6.003 11/2/2011

Rec 14

Lecture: Filtering

Today: Filtering & FS prop.

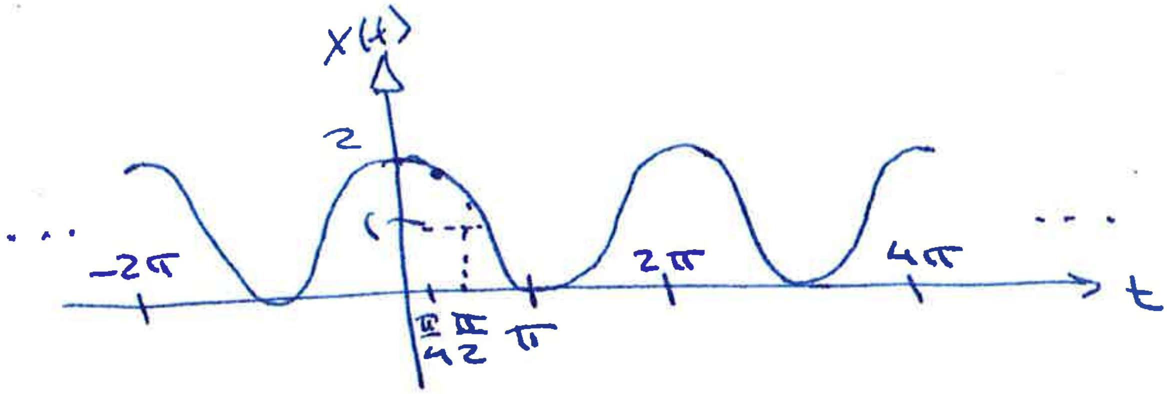
Recall ~~ee~~ from last Rec #13...

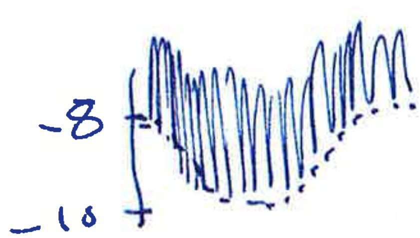
First, ... 12

$$x(t) = \overset{10}{1} + \frac{1}{2}e^{j\frac{2\pi}{2\pi}\cdot 1\cdot t} + \frac{1}{2}e^{j\frac{2\pi}{2\pi}(-1)\cdot t}$$

$$+ 5e^{j\frac{2\pi}{2\pi}\cdot 1000\cdot t} + 5e^{j\frac{2\pi}{2\pi}(-1000)t}$$

$$x(t) = 1 + \cos t + 10\cos(1\cdot 1000\cdot t)$$





Next, $y(t) = 1\cdot H(j\cdot 0) +$

$$\frac{1}{2}\cdot H(j\,1)e^{j\frac{2\pi}{2\pi}t} + \frac{1}{2}H(j(-1))\cdot e^{-j\frac{2\pi}{2\pi}t}$$

$$+ 5\cdot H(j\cdot 1000)e^{j\frac{2\pi}{2\pi}1000t}$$

$$+ 5\, H(j(-1000))e^{-j\frac{2\pi}{2\pi}1000t}$$

$$H(j\omega) = \frac{1}{1+j\omega}$$

$$H(j0) = 1 \measuredangle 0$$

$$H(j1) = \frac{1}{\sqrt{2}} \measuredangle \left(-\frac{\pi}{4}\right)$$

$$H(j(-1)) = \frac{1}{\sqrt{2}} \measuredangle \left(\frac{\pi}{4}\right)$$

$$H(j1000) \simeq 10^{-3} \measuredangle \left(-\frac{\pi}{2}\right)$$

$$H(j(-1000)) \simeq 10^{-3} \measuredangle \frac{\pi}{2}$$

so

$$y(t) = 1 + \frac{1}{\sqrt{2}} \left(\frac{e^{j(t-\pi/4)} + e^{-j(t-\frac{\pi}{4})}}{2} \right)$$

$$+ 10 \cdot 10^{-3} \frac{e^{j(1000t-\frac{\pi}{2})} + e^{-j(1000t-\frac{\pi}{2})}}{2}$$

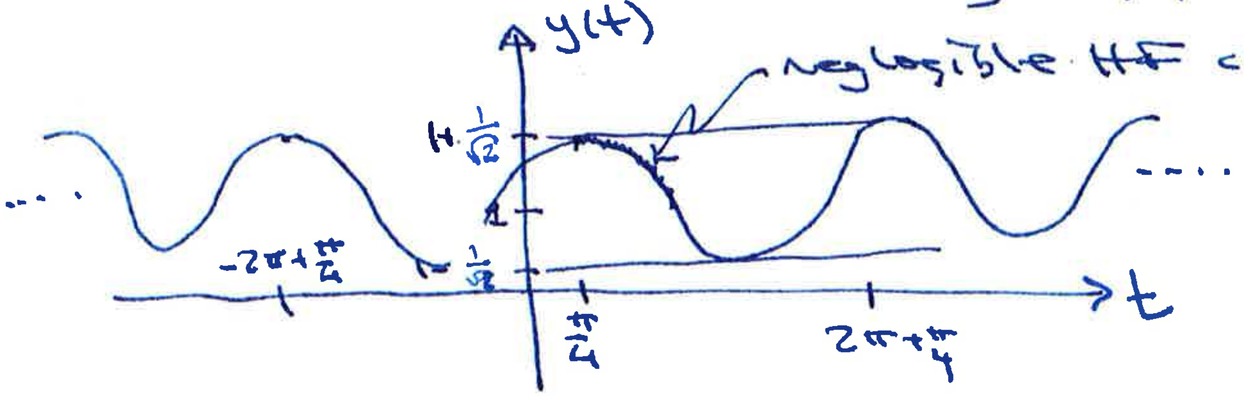
much smaller

$$y(t) \approx 1 + \frac{1}{\sqrt{2}} \cdot \cos\left(t - \frac{\pi}{4}\right) + 10^{-2} \cos\left(1000t - \frac{\pi}{2}\right)$$

DC unchanged

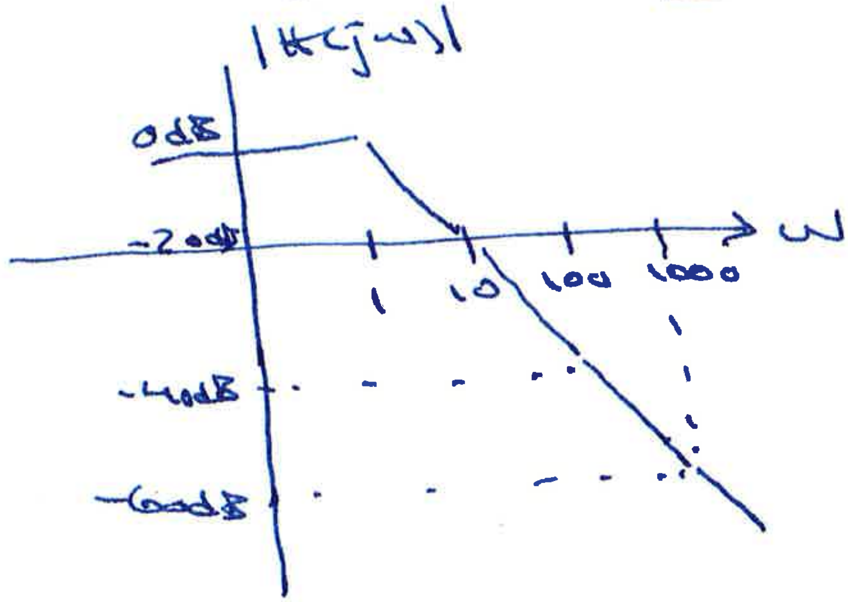
1st harmonic $\downarrow$ mag by $\sqrt{2}$

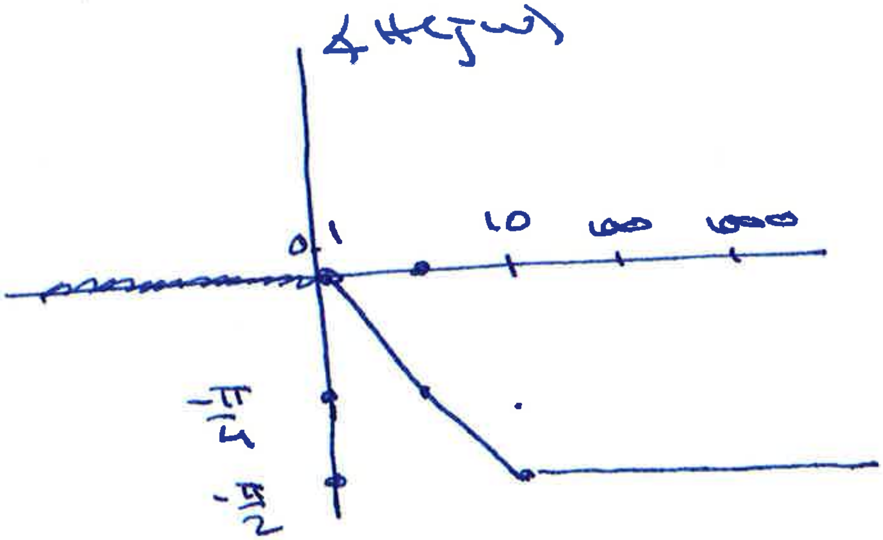
phase shift of 1st harmonic by $-\pi/4$



By Bode:

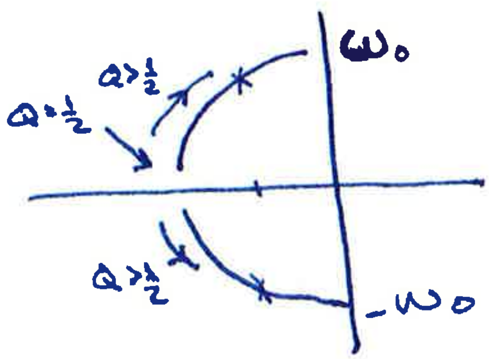
$$\cos\omega t \longrightarrow \boxed{H(j\omega)} \longrightarrow |H(j\omega)|\cos(\omega t + \angle H(j\omega))$$





Ex: Now, assume same input, but a 2nd order LTI system:

$$H(s)=\frac{\omega_0^2}{s^2+\frac{\omega_0}{Q}s+\omega_0^2}$$

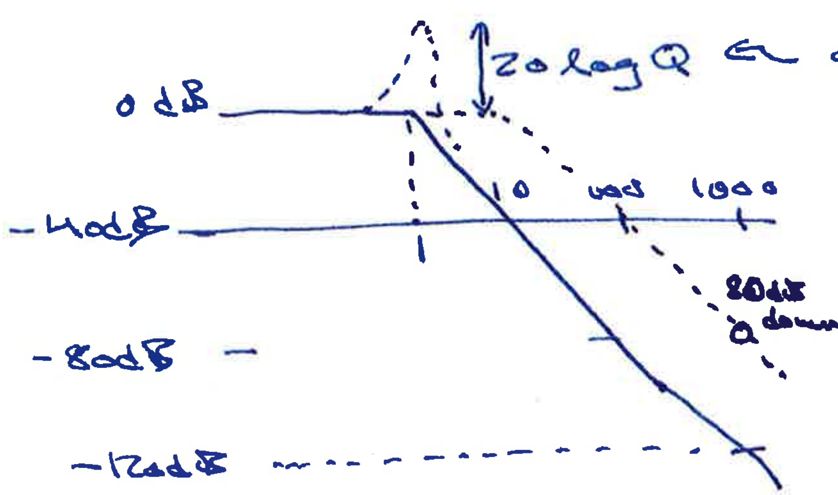


$$H(j\omega_0)=\frac{\omega_0^2}{(j\omega_0)^2+\frac{\omega_0 j\omega_0}{Q}+\omega_0^2}$$

$$H(j\omega_0) = -jQ$$

Use same critical freq $\omega_0 = 1$

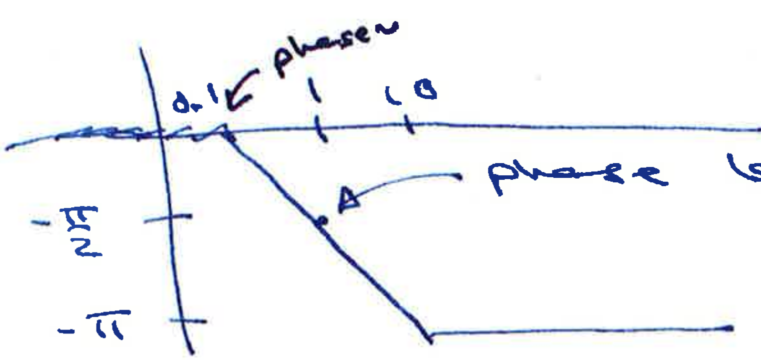
$$p_{1,2}=\omega_0\left(-\frac{1}{2Q}\pm\sqrt{\frac{1}{(2Q)^2}-1}\right)$$



could tune $Q = 1$ s.t. mag of 9^{st} harmonic is unchanged

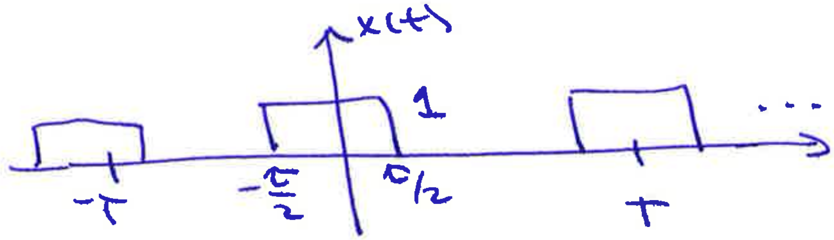
$$s^2+s+1 = \left(s-\left(-\tfrac{1}{2}+j\sqrt{3}/2\right)\right)\left(s-\left(-\tfrac{1}{2}-j\sqrt{3}/2\right)\right)$$

Factor 10^{-6} in magnitude



phase lag of $-\frac{\pi}{2}$ @ ω_0

Ex: Box Common



$\longleftrightarrow a_k$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

$k \neq 0$

$$= \frac{1}{T} \left. \frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi}{T}k}\right|_{-\tau/2}^{\tau/2} = \frac{e^{-j\frac{2\pi}{T}k\frac{\tau}{2}} - e^{+j\frac{2\pi}{T}k\frac{\tau}{2}}}{-j2\pi k}$$

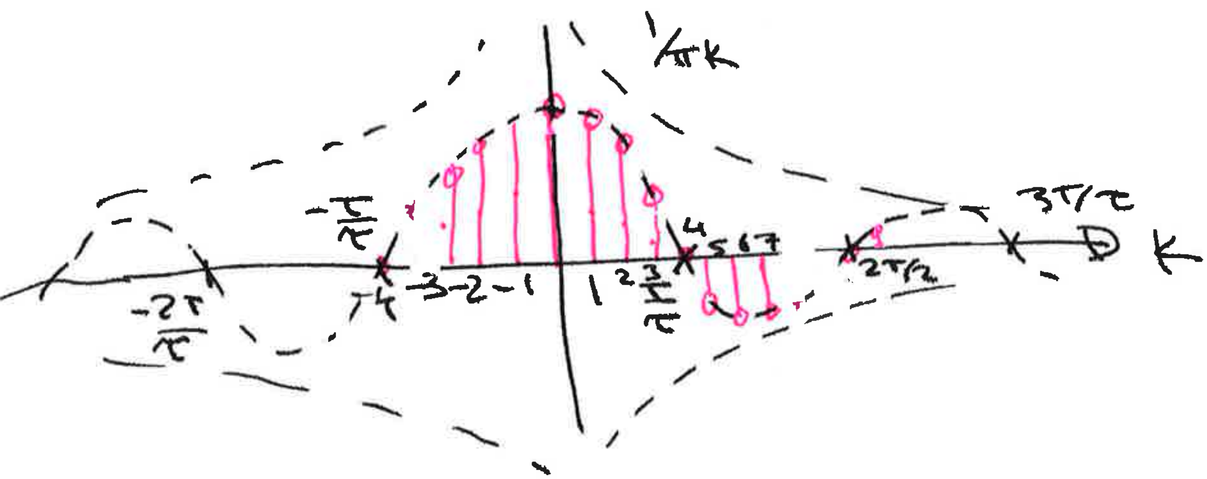
$$a_k = \frac{\sin \pi k \frac{\tau}{T}}{\pi k} \qquad k \neq 0$$

$k=0$, by integr.

$$a_k = \frac{\tau}{T} \quad \text{Also, limit}$$

Features: * Envelope $\propto \frac{1}{\pi k}$

* Zeros @ $k = \frac{T}{\tau} \cdot n$, $n = \pm 1, \pm 2, \ldots$



Symmetries:

$$x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$$

$$\therefore \quad x^*(t) = \sum_k a_k^* e^{j\frac{2\pi}{T}(-k)t}$$

i.e. if $x(t) \Leftrightarrow a_k$, then

$$\underset{\substack{\uparrow \\ \text{conjug}}}{x^*(t)} \Leftrightarrow \underset{\substack{\uparrow \\ \text{Flipped in } k \\ \text{\& conjug.}}}{a^*_{-k}}$$

When $x(t)$ is real, $x(t) = x^*(t)$,

so $\underbrace{a_k = a^*_{-k}}_{\Downarrow}$ "conjugate symmetry"

$$\alpha_k + j\beta_k = \alpha_{-k} - j\beta_{-k}$$

$\therefore$ real part even
imag part odd

$x(t)$ real $\Leftrightarrow$ $\begin{cases} a_0 \text{ is real} \\ |a_k| = |a_{-k}| \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$

Time reversal:

$$x(-t) = \sum_k a_k e^{j\frac{2\pi}{T}k(-t)}$$

$$\overset{m=-k}{=} \sum_m a_{-m} e^{j\frac{2\pi}{T}mt}$$

i.e. $x(t) \leftrightarrow a_k$
$x(-t) \leftrightarrow a_{-k}$

time reversal $\downarrow$ k-reversal $\downarrow$

i.e. $x(t)$ even $\Leftrightarrow$ a_k even $(a_k = a_{-k})$

$x(t)$ odd $\Leftrightarrow$ a_k odd. $(a_k = -a_{-k})$

Combined w/ conjug symm:

$x(t)$ real & even $\Leftrightarrow$ a_k real & even

$\Downarrow$ $a_k = a_{-k}^*$ $\quad$ $\Downarrow$ $a_k = a_{-k}$

$x(t)$ real & odd $\Leftrightarrow$ a_k imaginary & odd

$\Downarrow$ $a_k = a_{-k}^*$ $\quad$ $\Downarrow$ $a_k = -a_{-k}$