6.003 9/16/2011

* Shere out on scanners
* Hayden, Barker,

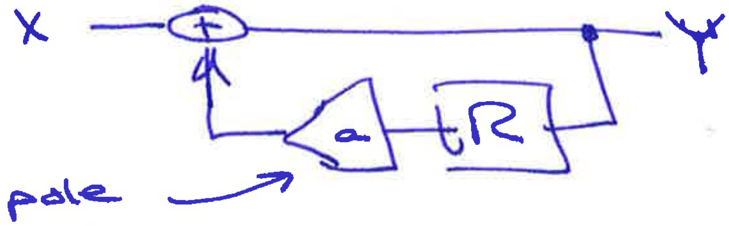
Rec 4

Ann: HW#2 Due Wed

Recap: Poles, Fundamental Modes, unit-sample response

Today: - System Representations
- 1st order
- 2nd order & PFE

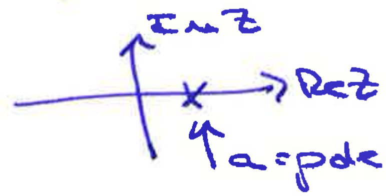
1^{st}-order system



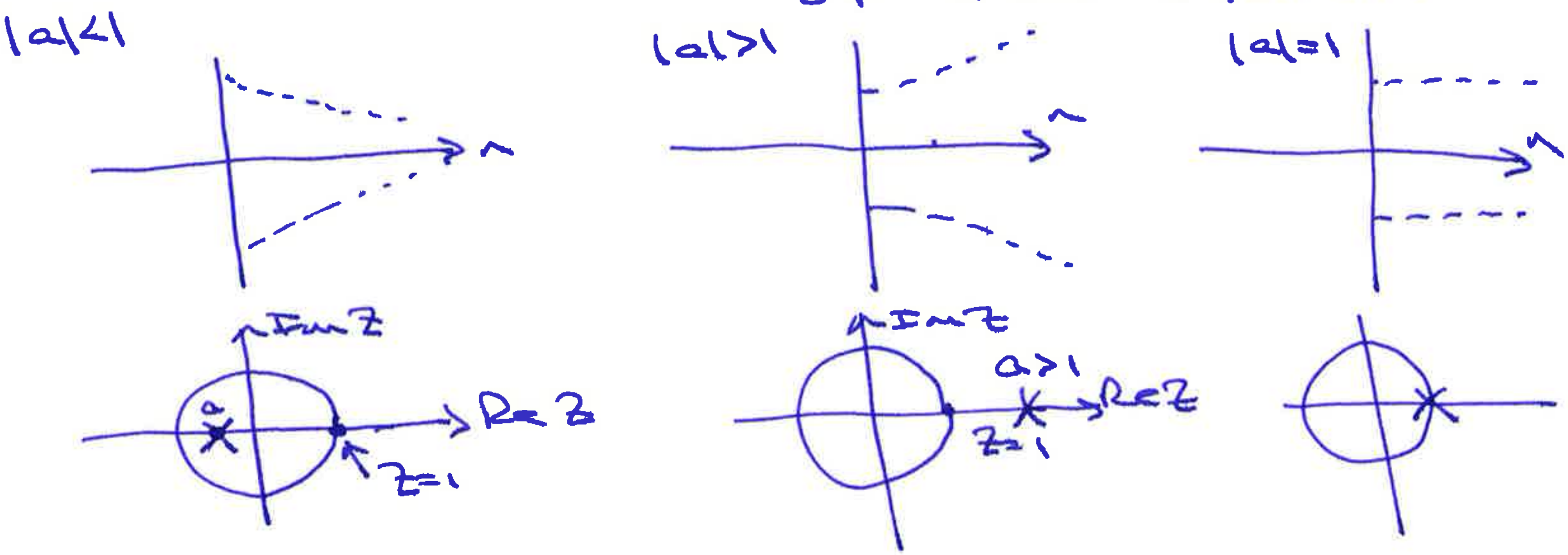
$$Y = aRY + X \quad \therefore \quad \frac{Y}{X} = \frac{1}{1-aR}$$

pole

$$\left.\frac{Y}{X}\right|_{R\to\frac{1}{z}} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$



a^n, $n \ge 0$ is fundamental mode of the system



"stable" cf HW#2

Understand 1st order, then tackle higher order:

PFE: Tool for high-order sys

$$= \sum (\text{low-order sys})$$

Two Methods: Match coeff; App A, O & W (staring)

1: Say $\frac{Y}{X} = \frac{\alpha\beta}{(\alpha+R)(\beta+R)}$, $\alpha \neq \beta$ $= \frac{1}{(1+\frac{1}{\alpha}R)(1+\frac{1}{\beta}R)}$ sometimes form

Match coeff in

$$\frac{\alpha\beta}{(\alpha+R)(\beta+R)} = \frac{A}{\alpha+R} + \frac{B}{\beta+R}$$

i.e.

$$\frac{Y}{X} = \frac{A(\beta+R) + B(\alpha+R)}{(\alpha+R)(\beta+R)}$$

$$= \frac{(A\beta + B\alpha) + R(A+B)}{(\alpha+R)(\beta+R)}$$

Match coeff:

$\therefore A + B = 0$ or $B = -A$

$A\beta + B\alpha = \alpha\beta$

$A\beta - A\alpha = \alpha\beta$ $\therefore A = \frac{\alpha\beta}{\beta-\alpha}$

2: O & W, App A:

$$\frac{Y}{X} = \frac{\alpha\beta}{(\alpha+R)(\beta+R)} = \frac{A}{\alpha+R} + \frac{B}{\alpha+R}$$

plug in PFE machinery

$$A = (\alpha+R)\frac{Y}{X}\bigg|_{R=-\alpha} = \frac{(\alpha+R)\,\alpha\beta}{(\alpha+R)(\beta+R)}\bigg|_{R=-\alpha} = \frac{\alpha\beta}{\beta-\alpha}$$

$$B = (\beta+R)\frac{Y}{X}\bigg|_{R=-\beta} = \frac{\alpha\beta}{\alpha+R}\bigg|_{R=-\beta} = \frac{\alpha\beta}{\alpha-\beta}$$

Note: If poles repeat, methods still work, but fractions contain e.g. $\frac{A}{\alpha+R}, \frac{B}{(\alpha+R)^2}, \ldots$

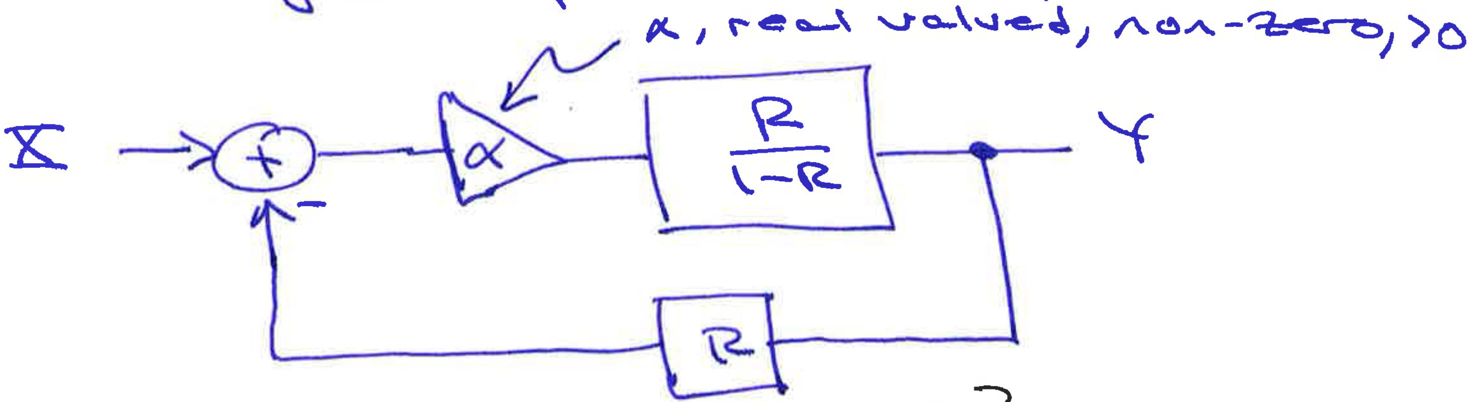
In Method 2: highest power

$$B = (\alpha+R)^2\frac{Y}{X}\bigg|_{R=-\alpha}$$ ← same as before

next lower: $\frac{d}{dR}$

$$A = (\alpha)\frac{d}{dR}\left((\alpha+R)^2\frac{Y}{X}\right)\bigg|_{R=-\alpha}$$ ← Different

Ex: System with variable gain parameter, α.



Q: Fundamental mode(s)?

Need system functional

$$Y = \alpha \cdot \frac{R}{1-R}(X - RY)$$

$$\therefore \quad Y\left(1 + \frac{\alpha R^2}{1-R}\right) = \frac{\alpha R}{1-R}$$

i.e. $$\frac{Y}{X} = \frac{\alpha R / 1-R}{1 + \frac{\alpha R^2}{1-R}} = \frac{\alpha R}{1 - R + \alpha R^2}$$

Order = 2, i.e. 2 fundamental modes

poles in z-plane are roots of denominator w/ $R \to 1/z$.

$$\alpha z^{-2} - z^{-1} + 1 = 0$$

or $$z^2 - z + \alpha = 0$$

poles: $$p_0, p_1 = \frac{1 \pm \sqrt{1-4\alpha}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha}$$

i.e.
$$\frac{Y}{X} = \frac{\alpha R}{(1-p_0R)(1-p_1R)}$$

when $p_0 \neq p_1$,

via PFE

$$\frac{Y}{X} = \frac{A}{1-p_0R} + \frac{B}{1-p_1R}$$

two first-order terms ∴ know fundamental modes

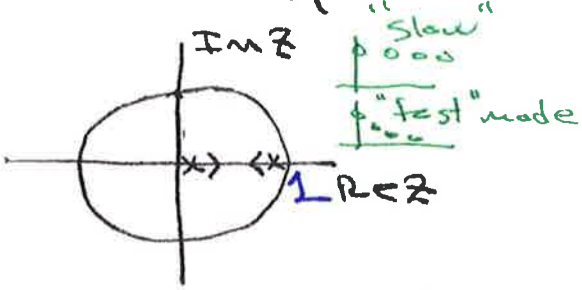
A, B by PFE.

and mode 0 is p_0^n

— 1 is p_1^n

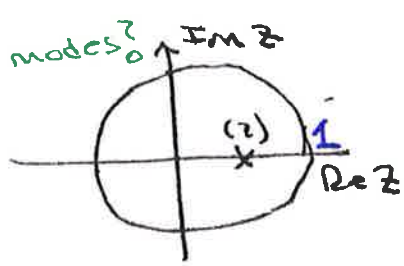
Three interesting ranges of α:

1. $0 < \alpha < \frac{1}{4}$



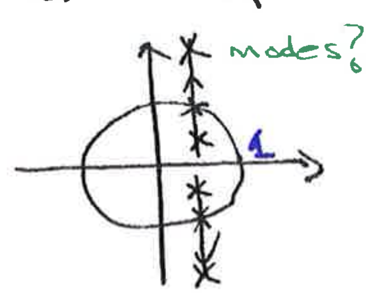
$$p_0, p_1 = \frac{1}{2} \pm \frac{1}{2}(1-4\alpha)^{\frac{1}{2}}$$
$$\approx \frac{1}{2} \pm \frac{1}{2}\left(1 - \frac{1}{2}\cdot 4\alpha + \cdots\right)$$
$$p_0, p_1 = \begin{cases} 1-\alpha \\ \alpha \end{cases}$$

2. $\alpha = \frac{1}{4}$



$$p_0 = p_1 = \frac{1}{2}$$

3. $\alpha > \frac{1}{4}$



$$p_0, p_1 = \frac{1}{2} \pm j\sqrt{\alpha - \frac{1}{4}}$$

e.g. @ unit circle when $\alpha = 1$

$$p_0, p_1 = \frac{1}{2} \pm j\,\sqrt{3}/2$$

Now, go back to example w/ $\alpha = \frac{1}{4}$.

From here can tell modes are $(\frac{1}{2})^n$, $(n+1)(\frac{1}{2})^n$

Here,

$$\frac{Y}{X} = \frac{\alpha R}{(1-p_0R)(1-p_1R)} = \frac{\frac{1}{4} R}{(1-\frac{1}{2}R)^2}$$

Could break into two modes w/ PFE:

$$\frac{\frac{1}{4} R}{(1-\frac{1}{2}R)^2} = \frac{A}{1-\frac{1}{2}R} + \frac{B}{(1-\frac{1}{2}R)^2}$$

where

$$B = (1-\tfrac{1}{2}R)^2 \cdot \frac{Y}{X}\Big|_{R=2} = \frac{\frac{1}{4}\cdot 2}{1} = \frac{1}{2}$$

$$A = (-2)\,\frac{d}{dR}\left\{(1-\tfrac{1}{2}R)^2\frac{Y}{X}\right\}\Big|_{R=2} = -2\cdot\frac{d}{dR}\,\tfrac{1}{4}R\Big|_{R=2}$$

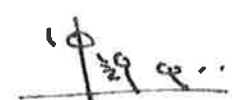
$A = -\frac{1}{2}$

Check

$$\frac{-\frac{1}{2}(1-\frac{1}{2}R)+\frac{1}{2}}{(1-\frac{1}{2}R)^2} = \frac{+\frac{1}{4}R}{(1-\frac{1}{2}R)^2}$$

These are drawn on lecture slide

i.e. $$\frac{Y}{X} = \frac{-\frac{1}{2}}{1-\frac{1}{2}R} + \frac{\frac{1}{2}}{(1-\frac{1}{2}R)^2}$$

so mode 0 is $(\frac{1}{2})^n$ 

mode 1 is $(n+1)(\frac{1}{2})^n$ 

unit sample

$y[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}(n+1)\left(\frac{1}{2}\right)^n u[n]$

What's the unit sample resp for $\frac{Y}{X} = \frac{1}{(1-\frac{1}{2}R)^2}$?

Try DE, i.e. write as

$$y[n] - 2\cdot\tfrac{1}{2}\, y[n-1] + \left(\tfrac{1}{2}\right)^2 y[n-2] = x[n]$$

Then, assuming initial rest; & $x[n] = \delta[n]$

$$y[n] = 0 \qquad n < 0$$

$$y[0] = 1 + 2\cdot\tfrac{1}{2}\cdot 0 - \left(\tfrac{1}{2}\right)^2\cdot 0 = 1$$

$$y[1] = 0 + 2\cdot\tfrac{1}{2}\cdot 1 - \left(\tfrac{1}{2}\right)^2\cdot 0 = 2\cdot\tfrac{1}{2}$$

$$y[2] = 0 + 2\cdot\tfrac{1}{2}\cdot 2\cdot\tfrac{1}{2} - \left(\tfrac{1}{2}\right)^2 1 = 3\left(\tfrac{1}{2}\right)^2$$

$\vdots$

$$y[n] = \begin{cases} (n+1)\left(\tfrac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$