6.003 Recitation 11-4-11.

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Today: Continuous Time Fourier Transform (CTFT)

CTFT is just the Laplace transform on the $j\omega$-axis.

$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ $\qquad$ $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt.$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

So what's the big deal?

1) FT has a (useful) inverse transform.
2) We'll define the FT for some signals where the integral does not converge. (is not finite).

Inverse transforms.

Recall inverse Laplace.

Eq: is an integral over complex domain.

we never used it.

Instead we did template matching.

$$x(t) = e^{-at} u(t) \longrightarrow X(s) = \frac{1}{s+a}, \ Re\{s\} > a$$

$$X_2(s) = \frac{A}{s+p_1} + \frac{B}{s+p_2} + \frac{C}{s+p_3} + \cdots$$

$$\Rightarrow x_2(t) = \_\_\, e^{-p_1 t} u(t) + \_\_\, e^{-p_2 t} u(t) + \cdots$$

Inverse FT is easy!

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$$

Note the similarity to

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

flip ω-axis.

scale by 2π.

"Duality": relationships for time → freq also hold for freq → time.

Inverse FT motivates FT for signals which don't converge.

$$x(t) = 1.$$

$$X(s) = ? = \cancel{\int_{-\infty}^{\infty} e^{-st}\, dt}.$$

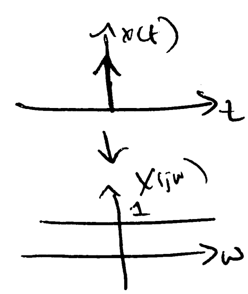
doesn't converge.

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = \begin{cases} \infty & \omega = 0. \\ 0 & \omega \neq 0. \end{cases}$$

So doesn't converge, but might be able to use $\delta(\omega)$.

Let's try $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j0\omega} = 1 \quad \forall \omega.$$



By duality. expect δ in freq $\Rightarrow$ 1 in time.

$$X(j\omega) = \delta(\omega).$$

$$x(t) = \frac{1}{2\pi}\int \delta(\omega)\, e^{j\omega t}\, d\omega. = \frac{1}{2\pi} e^{j0t} = \frac{1}{2\pi}.$$

So $\quad x(t) = 1 \quad \longleftrightarrow \quad X(j\omega) = 2\pi\,\delta(\omega).$

Try another:

$$x(t) = \cos(\omega_0 t).$$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

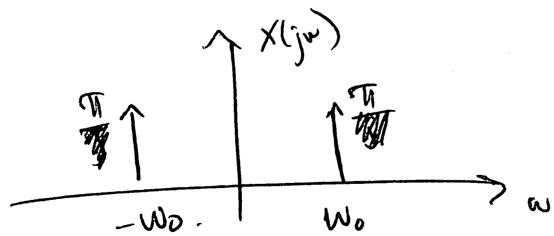
$$X(j\omega) = \frac{1}{2}\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega_0 t} e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega_0+\omega)t}\, dt.$$

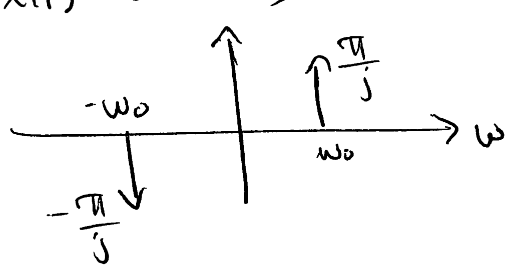
by argument above

$$= \frac{1}{2}\left[2\pi\,\delta(\omega_0-\omega) + 2\pi\,\delta(\omega_0+\omega)\right)$$

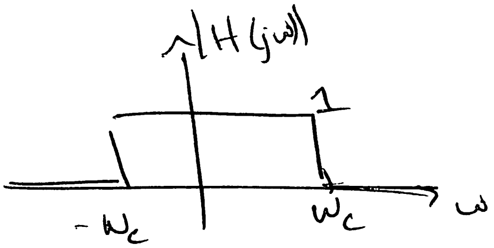
$x(t) = \cos(\omega_0 t) \rightarrow$



What about $x(t) = \sin(\omega_0 t)$



Ideal LPF



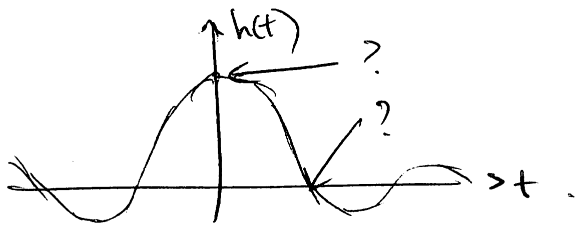
what happens if you push cos / sin through?

if $\omega_0 < \omega_c$, come through uneffected.

it $\omega_0 > \omega_c$, zero out.

What is $h(t)$ for ideal LPF?

Know from duality?



Hard to get from a physical system!

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}\,d\omega.$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{1}{jt}e^{j\omega t}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\frac{1}{jt}\left[e^{j\omega_c t} - e^{-j\omega_c t}\right] = \frac{\sin(\omega_c t)}{\pi t}.$$

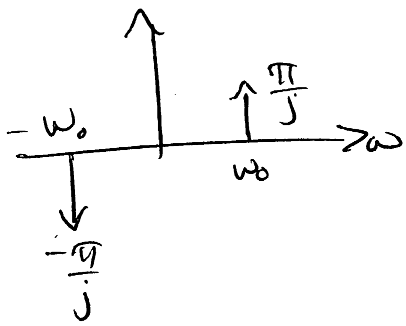
$h(t) = 0$ for $t = k\frac{\pi}{\omega_c}$ Integer k, $k \neq 0$.

$h(0) = \frac{\omega_c}{\pi}$ $\left[\text{at zero. } \sin(\omega_c t) \approx \omega_c t \Rightarrow \frac{\sin(\omega_c t)}{\pi t} \approx \frac{\omega_c \cancel{t}}{\pi \cancel{t}} = \frac{\omega_c}{\pi}.\right.$

or $\frac{1}{2\pi}$ · area in $H(j\omega)$.

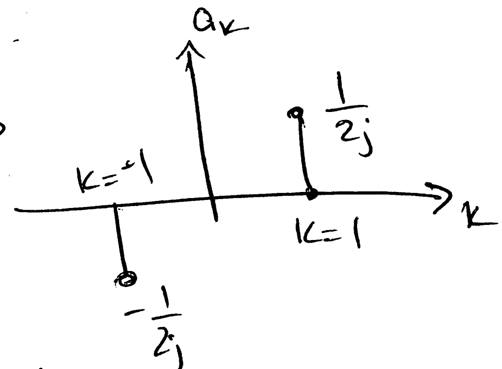
Something very cool happened. Did you notice.

$\sin(\omega_0 t) \longleftrightarrow$



Recall Fourier series.

$\sin(\omega_0 t) \rightarrow$



This is generally true.

Relationship to Fourier Series.

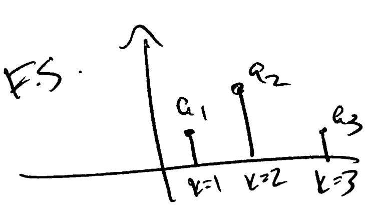
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} \, dt.$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{j(k\omega_0 - \omega)t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \, 2\pi \, \delta(k\omega_0 - \omega).$$

F.S.



F.T.