6.003: Signals and Systems

Discrete-Time Frequency Representations

November 8, 2011

Mid-term Examination #3

Wednesday, November 16, 7:30-9:30pm, Walker (50-340)

No recitations on the day of the exam.

Coverage: Lectures 1–18
Recitations 1–16
Homeworks 1–10

Homework 10 will not be collected or graded.
Solutions will be posted.

Closed book: 3 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

Review session Monday at 3pm (36-112) and at open office hours.

Prior term midterm exams have been posted on the 6.003 website.

Conflict? Contact freeman@mit.edu before Friday, Nov. 11, 5pm.

Signal Processing: From CT to DT

Signal-processing problems first conceived & addressed in CT:

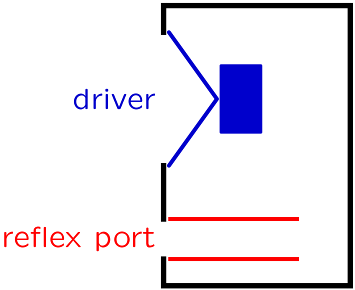
- audio
  - radio (noise/static reduction, automatic gain control, etc.)
  - telephone (equalizers, echo-suppression, etc.)
  - hi-fi (bass, treble, loudness, etc.)
- imaging
  - television (brightness, tint, etc.)
  - photography (image enhancement, gamma)
  - x-rays (noise reduction, contrast enhancement)
  - radar and sonar (noise reduction, object detection)

Such problems are increasingly solved with DT signal processing:

- MP3
- JPEG
- MPEG

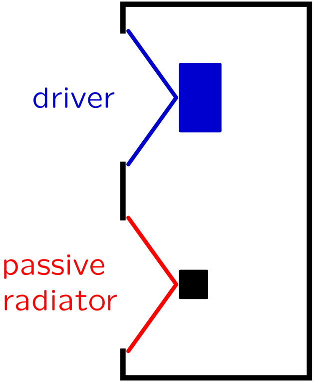
Signal Processing: Acoustical

Mechano-acoustic components to optimize frequency response of loudspeakers: e.g., “bass-reflex” system.



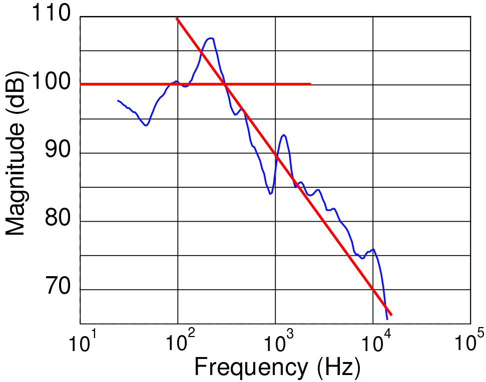
Signal Processing: Acoustico-Mechanical

Passive radiator for improved low-frequency preformance.



Signal Processing: Electronic

Low-cost electronics → new ways to overcome frequency limitations.

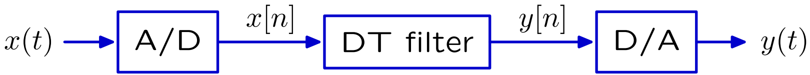


Small speakers (4 inch): eight facing wall, one facing listener.

Electronic “equalizer” compensated for limited frequency response.

Signal Processing

Modern audio systems process sounds digitally.



Signal Processing

Modern audio systems process sounds digitally.

Texas Instruments TAS3004

- 2 channels
- 24 bit ADC, 24 bit DAC
- 48 kHz sampling rate
- 100 MIPS
- $9.63 ($5.20 in bulk)

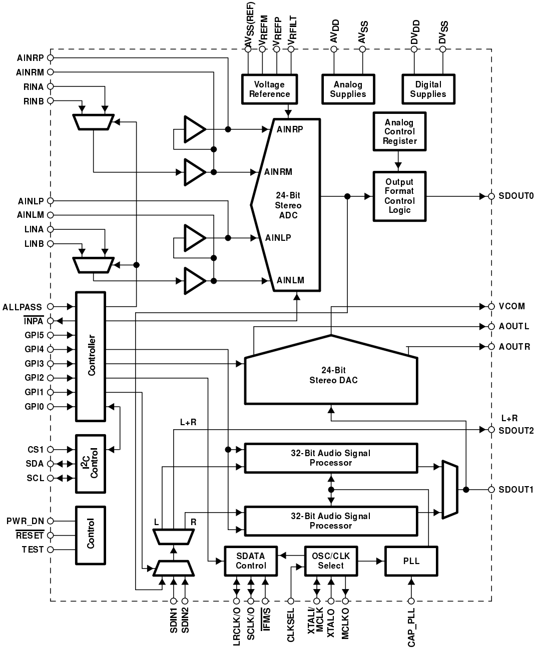


Figure 1-1. TAS3004 Block Diagram

DT Fourier Series and Frequency Response

Today: frequency representations for DT signals and systems.

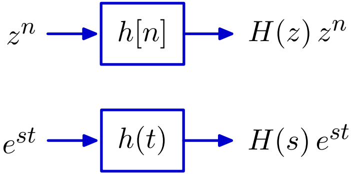
Review: Complex Geometric Sequences

Complex geometric sequences are eigenfunctions of DT LTI systems.

Find response of DT LTI system ($h[n]$) to input $x[n] = z^n$.

$$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z)\, z^n \,.$$

Complex geometrics (DT): analogous to complex exponentials (CT)



Review: Rational System Functions

A system described by a linear difference equation with constant coefficients $\rightarrow$ system function that is a ratio of polynomials in z.

Example:

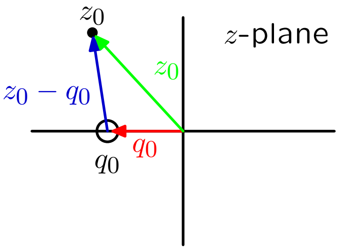
$$y[n-2] + 3y[n-1] + 4y[n] = 2x[n-2] + 7x[n-1] + 8x[n]$$

$$H(z) = \frac{2z^{-2} + 7z^{-1} + 8}{z^{-2} + 3z^{-1} + 4} = \frac{2 + 7z + 8z^2}{1 + 3z + 4z^2} \equiv \frac{N(z)}{D(z)}$$

DT Vector Diagrams

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(z_0) = K\frac{(z_0-q_0)(z_0-q_1)(z_0-q_2)\cdots}{(z_0-p_0)(z_0-p_1)(z_0-p_2)\cdots}$$



Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here q_0) to z_0, the point of interest in the z-plane.

Vector diagrams for DT are similar to those for CT.

DT Vector Diagrams

Value of $H(z)$ at $z = z_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(z_0) = K\frac{(z_0 - q_0)(z_0 - q_1)(z_0 - q_2)\cdots}{(z_0 - p_0)(z_0 - p_1)(z_0 - p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(z_0)| = |K|\,\frac{|(z_0 - q_0)||(z_0 - q_1)||(z_0 - q_2)|\cdots}{|(z_0 - p_0)||(z_0 - p_1)||(z_0 - p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(z_0) = \angle K + \angle(z_0 - q_0) + \angle(z_0 - q_1) + \cdots - \angle(z_0 - p_0) - \angle(z_0 - p_1) - \cdots$$

DT Frequency Response

Response to eternal sinusoids.

Let $x[n] = \cos \Omega_0 n$ (for all time):

$$x[n] = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) = \frac{1}{2}\left(z_0^n + z_1^n\right)$$

where $z_0 = e^{j\Omega_0}$ and $z_1 = e^{-j\Omega_0}$.

The response to a sum is the sum of the responses:

$$y[n] = \frac{1}{2}\left(H(z_0)\, z_0^n + H(z_1)\, z_1^n\right)$$

$$= \frac{1}{2}\left(H(e^{j\Omega_0})\, e^{j\Omega_0 n} + H(e^{-j\Omega_0})\, e^{-j\Omega_0 n}\right)$$

Conjugate Symmetry

For physical systems, the complex conjugate of $H(e^{j\Omega})$ is $H(e^{-j\Omega})$.

The system function is the Z transform of the unit-sample response:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

where $h[n]$ is a real-valued function of n for physical systems.

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

$$H(e^{-j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{j\Omega n} \equiv \left(H(e^{j\Omega})\right)^*$$

DT Frequency Response

Response to eternal sinusoids.

Let $x[n] = \cos \Omega_0 n$ (for all time), which can be written as

$$x[n] = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) .$$

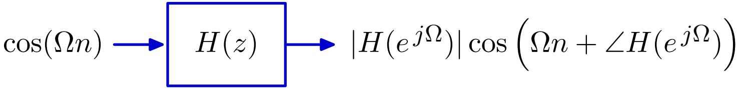
Then

$$\begin{aligned} y[n] &= \frac{1}{2}\left(H(e^{j\Omega_0})e^{j\Omega_0 n} + H(e^{-j\Omega_0})e^{-j\Omega_0 n}\right) \\ &= \mathrm{Re}\left\{H(e^{j\Omega_0})e^{j\Omega_0 n}\right\} \\ &= \mathrm{Re}\left\{|H(e^{j\Omega_0})|e^{j\angle H(e^{j\Omega_0})}e^{j\Omega_0 n}\right\} \\ &= |H(e^{j\Omega_0})|\mathrm{Re}\left\{e^{j\Omega_0 n + j\angle H(e^{j\Omega_0})}\right\} \end{aligned}$$

$$y[n] = \left|H(e^{j\Omega_0})\right| \cos\left(\Omega_0 n + \angle H(e^{j\Omega_0})\right)$$

DT Frequency Response

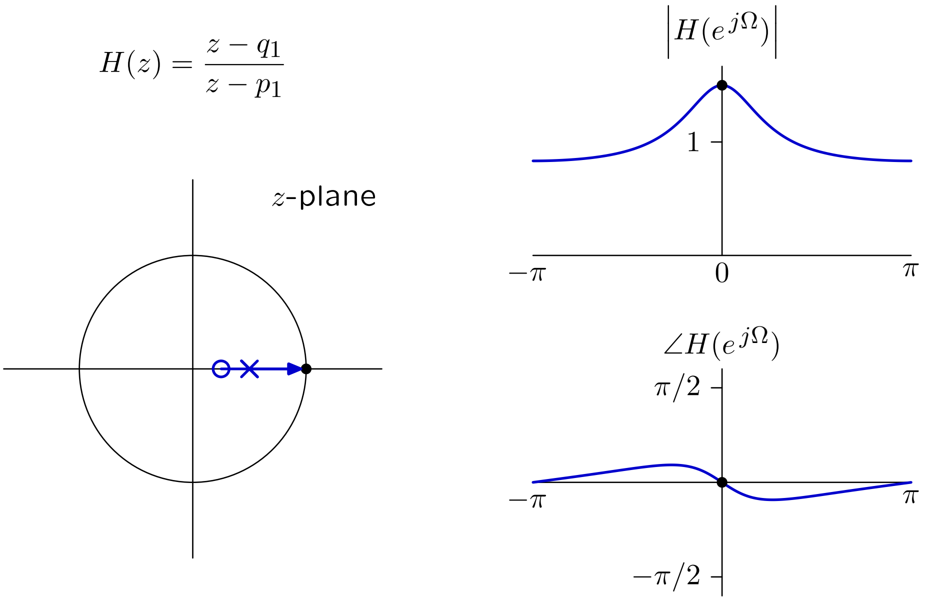
The magnitude and phase of the response of a system to an eternal cosine signal is the magnitude and phase of the system function evaluated on the unit circle.



$$H(e^{j\Omega}) = H(z)\big|_{z=e^{j\Omega}}$$

Finding Frequency Response with Vector Diagrams

$$H(z) = \frac{z - q_1}{z - p_1}$$



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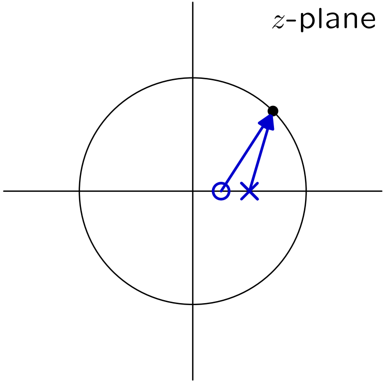

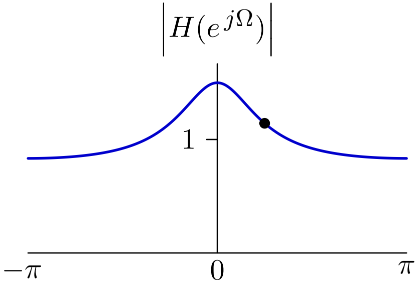

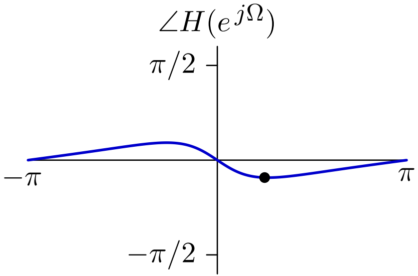

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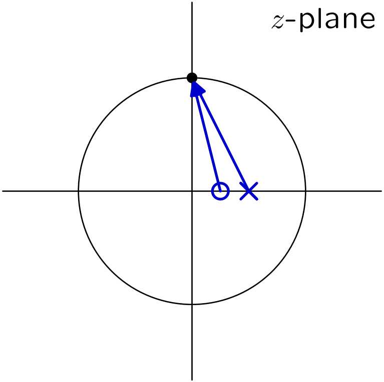

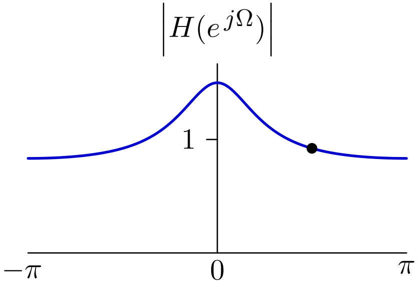

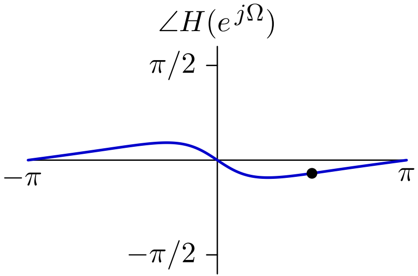

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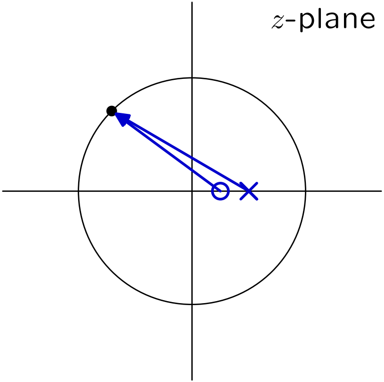

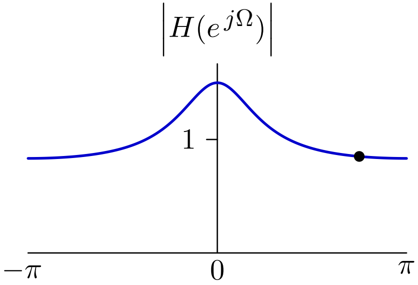

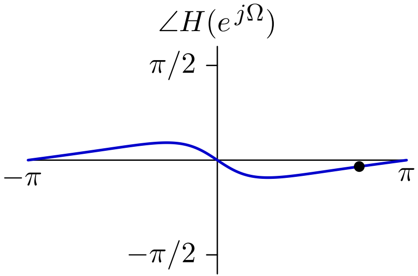

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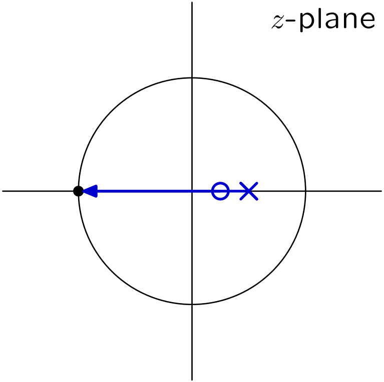

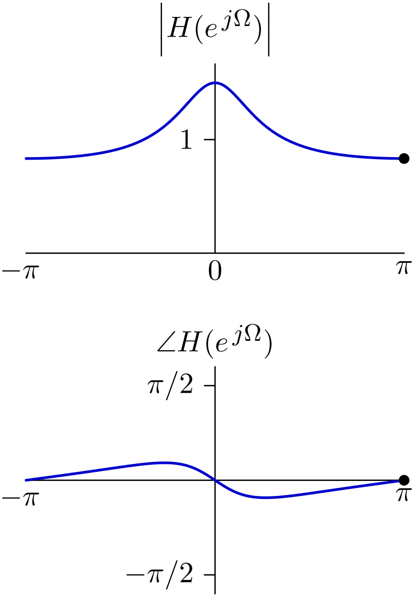

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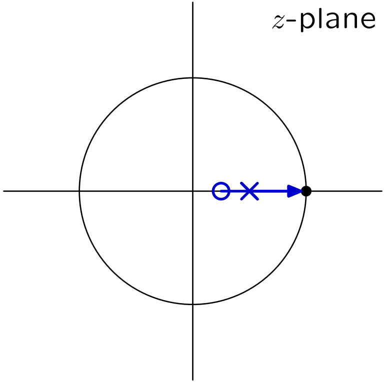

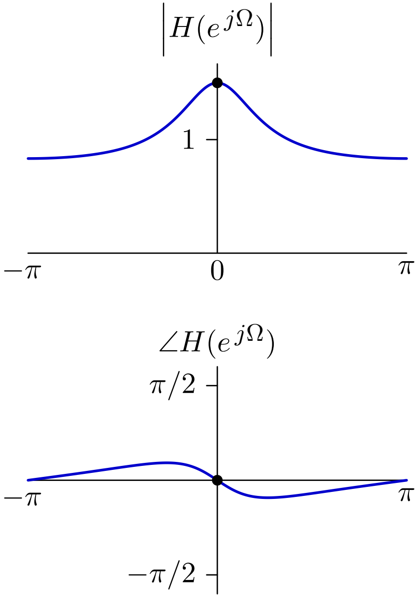

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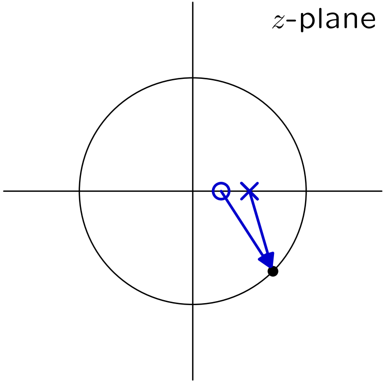

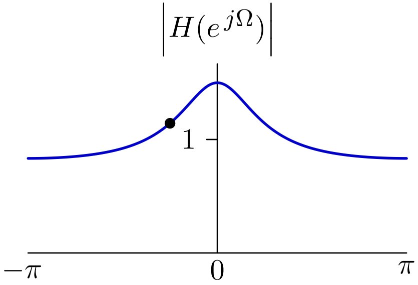

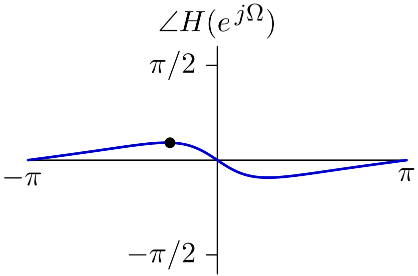

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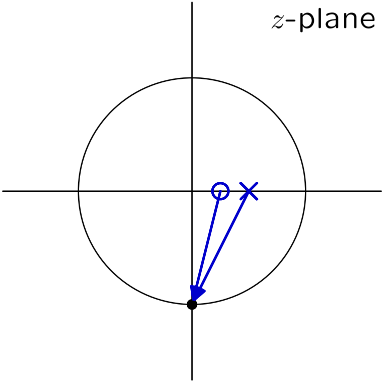

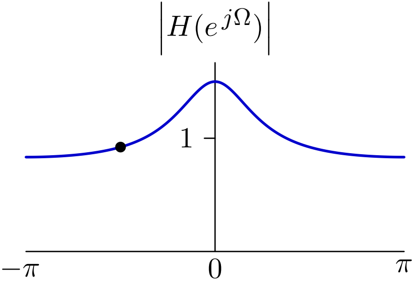

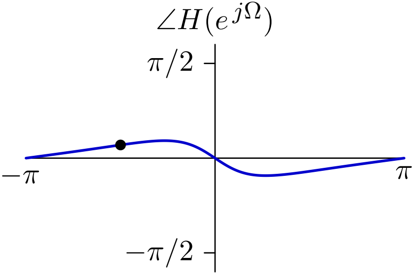

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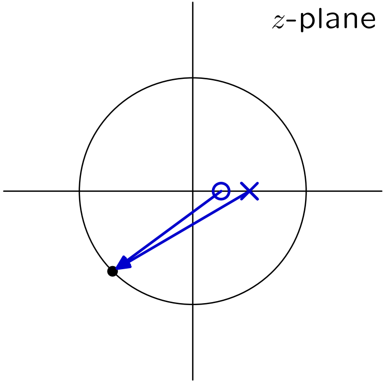

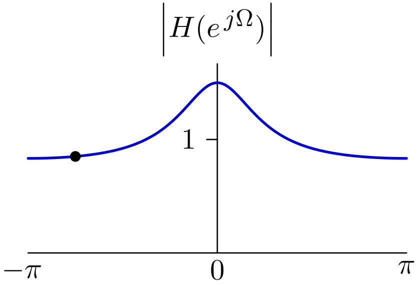

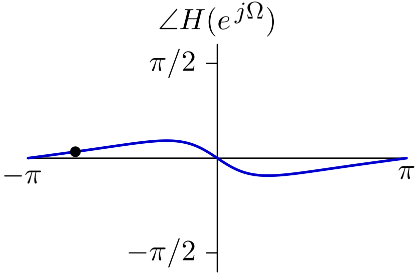

Finding Frequency Response with Vector Diagrams

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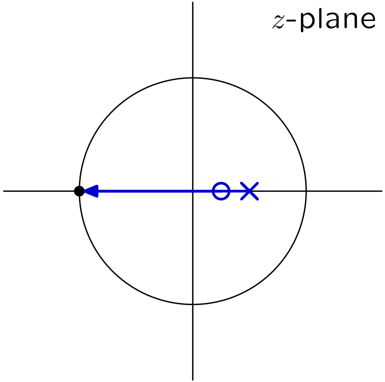
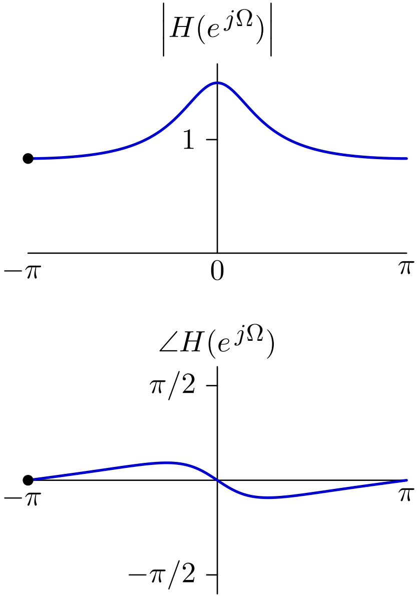
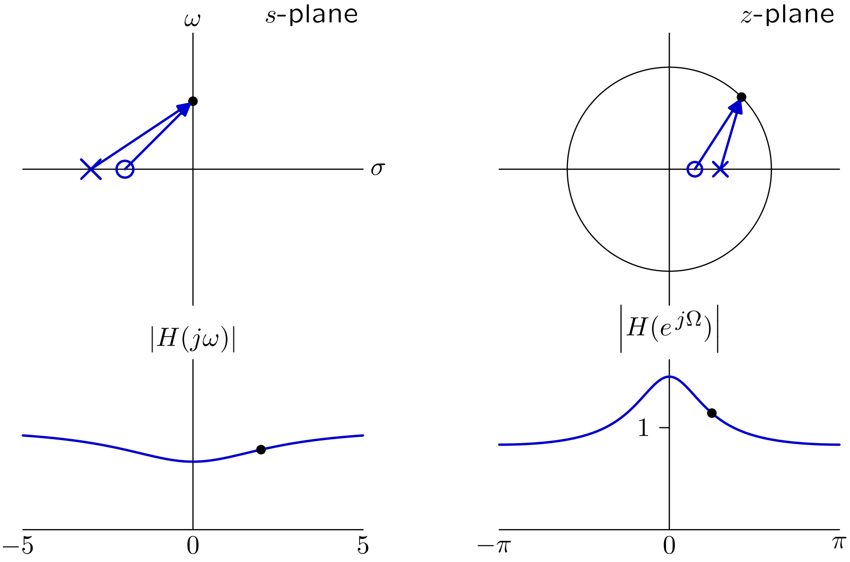




Comparision of CT and DT Frequency Responses

CT frequency response: $H(s)$ on the imaginary axis, i.e., $s = j\omega$.

DT frequency response: $H(z)$ on the unit circle, i.e., $z = e^{j\Omega}$.



DT Periodicity

DT frequency responses are periodic functions of Ω, with period 2π.

If $\Omega_2 = \Omega_1 + 2\pi k$ where k is an integer then

$$H(e^{j\Omega_2}) = H(e^{j(\Omega_1 + 2\pi k)}) = H(e^{j\Omega_1} e^{j2\pi k}) = H(e^{j\Omega_1})$$

The periodicity of $H(e^{j\Omega})$ results because $H(e^{j\Omega})$ is a function of $e^{j\Omega}$, which is itself periodic in Ω. Thus DT complex exponentials have many "aliases."

$$e^{j\Omega_2} = e^{j(\Omega_1 + 2\pi k)} = e^{j\Omega_1} e^{j2\pi k} = e^{j\Omega_1}$$

Because of this aliasing, there is a "highest" DT frequency: $\Omega = \pi$.

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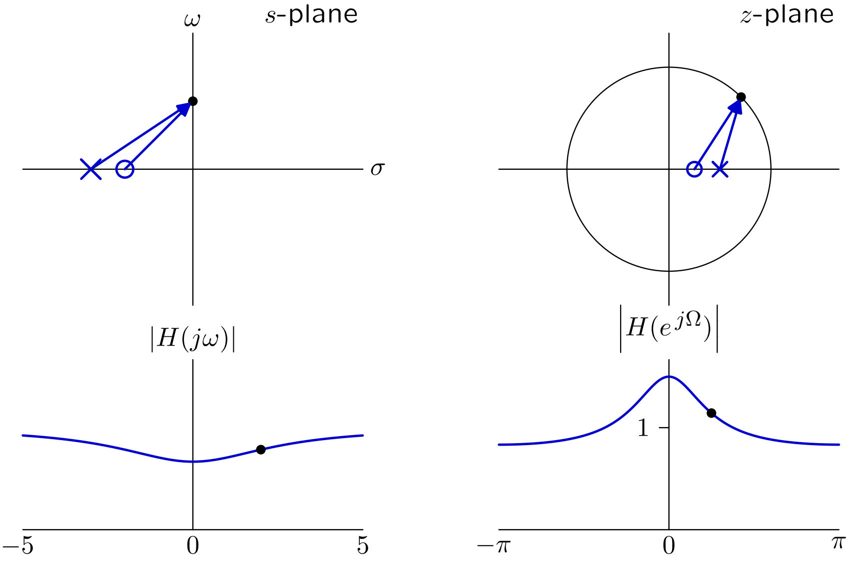


Check Yourself

Consider 3 CT signals:

$$x_1(t) = \cos(3000t) \quad ; \quad x_2(t) = \cos(4000t) \quad ; \quad x_3(t) = \cos(5000t)$$

Each of these is sampled so that

$$x_1[n] = x_1(nT) \quad ; \quad x_2[n] = x_2(nT) \quad ; \quad x_3[n] = x_3(nT)$$

where $T = 0.001$.

Which list goes from lowest to highest DT frequency?

0. $x_1[n]$ $x_2[n]$ $x_3[n]$
1. $x_1[n]$ $x_3[n]$ $x_2[n]$
2. $x_2[n]$ $x_1[n]$ $x_3[n]$
3. $x_2[n]$ $x_3[n]$ $x_1[n]$
4. $x_3[n]$ $x_1[n]$ $x_2[n]$
5. $x_3[n]$ $x_2[n]$ $x_1[n]$

Check Yourself

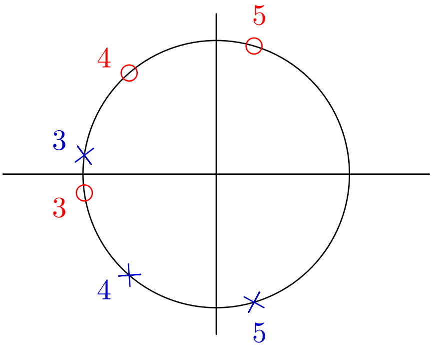
The discrete signals are

$$x_1[n] = \cos[3n]$$

$$x_2[n] = \cos[4n]$$

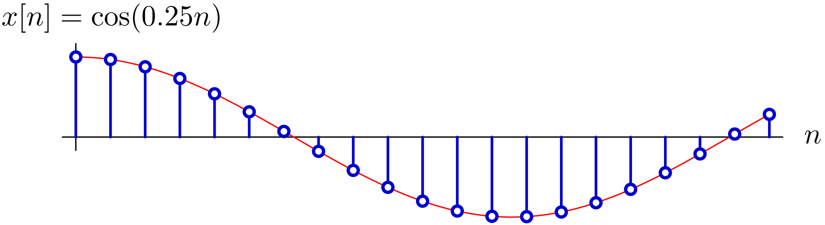
$$x_3[n] = \cos[5n]$$

and the corresponding discrete frequencies are $\Omega = 3, 4$ and 5, represented below with $\times$ marking $e^{j\Omega}$ and o marking $e^{-j\Omega}$).



Check Yourself

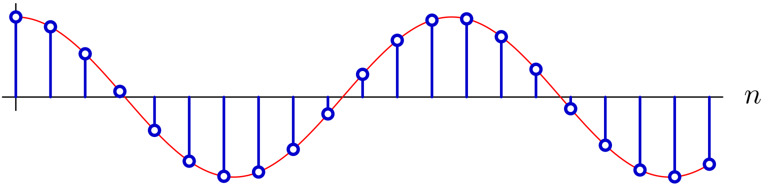
$\Omega = 0.25$



Check Yourself

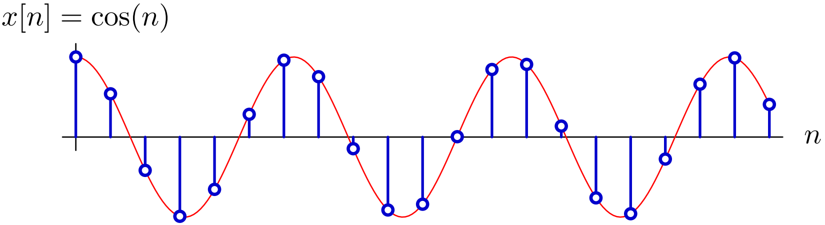
$\Omega = 0.5$

$x[n] = \cos(0.5n)$



Check Yourself

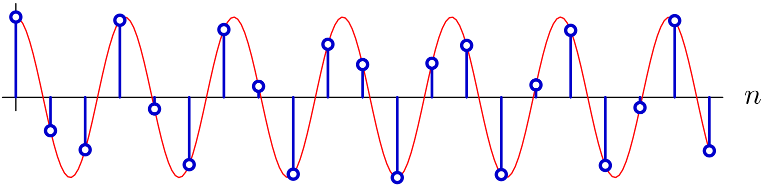
$\Omega = 1$



Check Yourself

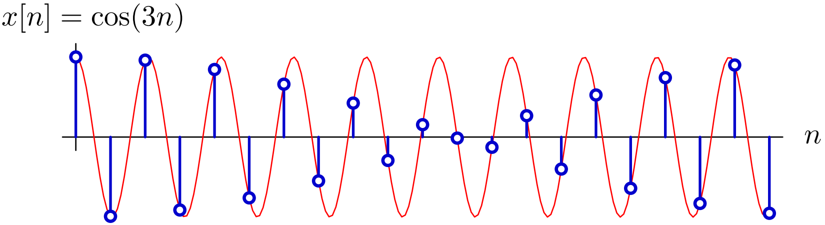
$\Omega = 2$

$x[n] = \cos(2n)$



Check Yourself

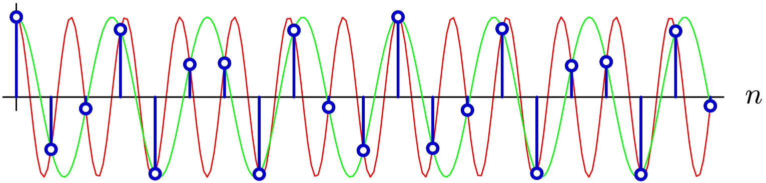
$\Omega = 3$



Check Yourself

$\Omega = 4$

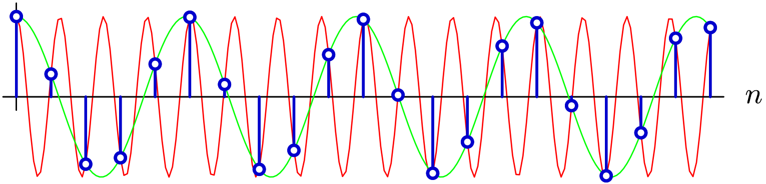
$$x[n] = \cos(4n) = \cos(2\pi - 4n) \approx \cos(2.283n)$$



Check Yourself

$\Omega = 5$

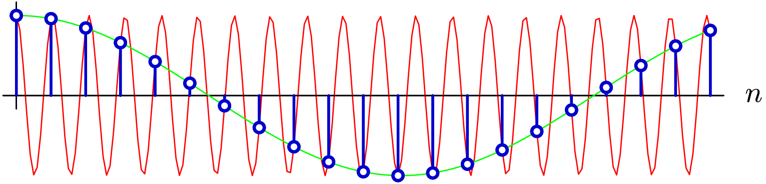
$$x[n] = \cos(5n) = \cos(2\pi - 5n) \approx \cos(1.283n)$$



Check Yourself

$\Omega = 6$

$$x[n] = \cos(6n) = \cos(2\pi - 6n) \approx \cos(0.283n)$$



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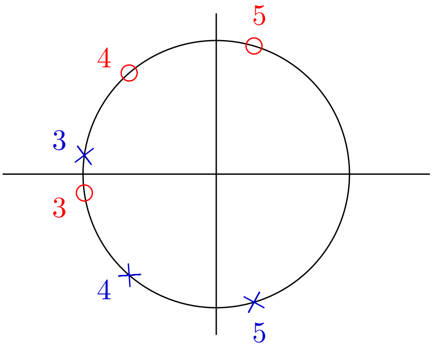
$$x_3[n] = \cos[5n]$$

and the corresponding discrete frequencies are $\Omega = 3, 4$ and 5, represented below with $\times$ marking $e^{j\Omega}$ and o marking $e^{-j\Omega}$).



Check Yourself

Consider 3 CT signals:

$$x_1(t) = \cos(3000t) \quad ; \quad x_2(t) = \cos(4000t) \quad ; \quad x_3(t) = \cos(5000t)$$

Each of these is sampled so that

$$x_1[n] = x_1(nT) \quad ; \quad x_2[n] = x_2(nT) \quad ; \quad x_3[n] = x_3(nT)$$

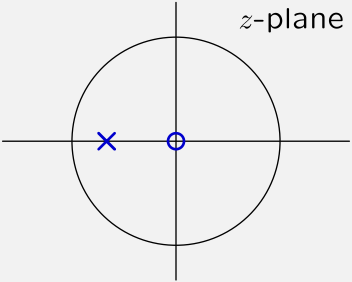
where $T = 0.001$.

Which list goes from lowest to highest DT frequency? 5

0. $x_1[n]$ $x_2[n]$ $x_3[n]$
1. $x_1[n]$ $x_3[n]$ $x_2[n]$
2. $x_2[n]$ $x_1[n]$ $x_3[n]$
3. $x_2[n]$ $x_3[n]$ $x_1[n]$
4. $x_3[n]$ $x_1[n]$ $x_2[n]$
5. $x_3[n]$ $x_2[n]$ $x_1[n]$

Check Yourself

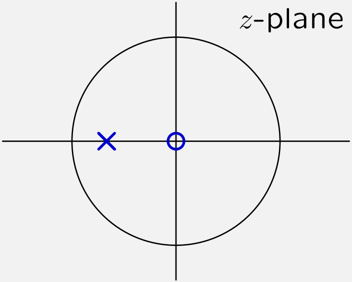
What kind of filtering corresponds to the following?



1. high pass
2. low pass
3. band pass
4. band stop (notch)
5. none of above

Check Yourself

What kind of filtering corresponds to the following? 1



1. high pass
2. low pass
3. band pass
4. band stop (notch)
5. none of above

DT Fourier Series

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

$$x[n] = \sum a_k e^{jk\Omega_0 n}$$

The period N of all harmonic components is the same (as in CT).

DT Fourier Series

There are (only) N distinct complex exponentials with period N.

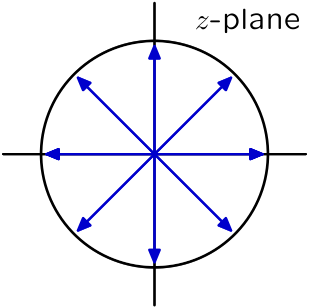
(There were an infinite number in CT!)

If $y[n] = e^{j\Omega n}$ is periodic in N then

$$y[n] = e^{j\Omega n} = y[n+N] = e^{j\Omega(n+N)} = e^{j\Omega n}e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1, and $e^{j\Omega}$ must be one of the N^{th} roots of 1.

Example: $N = 8$

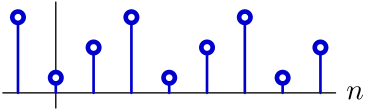


DT Fourier Series

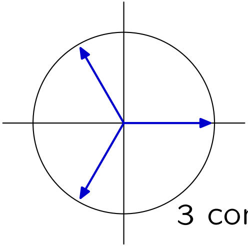
There are N distinct complex exponentials with period N.

These can be combined via Fourier series to produce periodic time signals with N independent samples.

Example: periodic in N=3

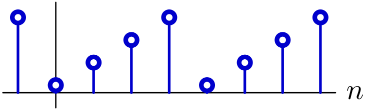


3 samples repeated in time

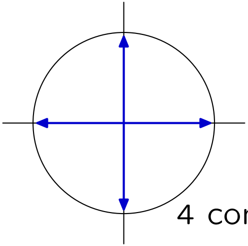


3 complex exponentials

Example: periodic in N=4



4 samples repeated in time



4 complex exponentials

DT Fourier Series

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

$$x[n] = x[n+N] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \quad ; \ \Omega_0 = \frac{2\pi}{N}$$

N equations (one for each point in time n) in N unknowns (a_k).

Example: $N = 4$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} e^{j\frac{2\pi}{N}0\cdot 0} & e^{j\frac{2\pi}{N}1\cdot 0} & e^{j\frac{2\pi}{N}2\cdot 0} & e^{j\frac{2\pi}{N}3\cdot 0} \\ e^{j\frac{2\pi}{N}0\cdot 1} & e^{j\frac{2\pi}{N}1\cdot 1} & e^{j\frac{2\pi}{N}2\cdot 1} & e^{j\frac{2\pi}{N}3\cdot 1} \\ e^{j\frac{2\pi}{N}0\cdot 2} & e^{j\frac{2\pi}{N}1\cdot 2} & e^{j\frac{2\pi}{N}2\cdot 2} & e^{j\frac{2\pi}{N}3\cdot 2} \\ e^{j\frac{2\pi}{N}0\cdot 3} & e^{j\frac{2\pi}{N}1\cdot 3} & e^{j\frac{2\pi}{N}2\cdot 3} & e^{j\frac{2\pi}{N}3\cdot 3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

DT Fourier Series

DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

$$x[n] = x[n+N] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} \quad ; \ \Omega_0 = \frac{2\pi}{N}$$

N equations (one for each point in time n) in N unknowns (a_k).

Example: $N = 4$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Orthogonality

DT harmonics are orthogonal to each other (as were CT harmonics).

$$\sum_{n=0}^{N-1} e^{j\Omega_0 kn} e^{-j\Omega_0 ln} = \sum_{n=0}^{N-1} e^{j\Omega_0 (k-l)n}$$

$$= \begin{cases} N & ;\ k = l \\ \frac{1-e^{j\Omega_0 (k-l)N}}{1-e^{j\Omega_0 (k-l)}} = \frac{1-e^{j\frac{2\pi}{N}(k-l)N}}{1-e^{j\frac{2\pi}{N}(k-l)}} = 0 & ;\ k \neq l \end{cases}$$

$$= N\delta[k-l]$$

Sifting

Use orthogonality property of harmonics to sift out FS coefficients.

Assume $x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}$

Multiply both sides by the complex conjugate of the l^{th} harmonic, and sum over time.

$$\sum_{n=0}^{N-1} x[n]e^{-jl\Omega_0 n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n} e^{-jl\Omega_0 n} = \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{jk\Omega_0 n} e^{-jl\Omega_0 n}$$

$$= \sum_{k=0}^{N-1} a_k N\delta[k-l] = Na_l$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$$

DT Fourier Series

Since both $x[n]$ and a_k are periodic in N, the sums can be taken over any N successive indices.

Notation. If $f[n]$ is periodic in N, then

$$\sum_{n=0}^{N-1} f[n] = \sum_{n=1}^{N} f[n] = \sum_{n=2}^{N+1} f[n] = \cdots = \sum_{n=<N>} f[n]$$

DT Fourier Series

$$a_k = a_{k+N} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\Omega_0 n} \quad ; \quad \Omega_0 = \frac{2\pi}{N} \qquad \text{(“analysis” equation)}$$

$$x[n] = x[n+N] = \sum_{k=<N>} a_k e^{jk\Omega_0 n} \qquad \text{(“synthesis” equation)}$$

DT Fourier Series

DT Fourier series have simple matrix interpretations.

$$x[n] = x[n+4] = \sum_{k=<4>} a_k e^{jk\Omega_0 n} = \sum_{k=<4>} a_k e^{jk\frac{2\pi}{4}n} = \sum_{k=<4>} a_k j^{kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$a_k = a_{k+4} = \frac{1}{4} \sum_{n=<4>} x[n] e^{-jk\Omega_0 n} = \frac{1}{4} \sum_{n=<4>} e^{-jk\frac{2\pi}{N}n} = \frac{1}{4} \sum_{n=<4>} x[n] j^{-kn}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

These matrices are inverses of each other.

Discrete-Time Frequency Representations

Similarities and differences between CT and DT.

DT frequency response

- vector diagrams (similar to CT)
- frequency response on unit circle in z-plane ($j\omega$ axis in CT)

DT Fourier series

- represent signal as sum of harmonics (similar to CT)
- finite number of periodic harmonics (unlike CT)
- finite sum (unlike CT)

The finite length of DT Fourier series make them especially useful for signal processing! (more on this next time)