6.003 (8th Monday Andrew) 10/28/2011

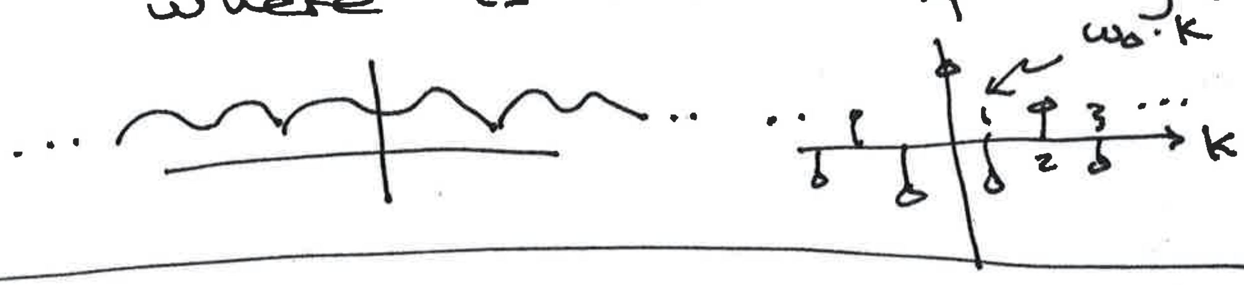
Recall: Orthog of complex exp:

$$\int_T e^{j\frac{2\pi}{T}(k-l)t}\,dt = T\,\delta[k-l]$$

Q: Time Domain | Freq Domain

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

where is the frequency?



Rec 13

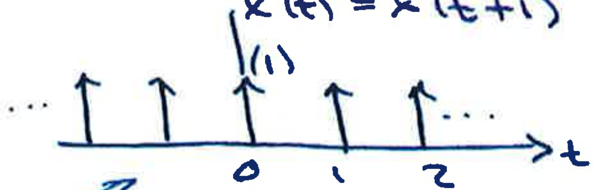
Lecture: Fourier Series & Filtering

$$x(t) = x(t+T) \overset{FS}{\longleftrightarrow} a_k$$

$$x(t) = \sum a_k e^{j\frac{2\pi}{T}kt} \qquad a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}\,dt$$

Today: Examples - FS, Filter

Ex: FS for impulse train



$T = 1$ $\longleftrightarrow$ $a_k = ?$

seems an unlikely candidate to fit FS synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

Find a_k by

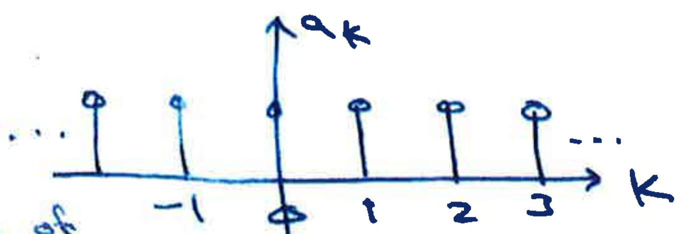
$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

any interval for T (0^- to 1^-, $\frac{1}{4}$ to $1\frac{1}{4}$, ...)

$$= 1\int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(t)\, e^{-j\frac{2\pi}{1}kt}\, dt$$

impulse area 1

$$a_k = 1 \quad \forall k$$



odd? i.e. strange

energy of error:

$$\int_T |e(t)|^2 dt = 0$$

$e(t) = x(t) - \sum_k a_k e^{j\frac{2\pi}{T}kt}$

"no energy in difference" (not: point-by-point the same)

Dirichlet: same, except where disc.

$\int_T |x| < \infty$

i) abs integr.

ii) finite # max min

iii) finite # disc.

works, Impulse train does not violate any of the 3 Dirichlet conditions.

$= \cos 2\pi \frac{t}{2}$ $\quad \therefore T = 2$

$$\boxed{\begin{aligned} &\frac{1}{2}\int_{-1}^{1} \frac{e^{j\pi t}+e^{-j\pi t}}{2}\, e^{-j\pi k t}\, dt \\ &= \frac{1}{2}\int_{-1}^{1} \frac{e^{-j\pi t(k-1)} + e^{-j\pi t(k+1)}}{2}\, dt \\ &= \frac{1}{2} \quad k = \pm 1, \quad 0 \text{ ow} \end{aligned}}$$

Ex: $x(t) = \cos \pi t \iff a_k$?

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} \cos \pi t \; e^{-j\frac{2\pi}{2}kt}\, dt$$

(ok by orthog, but ow hairy)

$$a_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ 0 & k \neq \pm 1 \end{cases} \quad \text{(by orthogonality)}$$

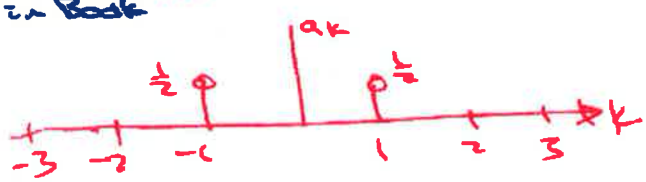
cf $x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$

or, directly

$$x(t) = \cos \pi t = \underset{a_1}{\frac{1}{2} e^{j\pi t}} + \underset{a_{-1}}{\frac{1}{2} e^{-j\pi t}}$$

$T = 2$

often used in Book $\quad \omega_0 = 2\pi/T = \pi$



⟵ Note $\omega_0 = \pi$

$x(t) = \cos \pi t$ from before.

Ex: $y(t) = x^2(t) \iff b_k = ?$

Simple by direct method:

$$y(t) = x^2(t) = \cos^2 \pi t = \frac{1}{2} + \frac{1}{2}\cos 2\pi t$$

$T = 1$

$\omega_0 = 2\pi$

$$y(t) = \frac{1}{2} e^{j0t} + \frac{1}{4} e^{j2\pi t} + \frac{1}{4} e^{-j2\pi t}$$

⟸ Note $\omega_0 = 2\pi$

$T = 1$

could also do by manipulating series:

$$x(t) \leftrightarrow a_k$$

$$x^2(t) \leftrightarrow b_k = ?$$

multiply in time $\leftrightarrow$ conv series

$$b_k = \frac{1}{T}\int_T x^2(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

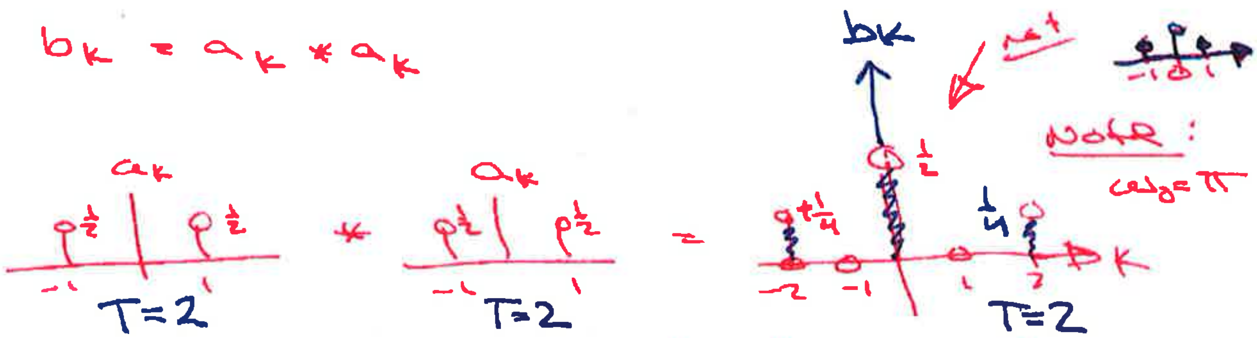
$$= \frac{1}{T}\int_T \sum_l a_l e^{j\frac{2\pi}{T}lt} \sum_m a_m e^{j\frac{2\pi}{T}mt} \cdot e^{-j\frac{2\pi}{T}kt}\, dt$$

$$= \sum_l \sum_m a_l a_m \underbrace{\frac{1}{T}\int_T e^{j\frac{2\pi}{T}(l+m-k)t}\, dt}_{\delta[l+m-k]}$$

sum over m zero except for $m = k-l$

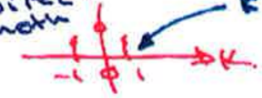
$$= \sum_l a_l a_{k-l} \text{, i.e. convolution.}$$

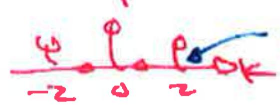
$$b_k = a_k * a_k$$



Note: Must use consistent periods

Direct meth

$\cos^2 \pi t \underset{T=1}{\longleftrightarrow}$  $k=1$ corresponds to actual freq of $\omega_0 = \frac{2\pi}{T} = 2\pi$

$\cos^2 \pi t \underset{T=2}{\longleftrightarrow}$  $k=2$ corresp to actual freq of $2\omega_0 = 2\cdot\frac{2\pi}{T} = 2\pi$

conv. method. SAME

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow e^{st} H(s)$$

Filtering:

$$x(t) \longrightarrow \underset{}{\overset{h(t)}{\boxed{\text{LTI}}}} \longrightarrow y(t)$$

$\parallel$

$$\sum_k a_k e^{j\frac{2\pi}{T}kt} \qquad\qquad \sum_k \underbrace{a_k H\left(j\tfrac{2\pi}{T}k\right)}_{b_k} e^{j\frac{2\pi}{T}kt}$$

LTI: "no new freq @ output"

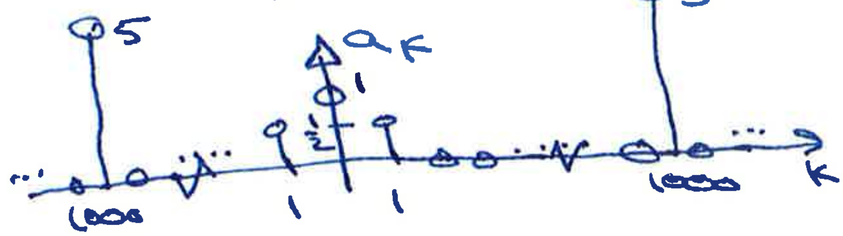
$\frac{1}{s+1}$, $\Re s > -1$

Ex: 1^{st} order: $h(t) = e^{-t}u(t)$

i.e. $H(j\omega) = \dfrac{1}{1+j\omega}$

Input: $x(t) = \sum a_k e^{j\frac{2\pi}{T}kt}$ where

where $T = 2\pi\,8$



Q: $x(t) = ?$

Q: $y(t) = ?$

How well does 1^{st} order filter remove high-freq term?