6.003 10/21/2011

Rec 12

Lecture: Feedback

↳ Trade off BW ⟺ Gain

↳ DC motor position control

Today: * Freq Response for 2nd-order terms

- Algebra ← less fun
- Graphics ← more intuitive

2nd order factors

Common notation for these factors

$$H(j\omega) = \frac{1}{(j\omega/\omega_0)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1}$$

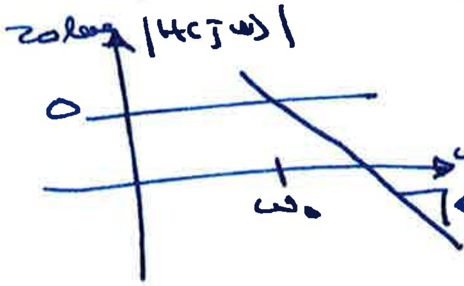
1 ← normalized. could be some gain.

ζ ← damping ratio

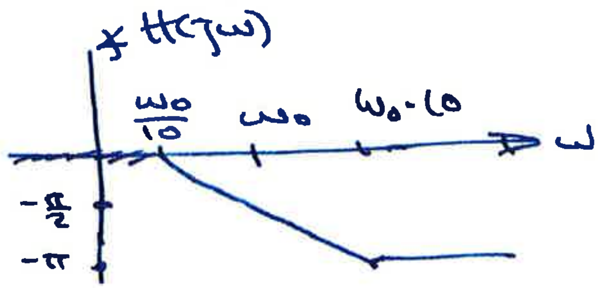
Also: $\frac{1}{Q}$ "Quality"

For such terms, the straight-line Bode is <u>still</u> simple.

$$\lim_{\omega \to 0} = 1 \qquad \lim_{\omega \to \infty} = \frac{1}{\left(j\frac{\omega}{\omega_0}\right)^2}$$



← −40 dB/decade ← 2X slope of 1st order



Difference from 1st order factors?

- How well does the actual response follow the asymptotes — ζ (or Q)

Q: where is neg freq on Bode plot?
or where is zero freq?

Vector Diagrams & 2^{nd} order factors

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0}\cdot\frac{1}{Q} + 1} = \frac{\omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

poles: $$p_{1,2} = \frac{-\frac{\omega_0}{Q} \pm \sqrt{\frac{\omega_0^2}{Q^2} - 4\omega_0^2}}{2}$$

$$p_{1,2} = \omega_0\left(-\frac{1}{2Q} \pm \sqrt{\frac{1}{(2Q)^2} - 1}\right)$$

i.e. $$H(s) = \frac{\omega_0^2}{(s-p_1)(s-p_2)}$$

Cases for analysis:

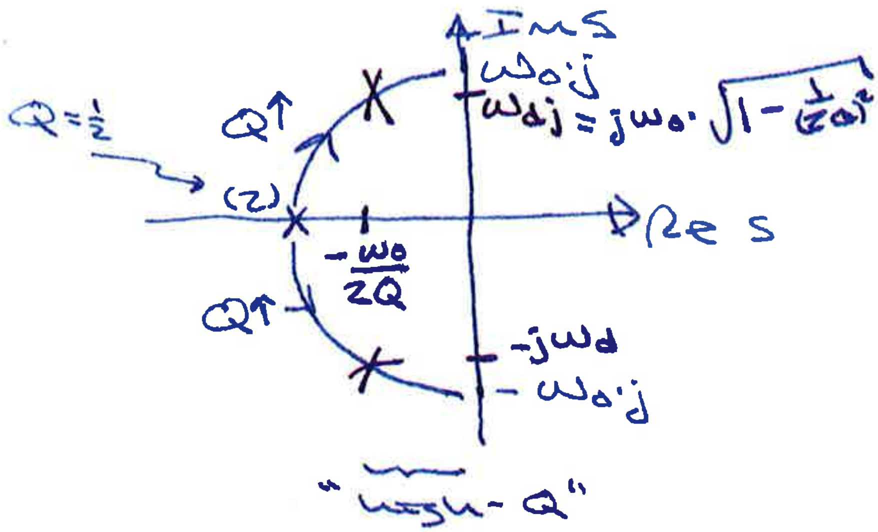
* complex pole $\longrightarrow$ high Q
* double pole ("critical damping")
* two real poles $\longrightarrow$ low Q

Start w/ $Q > \frac{1}{2}$, i.e. complex $p_{1,2}$:

$$p_{1,2} = \omega_0\Big(\underbrace{-\frac{1}{2Q}}_{\text{real}} \pm j\underbrace{\sqrt{1 - \frac{1}{(2Q)^2}}}_{\text{imag}}\Big)$$

Note: $(\text{real})^2 + (\text{imag})^2 = \omega_0^2\left(\left(\frac{1}{2Q}\right)^2 + 1 - \frac{1}{(2Q)^2}\right) = \omega_0^2$

i.e. $p_{1,2}$ are on a circle, radius ω_0, origin $s=0$.



Root locus as a func of Q.

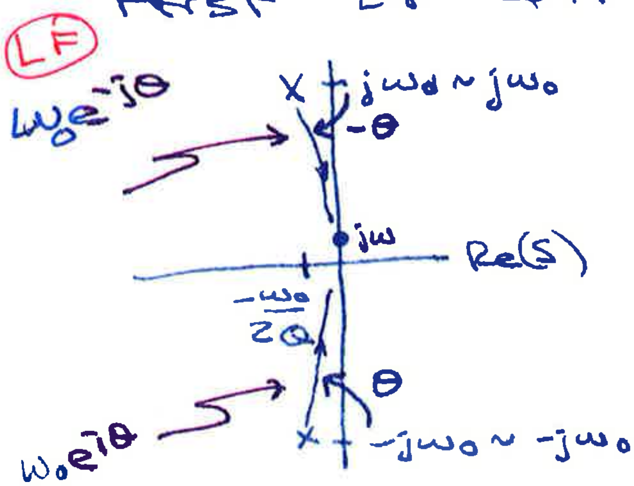
As $Q \to \infty$, $p \to \pm j \cdot \omega_0$

Now, examine Freq Resp for large Q:

First LF & HF limits:

$$H(j\omega) = \frac{\omega_0^2}{(s-p_1)(s-p_2)}$$

LF



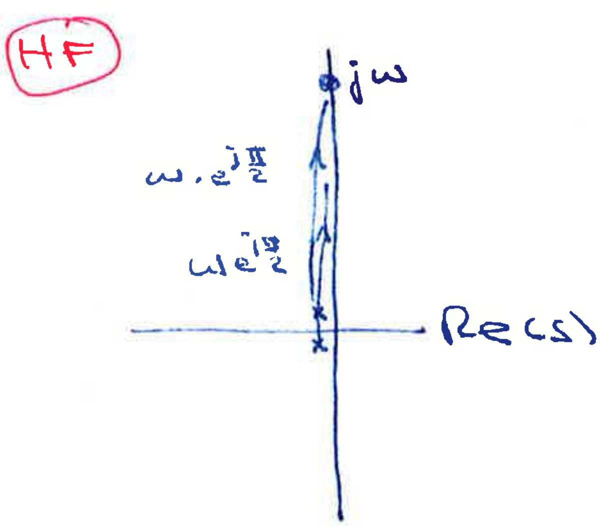
$$H(j\omega) \sim \frac{\omega_0^2}{\omega_0 \cdot e^{-j\theta} \cdot \omega_0 e^{j\theta}} = \frac{\omega_0^2}{\omega_0^2} e^{j0}$$

same as before

$$= 1 \cdot e^{j0}$$

LF limits from before

HF



$$H(j\omega) \sim \frac{\omega_0^2}{\omega e^{j\frac{\pi}{2}} \, \omega e^{j\frac{\pi}{2}}} = \frac{\omega_0^2}{\omega^2} e^{-j\pi}$$

$$= \left(\frac{\omega_0}{\omega}\right)^2 e^{-j\pi}$$

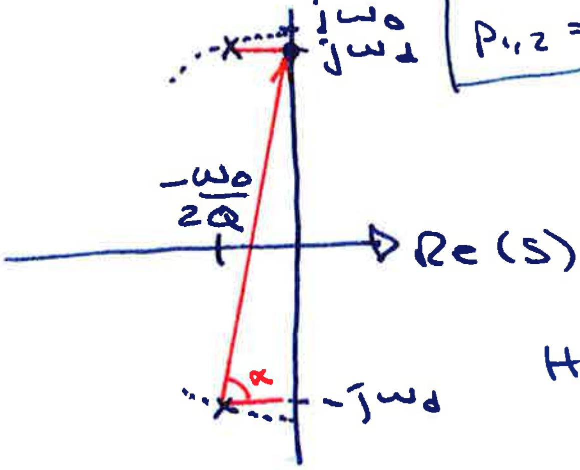
HF limits from before

Freq near ω_0

For $\omega = \omega_d$

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0}\cdot\frac{1}{Q} + 1} = \frac{\omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

$$p_{1,2} = \omega_0 \cdot \left(-\frac{1}{2Q} \pm j\sqrt{1-\left(\frac{1}{2Q}\right)^2}\right)$$



$$H(j\omega_d) = \frac{\omega_0^2}{2\omega_d e^{j\alpha} \cdot \frac{\omega_0}{2Q} e^{j0}}$$

$$H(j\omega_d) = Q\left(\frac{\omega_0}{\omega_d}\right) e^{-j\alpha}$$

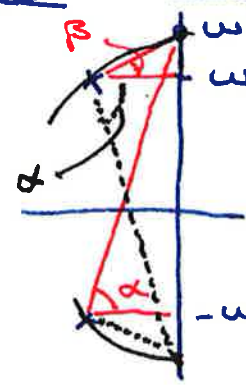
>1, but $\to 1$ as $Q\uparrow$

$\alpha < \frac{\pi}{2}$ but $\to \pi/2$ as $Q\uparrow$

Note: we know from the algebra (easy) that

$$H(j\omega_0) = Q e^{-j\frac{\pi}{2}}$$

$\uparrow$ exact.



$\alpha + \beta = \frac{\pi}{2}$ by geom. inscribed triangle.

Note: Where is the ω-location of max of $|H(j\omega)|$?

Approach 1: could use algebra to calculate

$$\omega_{max}^2 = \omega_0^2\left(1 - \frac{1}{2Q^2}\right) < \omega_0^2\left(1 - \frac{1}{(2Q)^2}\right) = \omega_d^2$$

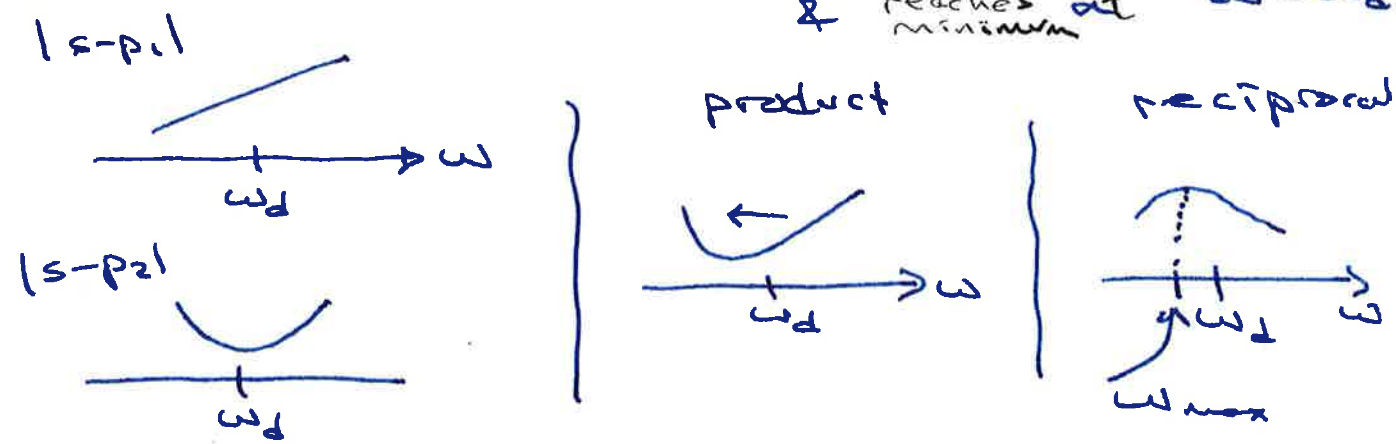
Approach 2: Qualitative statement: (suggested dwei):

p_1 @ $\left(-\frac{\omega_0}{2Q}, -j\omega_d\right)$

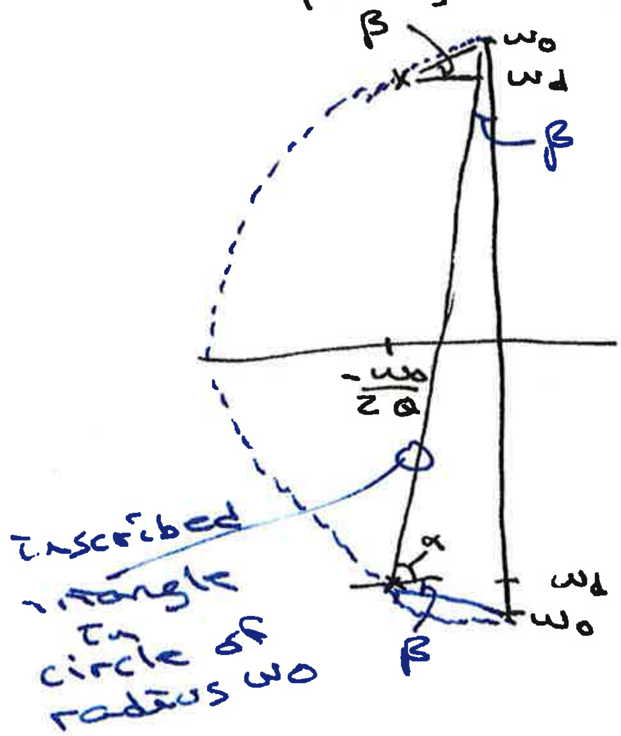
p_2 @ $\left(-\frac{\omega_0}{2Q}, +j\omega_d\right)$

for $\omega > 0$

* Magnitude of $s-p_1$ is incr as $\omega \uparrow$
* — — $s-p_2$ is symmetric @ ω_d & reaches minimum at $\omega = \omega_d$



<u>Note:</u> Geometric argument for $|H(j\omega_0)| = Q$:



length of short vector

$$\frac{\omega_0}{2Q} \cdot \frac{1}{\cos\beta}$$

length of long vector

$$2\omega_0 \cdot \cos\beta$$

$$\text{Product} = \frac{\omega_0^2}{Q}$$

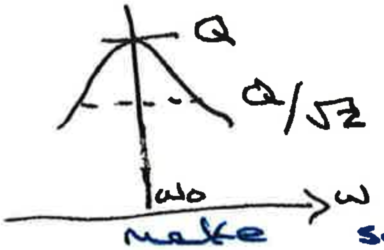
so

$$|H(j\omega_0)| = \frac{\omega_0^2}{\omega_0^2/Q} = Q$$

For what range of freqs (BW) is $|H(j\omega)|$ large?

$|H(j\omega)| > \frac{Q}{\sqrt{2}}$ ← From −3dB down.

small changes in freq from $\omega = \omega_0$

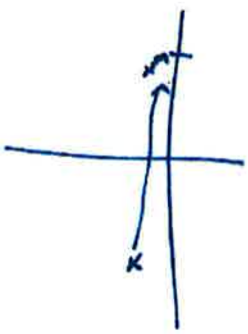


- small change in large vect
- large change in small vector

make small vector bigger by $\sqrt{2}$ at $\omega \pm \frac{\omega_0}{2Q}$

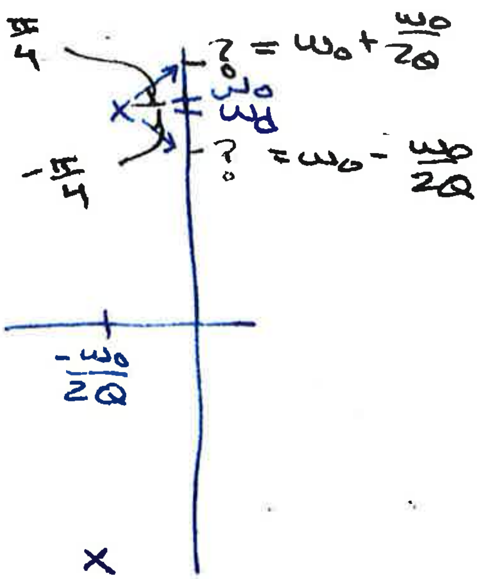
3dB BW: $\omega_0 - \frac{\omega_0}{2Q} < \omega < \omega_0 + \frac{\omega_0}{2Q}$

$\therefore BW \cong \frac{\omega_0}{Q}$

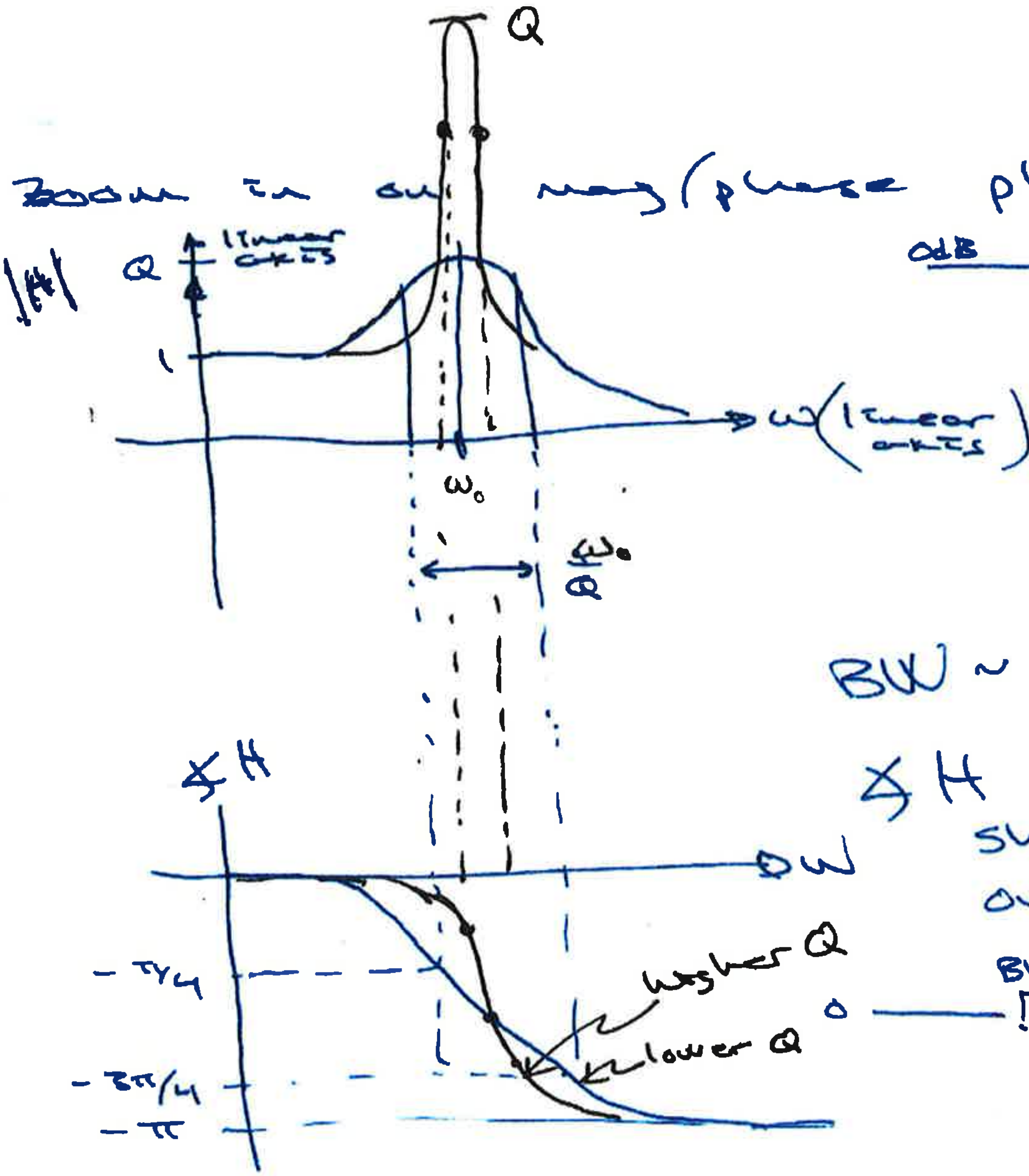


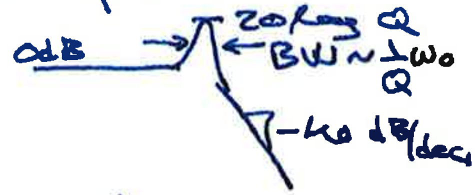
$$H(j\omega) \sim \frac{\omega_0^2}{2\omega_0 e^{j\frac{\pi}{2}} \left(\frac{\sqrt{2}\,\omega_0}{2Q} e^{\pm j\frac{\pi}{4}} \right)}$$

$$= \begin{cases} \frac{Q}{\sqrt{2}} e^{j\left(-\frac{3\pi}{4}\right)} \\ \frac{Q}{\sqrt{2}} e^{j\left(\frac{\pi}{4}\right)} \end{cases}$$



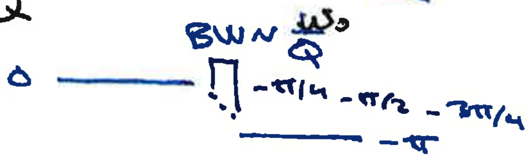
Zoom in on mag/phase plots:





$BW \sim \frac{\omega_0}{Q}$

$\angle H$ changes sharply over $\frac{1}{Q}$



$Q > \frac{1}{2}$ underdamped

$Q = \frac{1}{2}$ critically damped

$Q < \frac{1}{2}$ overdamped.