DT Fourier Representations
Mid-term Examination #3

Wednesday, November 16, 7:30-9:30pm, Walker (50-340)

No recitations on the day of the exam.

Coverage: Lectures 1–18
           Recitations 1–16
           Homeworks 1–10

Homework 10 will not be collected or graded.
Solutions will be posted.

Closed book: 3 pages of notes \((8 \frac{1}{2} \times 11\) inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

Review session Monday at 3pm (36-112) and at open office hours.

Prior term midterm exams have been posted on the 6.003 website.

Conflict? Contact freeman@mit.edu before Friday, Nov. 11, 5pm.
The frequency response of a DT LTI system is the value of the system function evaluated on the unit circle.

\[ H(z) = \frac{1}{z^n} \]

\[ H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} \]

\[ |H(e^{j\Omega})| \cos \left( \Omega n + \angle H(e^{j\Omega}) \right) \]
Comparision of CT and DT Frequency Responses

CT frequency response: $H(s)$ on the imaginary axis, i.e., $s = j\omega$.

DT frequency response: $H(z)$ on the unit circle, i.e., $z = e^{j\Omega}$.
A system \( H(z) = \frac{1 - az}{z - a} \) has the following pole-zero diagram.

Classify this system as one of the following filter types.

1. high pass  
2. low pass  
3. band pass  
4. all pass  
5. band stop  
0. none of the above
Check Yourself

Classify the system ...

\[ H(z) = \frac{1 - az}{z - a} \]

Find the frequency response:

\[ H(e^{j\Omega}) = \frac{1 - ae^{j\Omega}}{e^{j\Omega} - a} = e^{j\Omega} \frac{e^{-j\Omega} - a}{e^{j\Omega} - a} \leftarrow \text{complex} \]

\[ \left| H(e^{j\Omega}) \right| = 1. \leftarrow \text{conjugates} \]

Because complex conjugates have equal magnitudes, \[ H(e^{j\Omega}) \] = 1.

→ all-pass filter
A system \( H(z) = \frac{1 - az}{z - a} \) has the following pole-zero diagram.

Classify this system as one of the following filter types.

1. high pass
2. low pass
3. band pass
4. all pass
5. band stop
0. none of the above
Effects of Phase

\[ H(z) = \frac{1 - az}{z - a} \]

\[ |H(e^{j\Omega})| \]

\[ \angle H(e^{j\Omega}) \]
Effects of Phase

\[ x[n] \rightarrow H(z) = \frac{1 - az}{z - a} \rightarrow y[n] \]
Effects of Phase

\[ x[n] \rightarrow H(z) = \frac{1 - az}{z - a} \rightarrow y[n] \]

- $x[n]$ to $y[n]$ through $H(z)$
- Graphs of $x[n]$ and $y[n]$ shown with $n$ on the horizontal axis.
Effects of Phase

\[ x[n] \rightarrow H(z) = \frac{1 - az}{z - a} \rightarrow y[n] \]

artificial speech synthesized by Robert Donovan
Effects of Phase

\[ x[n] \rightarrow ??? \rightarrow y[n] = x[-n] \]

artificial speech synthesized by Robert Donovan
Effects of Phase

$x[n] \rightarrow ??? \rightarrow y[n] = x[-n]$ 

How are the phases of $X$ and $Y$ related?
Effects of Phase

How are the phases of $X$ and $Y$ related?

$$a_k = \sum_n x[n] e^{-jk\Omega_0 n}$$

$$b_k = \sum_n x[-n] e^{-jk\Omega_0 n} = \sum_m x[m] e^{jk\Omega_0 m} = a_{-k}$$

Flipping $x[n]$ about $n = 0$ flips $a_k$ about $k = 0$.

Because $x[n]$ is real-valued, $a_k$ is conjugate symmetric: $a_{-k} = a_k^*$.

$$b_k = a_{-k} = a_k^* = |a_k| e^{-j\angle a_k}$$

The angles are negated at all frequencies.
**Review: Periodicity**

DT frequency responses are periodic functions of $\Omega$, with period $2\pi$.

If $\Omega_2 = \Omega_1 + 2\pi k$ where $k$ is an integer then

$$H(e^{j\Omega_2}) = H(e^{j(\Omega_1+2\pi k)}) = H(e^{j\Omega_1}e^{j2\pi k}) = H(e^{j\Omega_1})$$

The periodicity of $H(e^{j\Omega})$ results because $H(e^{j\Omega})$ is a function of $e^{j\Omega}$, which is itself periodic in $\Omega$. Thus DT complex exponentials have many “aliases.”

$$e^{j\Omega_2} = e^{j(\Omega_1+2\pi k)} = e^{j\Omega_1}e^{j2\pi k} = e^{j\Omega_1}$$

Because of this aliasing, there is a “highest” DT frequency: $\Omega = \pi$. 
There are (only) $N$ distinct complex exponentials with period $N$. (There were an infinite number in CT!)

If $y[n] = e^{j\Omega n}$ is periodic in $N$ then

$$y[n] = e^{j\Omega n} = y[n + N] = e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1, and $e^{j\Omega}$ must be one of the $N^{th}$ roots of 1.

Example: $N = 8$
DT Fourier series represent DT signals in terms of the amplitudes and phases of harmonic components.

**DT Fourier Series**

\[ a_k = a_{k+N} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j k \Omega_0 n} ; \quad \Omega_0 = \frac{2\pi}{N} \]  

(“analysis” equation)

\[ x[n] = x[n+N] = \sum_{k=\langle N \rangle} a_k e^{j k \Omega_0 n} \]  

(“synthesis” equation)
DT Fourier Series

DT Fourier series have simple matrix interpretations.

\[
x[n] = x[n+4] = \sum_{k=\langle 4 \rangle} a_k e^{j\Omega_0 n} = \sum_{k=\langle 4 \rangle} a_k e^{j\frac{2\pi}{4} n} = \sum_{k=\langle 4 \rangle} a_k j^{kn}
\]

\[
\begin{bmatrix}
  x[0] \\
  x[1] \\
  x[2] \\
  x[3]
\end{bmatrix}
= \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & j & -1 & -j \\
  1 & -1 & 1 & -1 \\
  1 & -j & -1 & j
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix}
\]

\[
a_k = a_{k+4} = \frac{1}{4} \sum_{n=\langle 4 \rangle} x[n] e^{-j\Omega_0 n} = \frac{1}{4} \sum_{n=\langle 4 \rangle} e^{-j\frac{2\pi}{4} n} = \frac{1}{4} \sum_{n=\langle 4 \rangle} x[n] j^{-kn}
\]

\[
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix}
= \frac{1}{4} \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & -j & -1 & j \\
  1 & -1 & 1 & -1 \\
  1 & j & -1 & -j
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[1] \\
  x[2] \\
  x[3]
\end{bmatrix}
\]

These matrices are inverses of each other.
Scaling

DT Fourier series are important computational tools. However, the DT Fourier series do not scale well with the length $N$.

$$a_k = a_{k+2} = \frac{1}{2} \sum_{n=\langle 2 \rangle} x[n] e^{-jk\Omega_0 n} = \frac{1}{2} \sum_{n=\langle 2 \rangle} e^{-jk\frac{2\pi}{2} n} = \frac{1}{2} \sum_{n=\langle 2 \rangle} x[n] (-1)^{-kn}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

$$a_k = a_{k+4} = \frac{1}{4} \sum_{n=\langle 4 \rangle} x[n] e^{-jk\Omega_0 n} = \frac{1}{4} \sum_{n=\langle 4 \rangle} e^{-jk\frac{2\pi}{4} n} = \frac{1}{4} \sum_{n=\langle 4 \rangle} x[n] j^{-kn}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Number of multiples increases as $N^2$. 
Fast Fourier “Transform”

Exploit structure of Fourier series to simplify its calculation.

Divide FS of length $2N$ into two of length $N$ (divide and conquer).

Matrix formulation of 8-point FS:

$$
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6 \\
c_7 
\end{bmatrix} =
\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 
\end{bmatrix} \begin{bmatrix}
x[0] \\
x[1] \\
x[2] \\
x[3] \\
x[4] \\
x[5] \\
x[6] \\
x[7] 
\end{bmatrix}
$$

where $W_N = e^{-j\frac{2\pi}{N}}$

$8 \times 8 = 64$ multiplications
FFT

Divide into two 4-point series (divide and conquer).

Even-numbered entries in $x[n]$:  
\[
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix} =
\begin{bmatrix}
W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
W_4^0 & W_4^3 & W_4^2 & W_4^1 \\
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[2] \\
x[4] \\
x[6] \\
\end{bmatrix}
\]

Odd-numbered entries in $x[n]$:  
\[
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix} =
\begin{bmatrix}
W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
W_4^0 & W_4^3 & W_4^2 & W_4^1 \\
\end{bmatrix}
\begin{bmatrix}
x[1] \\
x[3] \\
x[5] \\
x[7] \\
\end{bmatrix}
\]

Sum of multiplications $= 2 \times (4 \times 4) = 32$: fewer than the previous 64.
Break the original 8-point DTFS coefficients $c_k$ into two parts:

$$c_k = d_k + e_k$$

where $d_k$ comes from the even-numbered $x[n]$ (e.g., $a_k$) and $e_k$ comes from the odd-numbered $x[n]$ (e.g., $b_k$)
The 4-point DTFS coefficients $a_k$ of the even-numbered $x[n]$ contribute to the 8-point DTFS coefficients $d_k$:

$$
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= \begin{bmatrix}
W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
W_4^0 & W_4^3 & W_4^2 & W_4^1
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[2] \\
x[4] \\
x[6]
\end{bmatrix}
= \begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^4 & W_8^0 & W_8^2 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[2] \\
x[4] \\
x[6]
\end{bmatrix}
$$
The 4-point DTFS coefficients $a_k$ of the even-numbered $x[n]$ contribute to the 8-point DTFS coefficients $d_k$:

$$
\begin{align*}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
\end{bmatrix}
  &=
\begin{bmatrix}
  W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
  W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
  W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
  W_4^0 & W_4^3 & W_4^2 & W_4^1 \\
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[2] \\
  x[4] \\
  x[6] \\
\end{bmatrix}
  &=
\begin{bmatrix}
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
  W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
  W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[2] \\
  x[4] \\
  x[6] \\
\end{bmatrix}
\end{align*}
$$
The 4-point DTFS coefficients $a_k$ of the even-numbered $x[n]$ contribute to the 8-point DTFS coefficients $d_k$:
The 4-point DTFS coefficients $a_k$ of the even-numbered $x[n]$:

\[
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} =
\begin{bmatrix}
  W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
  W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
  W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
  W_4^0 & W_4^3 & W_4^2 & W_4^1
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[2] \\
  x[4] \\
  x[6]
\end{bmatrix} =
\begin{bmatrix}
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
  W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
  W_8^0 & W_8^6 & W_8^4 & W_8^2
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[2] \\
  x[4] \\
  x[6]
\end{bmatrix}
\]

contribute to the 8-point DTFS coefficients $d_k$:

\[
\begin{bmatrix}
  d_0 \\
  d_1 \\
  d_2 \\
  d_3 \\
  d_4 \\
  d_5 \\
  d_6 \\
  d_7
\end{bmatrix} =
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_0 \\
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} =
\begin{bmatrix}
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
  W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
  W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
  W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
  W_8^0 & W_8^6 & W_8^4 & W_8^2
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[2] \\
  x[4] \\
  x[6] \\
  x[0] \\
  x[2] \\
  x[4] \\
  x[6]
\end{bmatrix}
\]
The $e_k$ components result from the odd-number entries in $x[n]$. 

$$
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3
\end{bmatrix}
= 
\begin{bmatrix}
W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
W_4^0 & W_4^3 & W_4^2 & W_4^1
\end{bmatrix}
\begin{bmatrix}
x[1] \\
x[3] \\
x[5] \\
x[7]
\end{bmatrix}
= 
\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2
\end{bmatrix}
\begin{bmatrix}
x[1] \\
x[3] \\
x[5] \\
x[7]
\end{bmatrix}
$$

$$
\begin{bmatrix}
e_0 \\
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6 \\
e_7
\end{bmatrix}
= 
\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^4 & W_8^0 & W_8^4 & W_8^4 \\
W_8^0 & W_8^5 & W_8^0 & W_8^2 & W_8^7 & W_8^1 & W_8^6 & W_8^3 \\
W_8^0 & W_8^6 & W_8^4 & W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 \\
W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[1] \\
x[2] \\
x[3] \\
x[4] \\
x[5] \\
x[6] \\
x[7]
\end{bmatrix}
$$
The $e_k$ components result from the odd-number entries in $x[n]$. 

\[
\begin{bmatrix}
    b_0 \\
    b_1 \\
    b_2 \\
    b_3 \\
\end{bmatrix} = \begin{bmatrix}
    W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
    W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
    W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
    W_4^0 & W_4^3 & W_4^2 & W_4^1 \\
\end{bmatrix} \begin{bmatrix}
    x[1] \\
    x[3] \\
    x[5] \\
    x[7] \\
\end{bmatrix} = \begin{bmatrix}
    W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
    W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
    W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
    W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
\end{bmatrix} \begin{bmatrix}
    x[1] \\
    x[3] \\
    x[5] \\
    x[7] \\
\end{bmatrix}
\]
The $e_k$ components result from the odd-number entries in $x[n]$.

$$
\begin{bmatrix}
  b_0 \\
  b_1 \\
  b_2 \\
  b_3 
\end{bmatrix}
= \begin{bmatrix}
  W_4^0 & W_4^0 & W_4^0 & W_4^0 \\
  W_4^0 & W_4^1 & W_4^2 & W_4^3 \\
  W_4^0 & W_4^2 & W_4^0 & W_4^2 \\
  W_4^0 & W_4^3 & W_4^2 & W_4^1 
\end{bmatrix}
\begin{bmatrix}
  x[1] \\
  x[3] \\
  x[5] \\
  x[7] 
\end{bmatrix}
= \begin{bmatrix}
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
  W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
  W_8^0 & W_8^6 & W_8^4 & W_8^2 
\end{bmatrix}
\begin{bmatrix}
  x[1] \\
  x[3] \\
  x[5] \\
  x[7] 
\end{bmatrix}
$$

$$
\begin{bmatrix}
  e_0 \\
  e_1 \\
  e_2 \\
  e_3 \\
  e_4 \\
  e_5 \\
  e_6 \\
  e_7 
\end{bmatrix}
= \begin{bmatrix}
  W_8^0 b_0 \\
  W_8^1 b_1 \\
  W_8^2 b_2 \\
  W_8^3 b_3 \\
  W_8^4 b_0 \\
  W_8^5 b_1 \\
  W_8^6 b_2 \\
  W_8^7 b_3 
\end{bmatrix}
= \begin{bmatrix}
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^1 & W_8^3 & W_8^5 & W_8^7 \\
  W_8^2 & W_8^6 & W_8^2 & W_8^6 \\
  W_8^3 & W_8^1 & W_8^7 & W_8^5 \\
  W_8^4 & W_8^4 & W_8^4 & W_8^4 \\
  W_8^5 & W_8^7 & W_8^1 & W_8^3 \\
  W_8^6 & W_8^2 & W_8^6 & W_8^2 \\
  W_8^7 & W_8^5 & W_8^3 & W_8^1 
\end{bmatrix}
\begin{bmatrix}
  x[1] \\
  x[3] \\
  x[5] \\
  x[7] 
\end{bmatrix}
$$
The $e_k$ components result from the odd-number entries in $x[n]$.

\[
\begin{bmatrix}
  b_0 \\
  b_1 \\
  b_2 \\
  b_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  W_4^0 & W_4^0 & W_4^0 \\
  W_4^0 & W_4^1 & W_4^2 \\
  W_4^0 & W_4^2 & W_4^0 \\
  W_4^0 & W_4^3 & W_4^1 \\
\end{bmatrix}
\begin{bmatrix}
  x[1] \\
  x[3] \\
  x[5] \\
  x[7] \\
\end{bmatrix}
= 
\begin{bmatrix}
  W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
  W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
  W_8^0 & W_8^4 & W_8^0 & W_8^2 \\
  W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
\end{bmatrix}
\begin{bmatrix}
  W_8^0 \\
  W_8^1 \\
  W_8^2 \\
  W_8^3 \\
\end{bmatrix}
\begin{bmatrix}
  e_0 \\
  e_1 \\
  e_2 \\
  e_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  W_8^0 \\
  W_8^1 \\
  W_8^2 \\
  W_8^3 \\
\end{bmatrix}
\begin{bmatrix}
  W_8^0 \\
  W_8^1 \\
  W_8^2 \\
  W_8^3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  W_8^0 \\
  W_8^1 \\
  W_8^2 \\
  W_8^3 \\
\end{bmatrix}
\begin{bmatrix}
  x[1] \\
  x[3] \\
  x[5] \\
  x[7] \\
\end{bmatrix}
\]
Combine $a_k$ and $b_k$ to get $c_k$.

$$
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6 \\
c_7 \\
\end{bmatrix}
= \begin{bmatrix}
d_0 + e_0 \\
d_1 + e_1 \\
d_2 + e_2 \\
d_3 + e_3 \\
d_4 + e_4 \\
d_5 + e_5 \\
d_6 + e_6 \\
d_7 + e_7 \\
\end{bmatrix}
= \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
+ \begin{bmatrix}
W_8^0 b_0 \\
W_8^1 b_1 \\
W_8^2 b_2 \\
W_8^3 b_3 \\
\end{bmatrix}
$$

FFT procedure:
- compute $a_k$ and $b_k$: $2 \times (4 \times 4) = 32$ multiplies
- combine $c_k = a_k + W_8^k b_k$: 8 multiplies
- total 40 multiplies: fewer than the original $8 \times 8 = 64$ multiplies
Scaling of FFT algorithm

How does the new algorithm scale?

Let $M(N) =$ number of multiplies to perform an $N$ point FFT.

- $M(1) = 0$
- $M(2) = 2M(1) + 2 = 2$
- $M(4) = 2M(2) + 4 = 2 \times 4$
- $M(8) = 2M(4) + 8 = 3 \times 8$
- $M(16) = 2M(8) + 16 = 4 \times 16$
- $M(32) = 2M(16) + 32 = 5 \times 32$
- $M(64) = 2M(32) + 64 = 6 \times 64$
- $M(128) = 2M(64) + 128 = 7 \times 128$

\[ \ldots \]

- $M(N) = (\log_2 N) \times N$

Significantly smaller than $N^2$ for $N$ large.
Fourier Transform: Generalize to Aperiodic Signals

An aperiodic signal can be thought of as periodic with infinite period.
An aperiodic signal can be thought of as periodic with infinite period. Let $x[n]$ represent an aperiodic signal DT signal.

"Periodic extension": $x_N[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$

Then $x[n] = \lim_{N \to \infty} x_N[n]$. 
Represent $x_N[n]$ by its Fourier series.

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x_N[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \frac{\sin \left( N_1 + \frac{1}{2} \right) \Omega}{\sin \frac{1}{2} \Omega}$$

$$\Omega_0 = \frac{2\pi}{N} \quad \Omega = k\Omega_0 = k\frac{2\pi}{N}$$
Fourier Transform

Doubling period doubles # of harmonics in given frequency interval.

\[ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x_N[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \frac{\sin \left( N_1 + \frac{1}{2} \right) \Omega}{\sin \frac{1}{2} \Omega} \]

\[ \Omega_0 = \frac{2\pi}{N} \]

\[ \Omega = k \Omega_0 = k \frac{2\pi}{N} \]
Fourier Transform

As $N \rightarrow \infty$, discrete harmonic amplitudes $\rightarrow$ a continuum $E(\Omega)$.

$$a_k = \frac{1}{N} \sum_{N} x_N[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \frac{\sin \left( N_1 + \frac{1}{2} \right) \Omega}{\sin \frac{1}{2} \Omega}$$

$$Na_k = \sin \frac{3}{2} \Omega$$

$$\Omega_0 = \frac{2\pi}{N}$$

$$\Omega = k \Omega_0 = k\frac{2\pi}{N}$$

$$Na_k = \sum_{n=<N>} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=<N>} x[n] e^{-j\Omega n} = E(\Omega)$$
Fourier Transform

As $N \to \infty$, synthesis sum $\to$ integral.

$$x_N[n]$$

$$N a_k = \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N} Kn} = \sum_{n=\langle N \rangle} x[n] e^{-j\Omega n} = E(\Omega)$$

$$x[n] = \sum_{k=\langle N \rangle} \left( \frac{1}{N} E(\Omega) e^{j\frac{2\pi}{N} Kn} \right) a_k = \sum_{k=\langle N \rangle} \frac{\Omega_0}{2\pi} E(\Omega) e^{j\Omega n} \to \frac{1}{2\pi} \int_{2\pi} E(\Omega) e^{j\Omega n} d\Omega$$
Fourier Transform

Replacing $E(\Omega)$ by $X(e^{j\Omega})$ yields the DT Fourier transform relations.

\[ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad \text{("analysis" equation)} \]

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \quad \text{("synthesis" equation)} \]
Relation between Fourier and Z Transforms

If the Z transform of a signal exists and if the ROC includes the unit circle, then the Fourier transform is equal to the Z transform evaluated on the unit circle.

Z transform:

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

DT Fourier transform:

\[ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = H(z) \bigg|_{z=e^{j\Omega}} \]
Relation between Fourier and Z Transforms

Fourier transform “inherits” properties of Z transform.

<table>
<thead>
<tr>
<th>Property</th>
<th>$x[n]$</th>
<th>$X(z)$</th>
<th>$X(e^{j\Omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$ax_1[n] + bx_2[n]$</td>
<td>$aX_1(s) + bX_2(s)$</td>
<td>$aX_1(e^{j\Omega}) + bX_2(e^{j\Omega})$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$x[n - n_0]$</td>
<td>$z^{-n_0}X(z)$</td>
<td>$e^{-j\Omega n_0}X(e^{j\Omega})$</td>
</tr>
<tr>
<td>Multiply by $n$</td>
<td>$nx[n]$</td>
<td>$-z \frac{d}{dz}X(z)$</td>
<td>$j \frac{d}{d\Omega}X(e^{j\Omega})$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$(x_1 * x_2)[n]$</td>
<td>$X_1(z) \times X_2(z)$</td>
<td>$X_1(e^{j\Omega}) \times X_2(e^{j\Omega})$</td>
</tr>
</tbody>
</table>
DT Fourier Series of Images

Magnitude

Angle
DT Fourier Series of Images

Magnitude

Uniform Angle
DT Fourier Series of Images

Uniform Magnitude

Angle
DT Fourier Series of Images

Different Magnitude

Angle
DT Fourier Series of Images

Magnitude

Angle
DT Fourier Series of Images

Magnitude

Angle

Magnitude

Angle
DT Fourier Series of Images
Fourier Representations: Summary

Thinking about signals by their frequency content and systems as filters has a large number of practical applications.