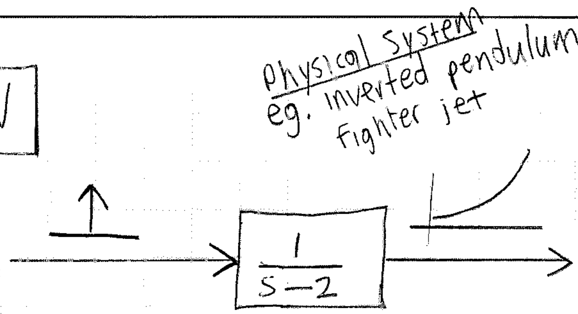
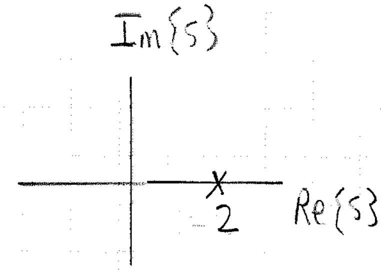
ROOT LOCUS

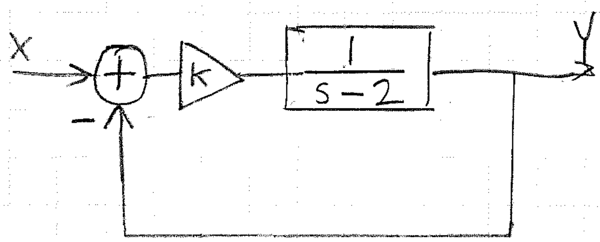
MOTIVATION

Unstable System



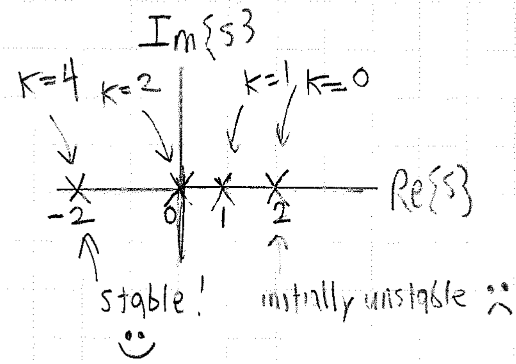


Unstable system in feedback w̄ gain K



K	pole location
0	2
1	1
2	0
4	−2

$$H(s) = \frac{\frac{K}{s-2}}{1+\frac{K}{s-2}} = \frac{K}{s-2+K}$$

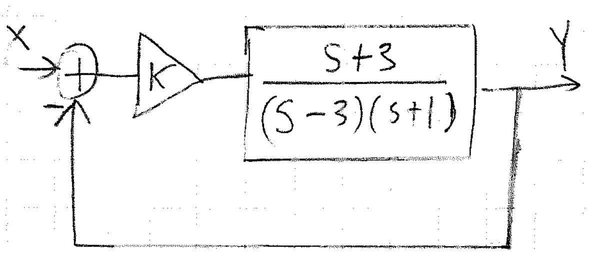


What is happening?

1) The poles of the overall system can be made "closed-loop stable" even though the underlying system is not!

2) The movement of the poles is called the root-locus (under the influence of a single parameter, eg. K)

How can we compute root locus?

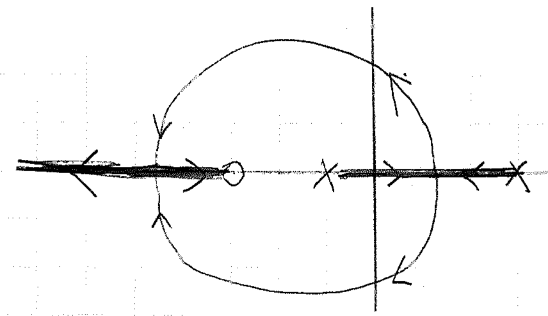


1) Analytical formula eg. $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$
↳ only have formulas for limited # of roots

2) "rlocus"
↳ need access to a computer eg. matlab

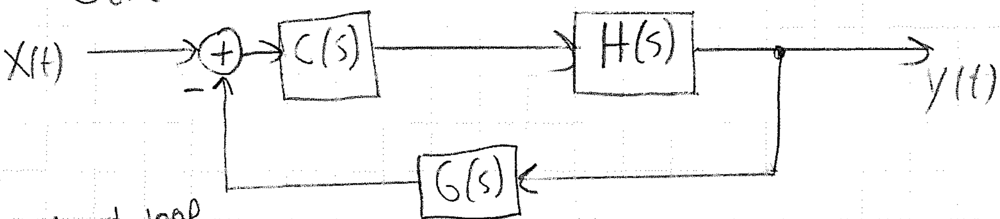
3) Simple rules derived from 2 fundamental conditions!

↳ quick way to gain insight into what the feedback is doing!



ROOT LOCUS CONDITIONS

General formulation of feedback:



$H(s)$ → physical system (no control)

$C(s)$ → controller
$G(s)$ → sensor
} designed w/ 1 free real parameter K

closed loop TF

$$Q(s) = \frac{CH}{1 + \underbrace{CHG}_{\text{open loop TF}}}$$

Q what are the poles of the overall closed loop system?

ANS The roots of the denominator of $Q(s)$

Let's assume $C(s)$ is just the free parameter "K"
↳ then the poles of the CLS (eg. roots of den. of $Q(s)$) are given by:

$$1 + kH(s)G(s) = \emptyset$$

Aside: The parameter K can be anywhere inside $C(s)$ or $G(s)$.
What matters for root locus is that the denominator of the CLS TF can be factored as: $1 + KA(s)$,
where A is a ratio of polynomials in s

Angle condition

$\angle kH(s)G(s) = \pi + 2\pi n,$

$\angle H(s)G(s) = \pi + 2\pi n$ n integer $K > 0$

→ most practical
→ can always find a K to satisfy the magnitude cond later!

Magnitude Condition

$|kH(s)G(s)| = 1$

General idea

* We will pick a point "s_0" in the s-plane and use vector diagrams to help us see which values of s_0 satisfy the conditions

ROOT LOCUS RULES

All of the rules follow from simple derivations based on the magnitude & angle conditions.

We will first look @ the limiting cases for $K \to 0$ & $K \to \infty$ and then connect the gaps.

For $K \to 0$: For mag. cond^n to be satisfied, $|H(s)\,G(s)| \to \infty$

$$\underbrace{|K|}_{\text{if } K \text{ approches } 0} \cdot \underbrace{|H(s) \cdot G(s)|}_{\text{then } |H(s)\cdot G(s)| \text{ must approach } \infty \text{ to satisfy cond}^n} = 1$$

> RULE: Roots start @ the poles of the open-loop TF for $K=0$

For $K \to \infty$: For mag cond^n to be satisfied, $|H(s)G(s)| \to 0$

$$\underbrace{|K|}_{\text{if } K \to \infty} \underbrace{|H(s)\,G(s)|}_{\text{then } |H(s)G(s)| \to 0 \text{ to satisfy cond}} = 1$$

> RULE: Roots end @ the zeros of the open-loop TF as $K \to \infty$

Real axis:

$$1 + K\underbrace{\frac{(s+3)}{(s-3)(s+1)} \cdot (1)}_{H \cdot G} = 0$$

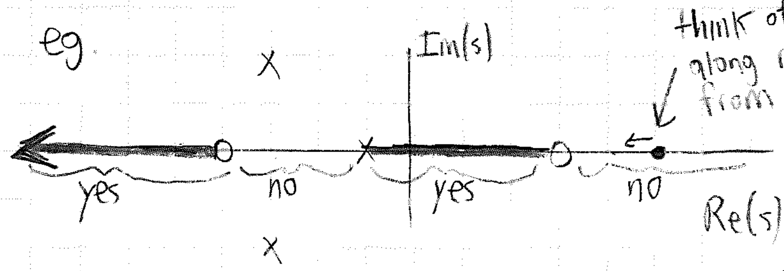
Angle cond:

$$\angle H(s)\,G(s) = \pi + 2\pi n$$

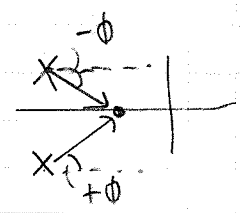
For positive K

> RULE: need odd # of p–z to the right of the RL
> "no–yes–no..." rule

eg.



think of moving 's' along real axis from right to left and drawing the vectors from each p/z

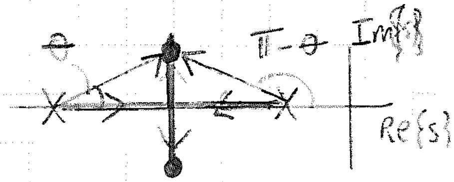


note: complex conj. pairs contribute nothing to the angle condition!

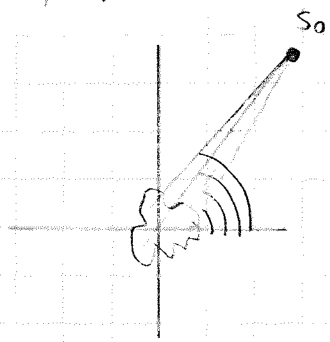
Breakaway points:

RULE: If 2 closed-loop poles collide on real axis → become complex

At the point of collision: $\angle H(s)G(s) = \pi + 2\pi n$

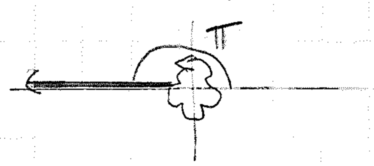


Asymptotes

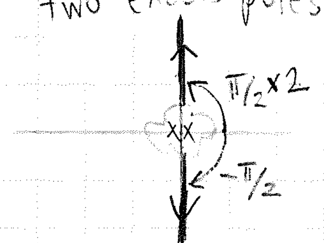


In the far limit, all angles are equal
↳ what matters is how many angle contributions from poles are cancelled by zeros
↳ only "excess" poles matter
↑
implied zeros are @ infinity

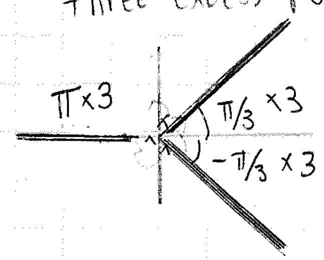
one "excess pole"



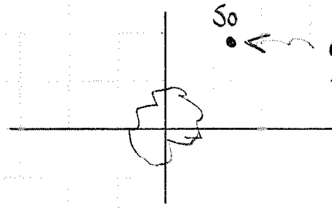
two "excess poles"



"three excess poles"



no excess poles



← no matter where you put So, you will never have total angle contribution of π

Angle cond:

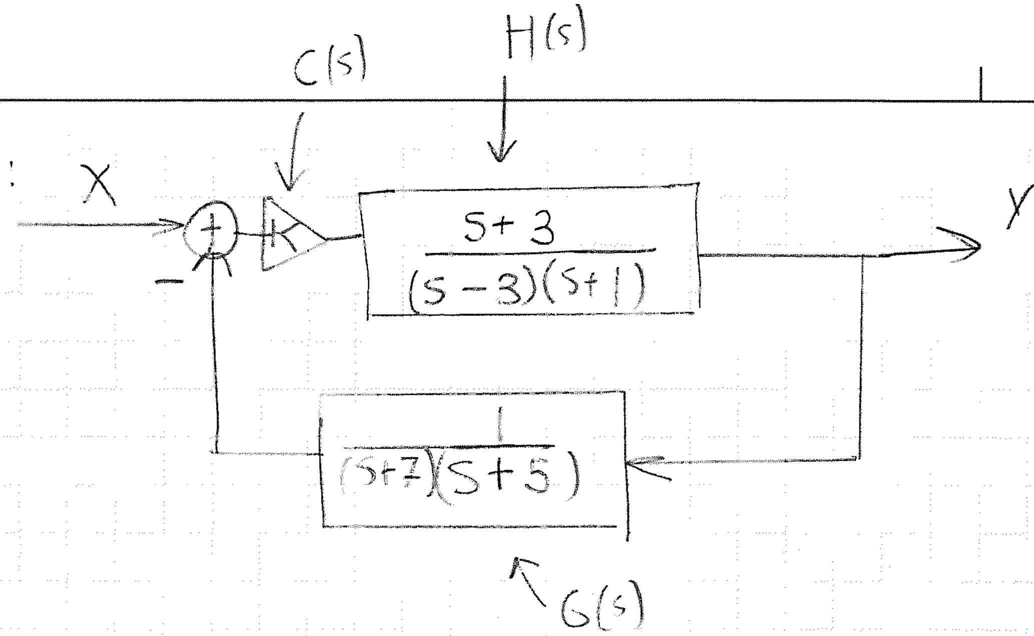
$$\angle G(s)H(s) = \angle S_0 \cdot (n_p - n_z) = \pi + 2\pi n$$

$$\rightarrow \angle S_0 = \frac{\pi + 2\pi n}{n_p - n_z}$$

RULE: Asymptotes are at:

$$\angle S_0 = \frac{\pi + 2\pi n}{n_{poles} - n_{zeros}}$$

EXAMPLE:



closed loop TF

$$Q(s) = \frac{\frac{K(s+3)}{(s-3)(s+1)}}{1 + K\frac{(s+3)}{(s-3)(s+1)(s+5)(s+7)}}$$

mag & angle condition comes from

$$1 + K\underbrace{\frac{(s+3)}{(s-3)(s+1)(s+5)(s+7)}}_{\text{called the open-loop TF}} = 0$$ ($\rightarrow$ gives poles of CL system)

3 excess poles

asymptotes @ $s_0 = \frac{\pi}{3} + \frac{2\pi}{3}n$