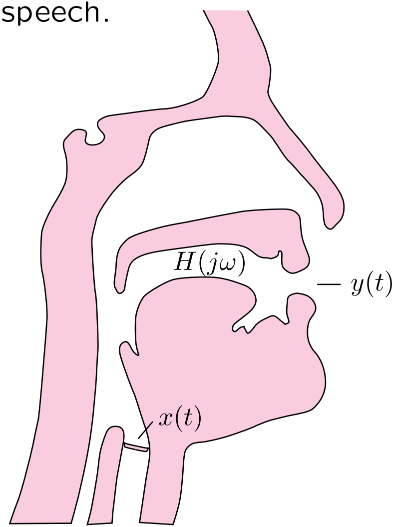
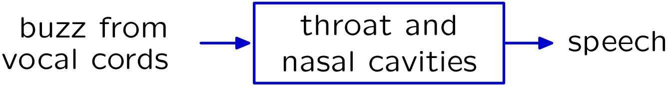


Source-Filter Model of Speech Production

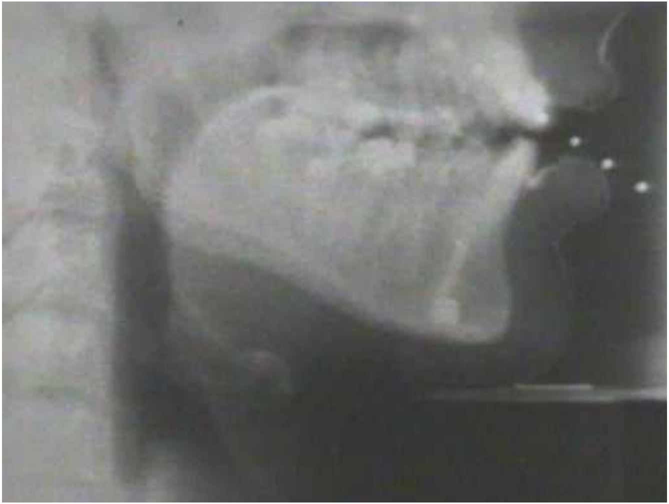
Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.





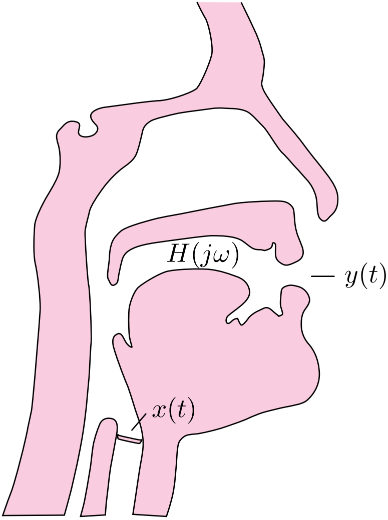
Speech Production

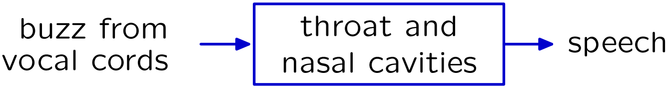
X-ray movie showing speech in production.



Demonstration

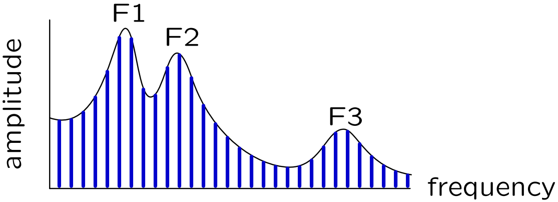
Artificial speech.





Formants

Resonant frequencies of the vocal tract.

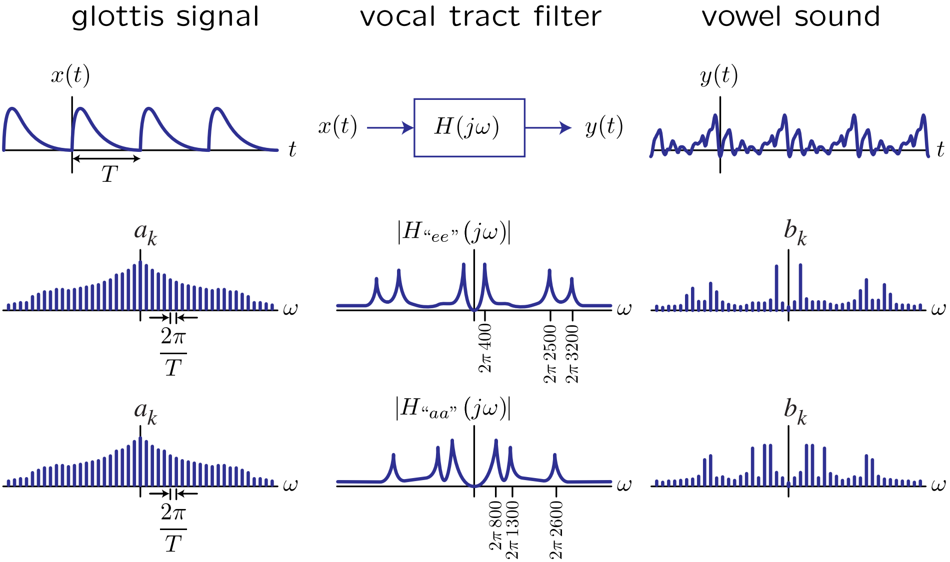


	Formant	heed	head	had	hod	haw'd	who'd
Men	F1	270	530	660	730	570	300
	F2	2290	1840	1720	1090	840	870
	F3	3010	2480	2410	2440	2410	2240
Women	F1	310	610	860	850	590	370
	F2	2790	2330	2050	1220	920	950
	F3	3310	2990	2850	2810	2710	2670
Children	F1	370	690	1010	1030	680	430
	F2	3200	2610	2320	1370	1060	1170
	F3	3730	3570	3320	3170	3180	3260

http://www.sfu.ca/sonic-studio/handbook/Formant.html

Speech Production

Same glottis signal + different formants → different vowels.



We detect changes in the filter function to recognize vowels.

Singing

We detect changes in the filter function to recognize vowels
... at least sometimes.

Demonstration.

"la" scale.

"lore" scale.

"loo" scale.

"ler" scale.

"lee" scale.

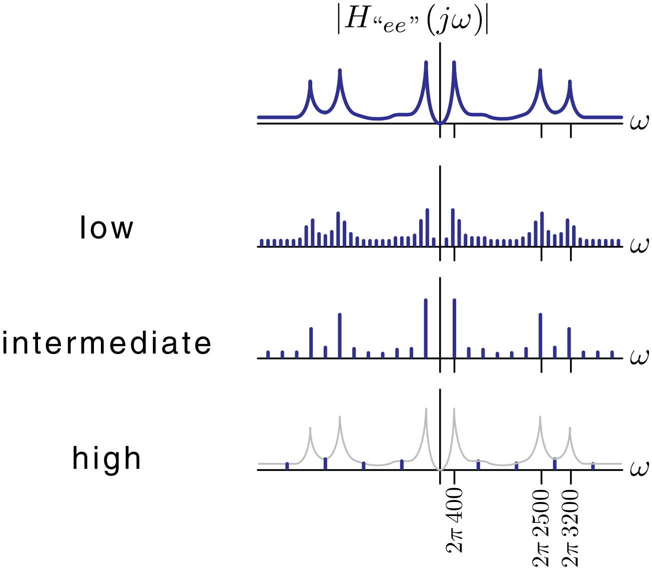
Low Frequency: "la" "lore" "loo" "ler" "lee".

High Frequency: "la" "lore" "loo" "ler" "lee".

http://www.phys.unsw.edu.au/jw/soprane.html

Speech Production

We detect changes in the filter function to recognize vowels.



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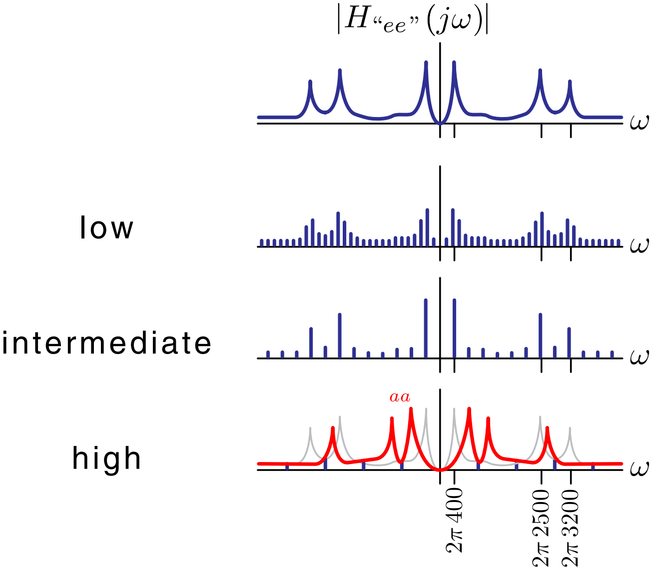

Continuous-Time Fourier Series: Summary

Fourier series represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., music.

Fourier series motivate a new representation of a system as a filter.

Representing a system as a filter is useful for many systems, e.g., speech synthesis.