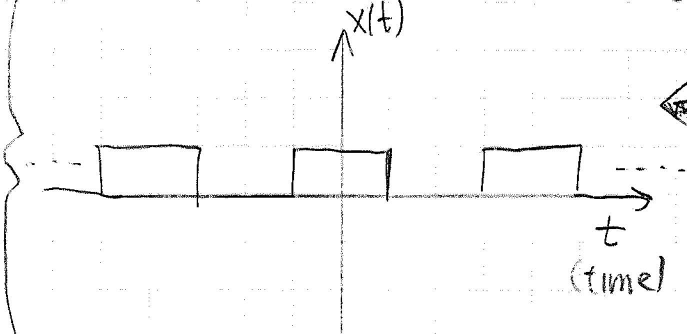
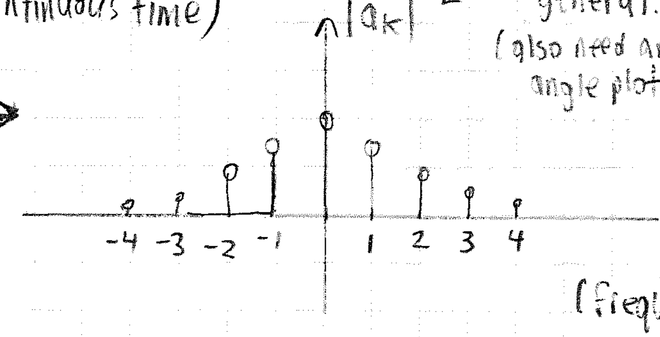
G.003 Review Quiz #3

Phillip Nadeau

WHAT IS FOURIER (series/transforms)?

Lets look @ the Fourier Series (continuous time)

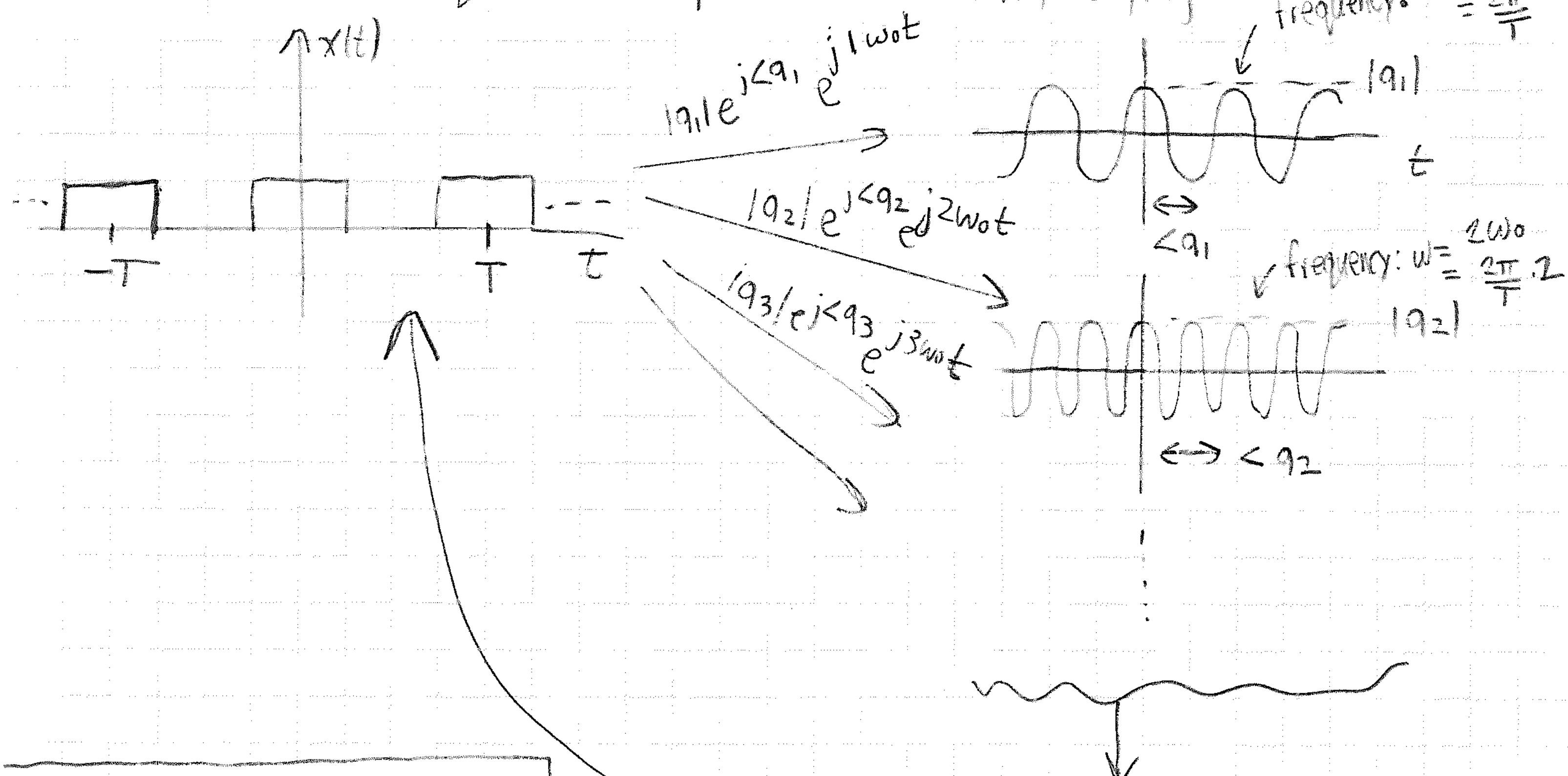




complex in general! (also need an angle plot $\angle a_k$)

PERMITS CONVERSION OF A SIGNAL BETWEEN THE TIME & FREQ. DOMAINS! (with no loss of information!)

What does the freq. domain representation really saying?



If you add these up, you will get back your original signal!

But not all time domain signals are continuous & periodic!

Hence we need different versions of the Fourier Series relationship/idea!

TIME DOMAIN	FREQ. DOMAIN	TOOL	ANALYSIS (time → freq)	SYNTHESIS (freq → time)
continuous/ periodic	aperiodic/ discrete	CTFS	$a_k = \frac{1}{T}\int_T x(t) e^{-j\omega_0 k t} dt$	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$
continuous/ aperiodic	aperiodic/ continuous	CTFT	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
discrete/ periodic	periodic/ discrete	DTFS	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-j\Omega_0 k n}$	$x[n] = \sum_{k=\langle N\rangle} a_k e^{j\Omega_0 k n}$
discrete/ aperiodic	periodic/ continuous	DTFT	$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega}) \cdot e^{j\Omega n} d\Omega$

MANY OBSERVATIONS ARE POSSIBLE!

TD Continuous ⟷ FD aperiodic
TD Discrete ⟷ FD periodic

TD periodic ⟷ FD discrete
TD aperiodic ⟷ FD continuous

continuous ⟷ integral
discrete ⟷ summation

periodic ⟷ finite limits
aperiodic ⟷ infinite limits

analysis:
basis fn's have minus sign

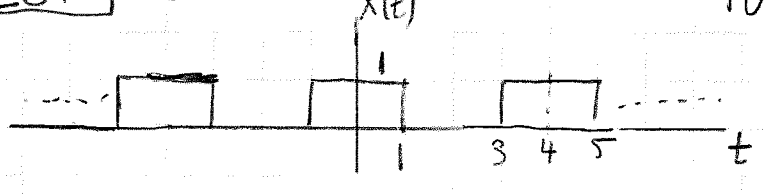
synthesis:
basis fn's have plus sign

	TIME VARIABLE		FREQUENCY VARIABLE
CTFS	t		K
	periodic in T	$\longleftrightarrow$	discrete interval: $\omega_0 = \frac{2\pi}{T}$ ↳ fundamental harmonic
CTFT	t		ω
DTFS	n		K
	periodic in N	$\longleftrightarrow$	discrete interval: $\Omega_0 = \frac{2\pi}{N}$ ↳ fundamental harmonic
	discrete interval: T_s ↳ sampling time	$\longleftrightarrow$	periodic in 2π
DTFT	n		Ω $(= \omega \cdot T_s)$
	discrete interval: T_s ↳ sampling time	$\longleftrightarrow$	periodic in 2π

CAN DIRECTLY COMPUTE FREQUENCY DOMAIN USING ANALYSIS FORMULA

CAN DIRECTLY COMPUTE TIME DOMAIN USING SYNTHESIS FORMULA

$\boxed{\text{EG\#1}}$ Calc. freq domain of: TD: continuos/periodic $\longrightarrow$ CTFS



$$a_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 k t}\,dt, \quad \omega_0 = \frac{2\pi}{T}$$

$T = 4$

$$a_k = \frac{1}{4}\int_{-1}^{1} 1\, e^{-j\frac{2\pi}{4}kt}$$
$$= \frac{-1}{j2\pi k}\left[e^{-j\frac{2\pi}{4}k} - e^{j\frac{2\pi}{4}k}\right]$$
$$= \frac{1}{\pi k}\left[\sin\left(\frac{2\pi}{4}k\right)\right]$$

↓

FD: aperiodic
discrete

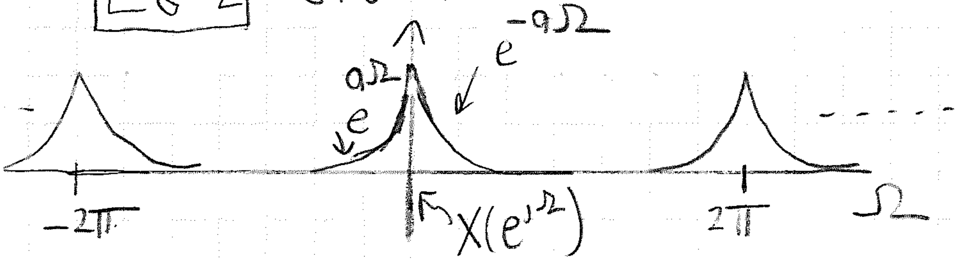
what happens @ $k=0$? (eg. DC?)

1) L'hopital's rule:
$$a_0 = \frac{2\pi/4}{\pi} = 1/2$$

2) small angle approx.
$$a_0 = \frac{2\pi/4 \cdot \cancel{k}}{\pi \cdot \cancel{k}} = 1/2$$

3) plug into original analysis eq.
$$a_0 = \frac{1}{4}\int_{-1}^{1} 1\, e^{-jk\omega_0 0}\, dt$$
$$= \frac{1}{4}[t]_{-1}^{1} = 1/2$$

$\boxed{\text{EG 2}}$ Calc. time domain of:

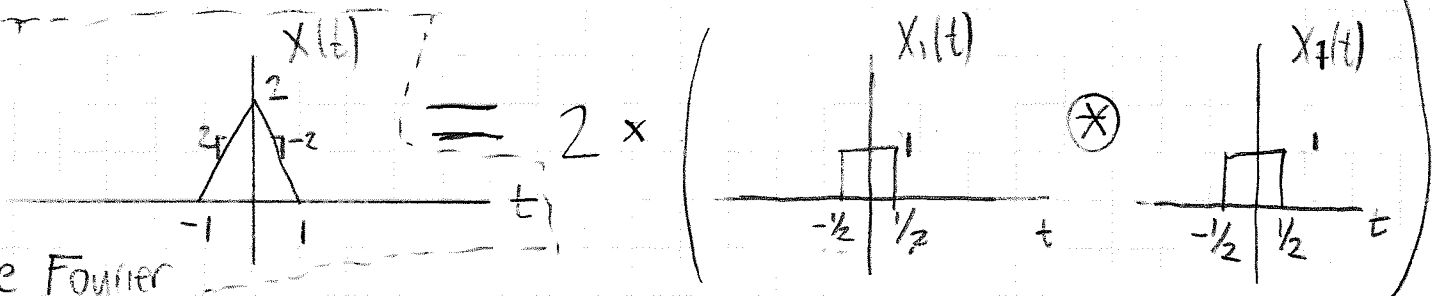


FD: periodic/ continuous → DTFT

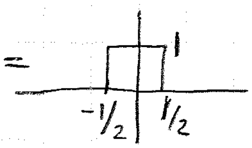
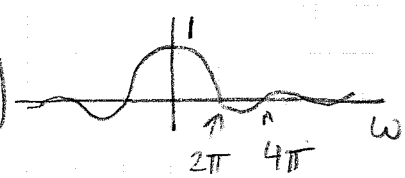
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{a\Omega}e^{j\Omega n}\,d\Omega + \int_{0}^{\pi} e^{-a\Omega}e^{j\Omega n}\,d\Omega\right]$$
$$= \frac{1}{2\pi}\frac{1}{a+jn}\left[e^{(a+jn)0} - \underbrace{e^{-(a+jn)\pi}}_{\text{small}}\right] + \frac{1}{2\pi}\frac{1}{-a+jn}\left[\underbrace{e^{(-a+jn)\pi}}_{\text{small}} - e^{(-a+jn)0}\right]$$
$$= \frac{1}{2\pi}\left[\frac{1}{a+jn} + \frac{1}{+a-jn}\right]$$
$$x[n] = \frac{1}{2\pi}\left[\frac{a}{a^2+n^2}\right]$$
→ TD: discrete
aperiodic

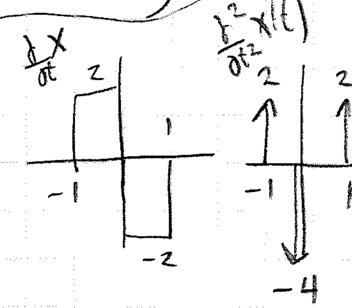
$\boxed{\text{CAN ALSO COMPUTE SERIES/TRANSFORMS GIVEN A BASIC SERIES/TRANSFORM AND SOME SIMPLE PROPERTIES}}$

$\boxed{\text{EG 3}}$



Calculate Fourier Transform of x(t)

time	frequency
$x_1(t) =$ 	$X_1(jw)$  $\dfrac{2\sin(w/2)}{w}$
$x(t) = 2 \cdot x_1(t) \circledast x_2(t)$	$X(jw) = 2 \cdot X_1(jw) \cdot X_2(jw)$
	$= 8\,\dfrac{\sin^2(w/2)}{w^2} = \dfrac{4}{w^2}(1-\cos(w))$
	FD: aperiodic/continuous



<u>Alternate method</u>

list of properties & transforms

time	freq
$\frac{d}{dt}x(t)$	$jw\,X(jw)$
$\frac{d^2}{dt^2}x(t)$	$-w^2 \cdot X(jw)$
$\delta(t)$	1
$y(t-t_0)$	$Y(jw)e^{-jwt_0}$
$\delta(t-t_0)$	$1 \cdot e^{-jwt_0}$
$\frac{d^2}{dt^2}x(t) = 2\delta(t+1) - 4\delta(t) + 2\delta(t-1)$	$2e^{jw} - 4 + 2e^{-jw} = -w^2 X(jw)$

transform we are looking for

$$X(jw) = \frac{4}{w^2}(1-\cos(w))$$

LINEAR TIME INVARIENT SYSTEMS

complex scaling factors

$$\text{CT} \quad x(t) = re^{j\omega t} \longrightarrow \boxed{H(s)} \longrightarrow y(t) = re^{j\omega t} \cdot H(j\omega)$$

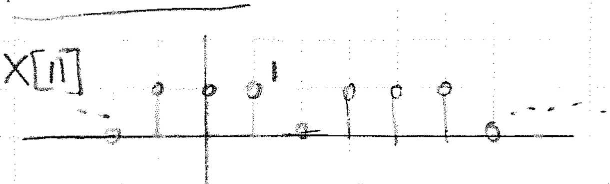
$$\text{DT} \quad x[n] = r\left(e^{j\Omega}\right)^n \longrightarrow \boxed{H(z)} \longrightarrow y[n] = r\left(e^{j\Omega}\right)^n \cdot H(e^{j\Omega})$$

INPUT SIGNAL	RELEVANT SYS FUNCTION	EVALUATE SYS FUNCTION @	INPUT SIGNAL CONV. TO FREQ. DOMAIN
continuos/ periodic	Laplace	$s = jk\omega_0$; $\omega_0 = \frac{2\pi}{T}$ T=period	CTFS
continuous/ aperiodic	Laplace	$s = j\omega$	CTFT
discrete/ periodic	Z-transform	$z = e^{jk\Omega_0}$; $\Omega_0 = \frac{2\pi}{N}$ N = period	DTFS
discrete/ aperiodic	Z-transform	$z = e^{j\Omega}$	DTFT

$\boxed{\text{EG4}}$ Given the periodic signal $x[n]$ & the system unit sample response $h[n]$, calculate the DTFS coefficients of the output $y[n]$

① INPUT SIGNAL

$x[n]$



$$a_k = \frac{1}{4}\sum_{4} x[n]e^{-j\Omega_0 kn} \qquad \Omega_0 = \frac{2\pi}{4} = \pi/2$$

$$= \frac{1}{4}\left(1 + 1e^{-j\pi/2 k} + 1e^{j\pi/2 k}\right)$$

$$= \frac{1}{4}\left(1 + 2\cos(\pi/2\, k)\right)$$

② SYSTEM

$$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n]$$

unit sample response

$$h[n] = \left(\frac{1}{2}\right)^n \left[u[n] - u[n-1]\right]$$

Sys function:

$$H(z) = \frac{z}{z+1/2} - \frac{1}{2}z^{-1}\frac{z}{z+1/2}$$

$$H(z) = \frac{z-1/2}{z+1/2}$$

③ EVALUATE SYS FUNCTION @

$z = e^{j\Omega_0 k}, \quad \Omega_0 = \frac{2\pi}{N},$

$$H(e^{j\Omega_0 k}) = \frac{e^{j\Omega_0 k} - 1/2}{e^{j\Omega_0 k} + 1/2}$$

④ FOURIER SERIES OF OUTPUT

$$b_k = a_k \cdot H(e^{j\Omega_0 k})$$

$$b_k = \frac{1}{4}\left(1 + 2\cos\left(\frac{\pi}{2}k\right)\right) \cdot \left(\frac{e^{j\Omega_0 k} - 1/2}{e^{j\Omega_0 k} + 1/2}\right)$$

DT FREQUENCY RESPONSE

- ↳ uses z-transform for the system function
- ↳ just like CT-frequency response but the system function is evaluated on the unit circle

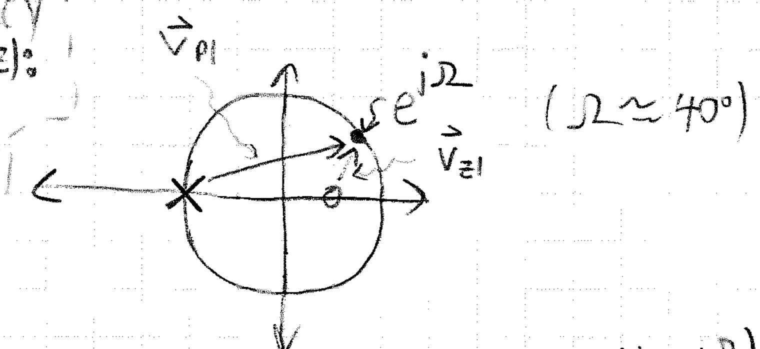
  eg. $z = e^{j\Omega}$
- ↳ idea of vector magnitudes & angles is otherwise exactly the same!

$$|H(z)| = \frac{|\vec{V}_{z1}| \cdot |\vec{V}_{z2}| \cdot |\vec{V}_{z3}| \cdots}{|\vec{V}_{p1}| \cdot |\vec{V}_{p2}| \cdot |\vec{V}_{p3}|} = \frac{\text{prod. of vectors from zeros}}{\text{prod. of vectors from poles}}$$

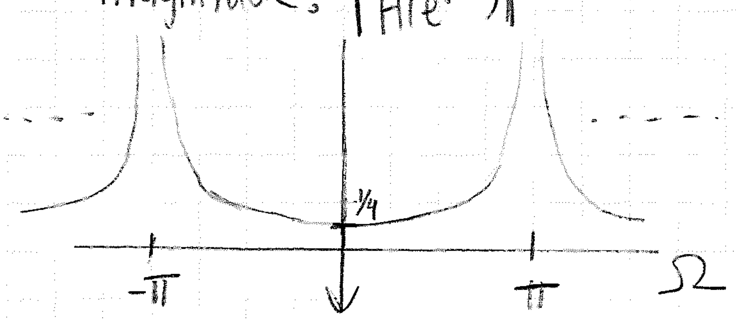
$$\angle H(z) = \left(\sum_{\text{zeros}} \angle(\vec{V}_z)\right) - \left(\sum_{\text{poles}} \angle(\vec{V}_p)\right) = \left(\text{sum of angl. from zeros}\right) - \left(\text{sum of angles from poles}\right)$$

$\boxed{EG5}$ Sketch the frequency response of $H(z)$:

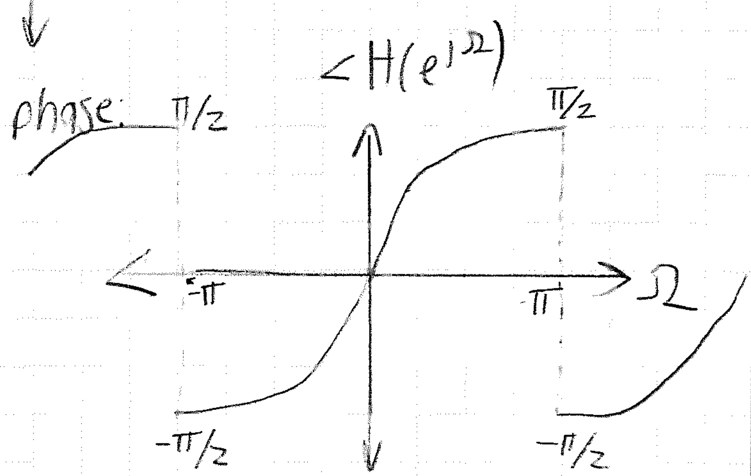
$$H(z) = \frac{(z - 1/2)}{(z+1)}$$



magnitude: $|H(e^{j\Omega})|$



phase:



unfortunately, we don't have Bode rules for DT ☹